Tutorial Solutions

1   Revision and General Optics

All questions here are general background optics which will be assumed in this course. Problem 3 requires technical details of the microscope which most of you will require to look up in a standard textbook.

1.1   Simple Lens

Use geometric optics and Snell's law to show that for a thin lens of focal length f the object and image plane distances are related by
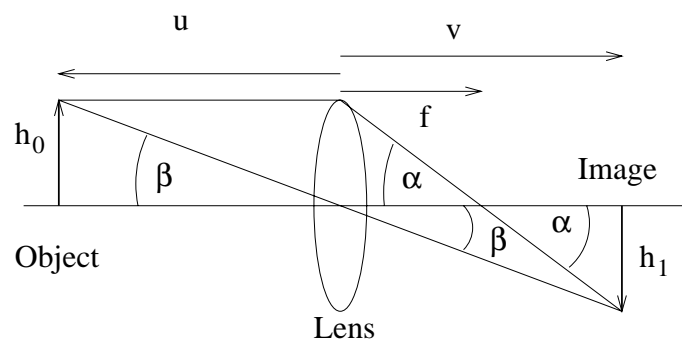
$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

If the lens is made of glass with refractive index n show that

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

where R_1 and R_2 are the radii of curvature of the lens surfaces.

Solution

Part A For a lens of focal length f, rays parallel to the optical axis will be focused a distance f behind the lens. Taking rays from an object we get,

Thus

$$\tan\alpha = \frac{h_1}{v - f} \quad \& \quad \tan\beta = \frac{h_0}{u} = \frac{h_1}{v}$$

so that

$$h_0 = \frac{h_1 f}{v - f} = \frac{h_1 u}{v} \quad \Rightarrow \quad \frac{h_0}{h_1} = \frac{u}{v}$$

This is the **magnification** of the system. We then have that that

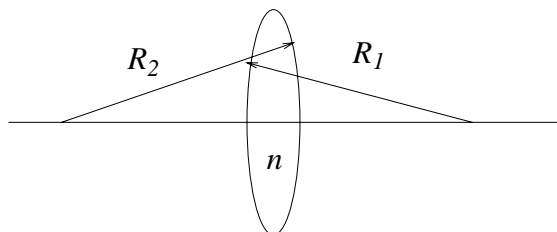$$\frac{f}{v - f} = \frac{u}{v} \Rightarrow vf = u(v - f)$$

Which can then be rearranged to give
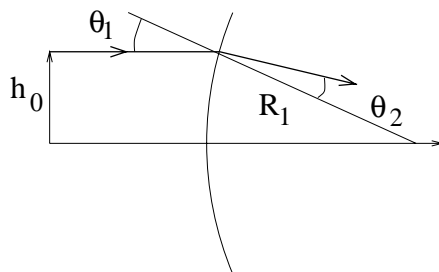
$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

the expected formula.

Note: This is the "real-is-positive" expression for a lens, see text-books for alternatives.

Part B Consider a lens with radii R_1 and R_2, made of glass with refractive index n,

Take a ray parallel to the optical axis at height h incident on the first surface.

If θ_1 is *small* then:

$$\theta_1 \approx \frac{h_0}{R_1}$$

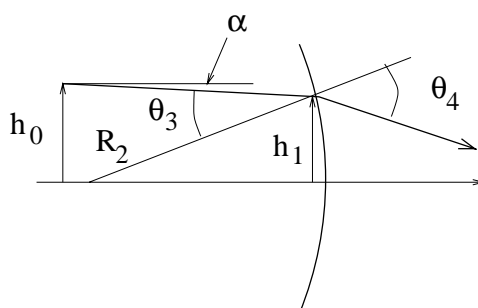From Snell's Law, we have that

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

where, in this case $n_1 = 1$ (air) and $n_2 = n$ (glass). Now, again if both θ_1 and θ_2 are *small* then

$$\theta_2 = \frac{1}{n}\theta_1$$

So the ray is deviated, downwards, by angle,

$$\alpha = \theta_1 - \theta_2 = \theta_1\left(1 - \frac{1}{n}\right) = \frac{h_0}{R_1}\left(1 - \frac{1}{n}\right)$$

This ray then falls on the second surface,

then, if the lens is *thin*, then $h_1 \approx h_0$, so that

$$\theta_3 \approx -\left(\frac{h_0}{R_2} + \alpha\right)$$

and from Snell's Law (noting that this is a glass→air interface),

$$\theta_4 = n\theta_3$$

So, on refraction, this ray is deviated by a further angle

$$\beta = \theta_3 - \theta_4 = (1-n)\theta_3 = -(1-n)\left(\frac{h_0}{R_2} + \alpha\right)$$

so while passing through the lens, the ray is deviated by the total angle

$$\begin{aligned}\gamma &= \alpha + \beta \\ &= (n-1)\left[\frac{h_0}{R_1} + \frac{h_0}{R_2}\right]\end{aligned}$$

A ray parallel to the optical axis is deviated by and angle γ, this will intersect the optical axis is distance f beyond the lens, where

$$\gamma \approx \frac{h_0}{f}$$

where, from Part A, f is the focal length of the lens. So we have that:

$$\frac{h_0}{f} = (n-1)\left[\frac{h_0}{R_1} + \frac{h_0}{R_2}\right]$$

so that the focal length f is given by:

$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} + \frac{1}{R_2}\right]$$

which is known as the *Lens Maker's Formula* which we will derive again from scalar diffraction theory in Lecture 4.

1.2   Binocular, and Telescopes

You are asked by a horse racing punter to supply them with a pair of binoculars that will allow them to read the starting prices from the far side of Musselburgh race course, which is approximately 400 m across. The starting prices are written in chalk in approximately 10 cm high letters. Specify a suitable magnification, and sketch a possible optical system for the binoculars.

Hint: Most people can easily obtain the minumum standard of vision required to drive a car (read a numberplate at 25 yards). The optical system in a binocular is similar to a telescope, but it is folded with a prism(s) that also re-inverts the image.

Solution

If you have a driving licence you must be able to read standard car number plate at 25 yards (22.75 m). These letters are 75 mm high so a reasonable estimate of the required angular size of readable letters is:

$$\Delta\theta = 3.3 \times 10^{-3}\text{Rad}$$

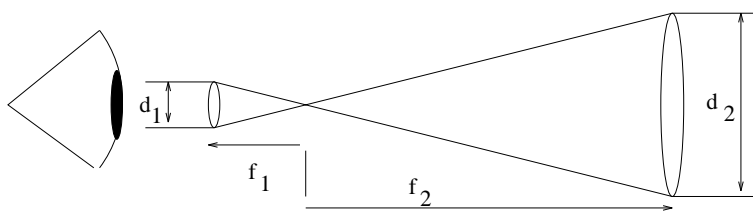We will compare this with the diffraction limited resolution of the eye later in the course.

In this case, the letters are 10 cm high at a distance of 400 m, so their angular extent is

$$\delta\theta = 2.5 \times 10^{-4}\text{Rad}$$

to in order to see the letters easily, we need an optical system with a magnification of

$$\frac{\Delta\theta}{\delta\theta} = 13.2\times$$

The optical system in a binocular is just a folded telescope, so take the system of:

For a telescope, the angular magnification

$$M = \frac{f_2}{f_1}$$

so by similar triangles, the lens sizes are also given by:

$$M = \frac{d_2}{d_1}$$

We need to match the d_1 to the size of the pupil of the eye. In normal daylight the pupil is $2 \to 4$ mm in diameter and to make the binoculars easy to use we require that $d_1 >$ pupil diameter. $d_1 = 4$ mm would be typical.

Want to keep the optics simple (and thus cheap), so we f_1 must not be too short. For example if

$$f_1 \approx 4d_1 \quad \text{F}_{\text{No}} = 4$$

then we would be able to use a simple doublet lens for the eyepiece.

Aside: if $f_1 < 16$ mm then a more complex optical system would be required to prevent lens aberrations making the system unusable. This is discussed in more detail in Lecture 8.

So for a magnification of $13.2\times$ we require

$$f_2 = 211\text{mm} \quad \text{and} \quad d_2 = 53\text{mm}$$

The binocular system is "folded" by use of a pair of silvered prisms. This has two effects, a) re-inverts the image so it is the correct way-up when viewed, b) reduces the length of the system to make it easier to hold.
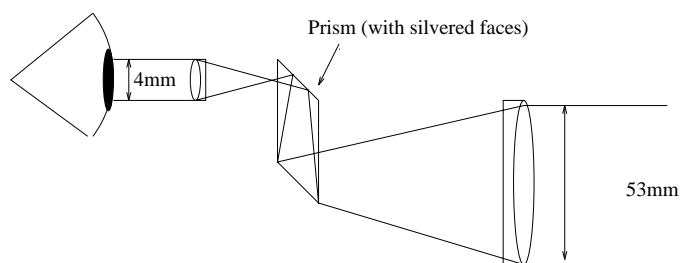
This would be a physically big and fairly expensive pair pair of binoculars (see below).

(Can also get "roof-prism" style binoculars that have the eyepiece and front objective in-line. These use a double dove prism and are more compact and rugged, however the optics are more complex and they are very much more expensive to make.)

Binoculars are specified by

$$Magnification \times Objective\ size\ in\ mm$$

Typical systems are

6×20	*Compact "pocket" binocular*
8×30	*Standard "Field Glasses"*
7×50	*Low light level binoculars*
10×40	*Bird watchers binocular*
12×50	*Horse racing / viewing sport*
20×72	*Bridge of ship or observation post*

A $10\times$ binocular is the highest magnification that you can easily hand hold. For larger magnifications you need a support to keep then still. Note: small movements in the binocular are also magnified.

1.3 Microscopes

A microscope objective is marked $\times 20, 0.35\text{NA}$. Explain what these terms mean. What is the focal length of this objective. Sketch an optical system to give a $\times 300$ visual magnification, and calculate the required focal length of the eye piece.

Hint: Assume a "tube length" of 160 mm and a visual near point of 254 mm. You will need to look-up these terms in a standard optics book. The easiest explanation is in Hecht's *Optics* page 190(ish).

Solution

This solution contains a lot of background information about microscopes that it not part of the course but people involved in optics *should* really know about!

If an objective is marked "$\times 20$ .35 NA" this gives the magnification (and thus focal length), and effective diameter of the lens as follows:

Numerical Aperture, (NA): If we have a lens of focal length f and diameter d
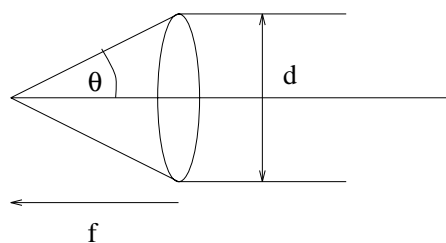
then the Numerical Aperture is defined as

$$NA = \sin\theta$$

So that NA = 0.35 means that the half-angle $\theta = 20.5°$, or the full exceptance angle of the objective $2\theta = 51°$.

For most good microscope objectives θ is **NOT** small, and good quality, high magnification objective can have NA ≈ 0.85 which gives $\theta \approx 30°$.

For very high magnification work the effective NA is increased by filling the space between the lens and the specimen with oil of refractive index n. The Numerical Aperture is then given by

$$NA = n\sin\theta$$

so with typical values of n $\approx$ 1.5 it is possible to have NA > 1. ("×100 1.25 NA oil" objectives are available at a price.)
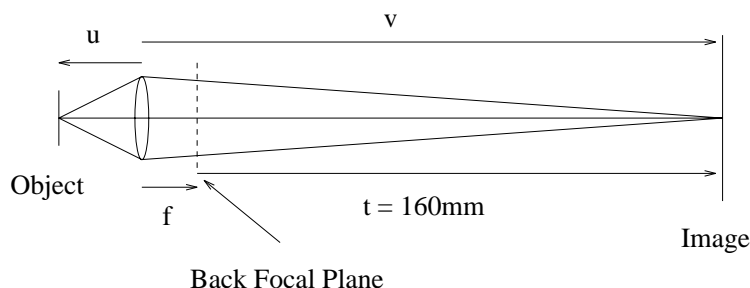
The alternative measure is F-Number F_{No}, which is defined by

$$F_{No} = \frac{f}{d}$$

if θ is *small* then we have the relation that

$$NA \approx \frac{1}{2F_{No}}$$

Magnification: defines the focal length when used in a *Standard Microscope* with a "tube length" of 160 mm.

The "tube length" is the distance from the back-focal-plane of the objective to the image plane. Magnification is

$$M = \frac{v}{u}$$

but we have that $t = v - f$, so from the lens formula

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f} \quad \Rightarrow \quad fv = u(v-f)$$

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so we have that

$$fv = ut \quad \Rightarrow \quad \frac{v}{u} = \frac{t}{f} = M$$

so for a $\times 20$ objective and a 160 mm "tube-length" the focal length $f = 8$mm.

The use of a standard "tube-length" allows microscope objectives to be moved between microscopes.

Most microscope objectives are multi-element systems, but they are all designed with a standard location for the back focal plane as shown:
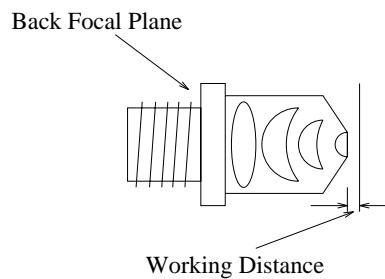
This allows the objective to be changed and the image to stay in focus. The other important parameter is "Working Distance" which is the physical distance from the object to the front element. This can be as small as 0.1 mm.

Microscope objectives come in a range of standard magnifications, these being:

Magnification	Typical NA	Working Distance	Focal Length
5		12 mm	32 mm
10		5 mm	16 mm
20		2 mm	8 mm
40	$0.45 \to 0.65$	0.5 mm	4 mm

Objectives with higher magnification and longer working distance for a given magnification *are* available but involve complex optics and tend to be very expensive.

Aside: The standard tube-length was a standard that has lasted for about 100 years, and is still used on most low and medium quality instruments. However the highest quality instruments have "infinity corrected" microscope objectives where the image plane is at infinity. This has been introduced to allow easy insertion of polarisers, beam-splitter, interference prisms etc., between the objective and the image without the need for re-focusing. However, just to make matters confusing, these objectives are still marked with the $\times$Mag notation, which is the "equivalent" magnification when combined with a suitable eyepiece. Objectives of this type will have the ∞ symbol beside the $\times$ and NA.

Simple Microscope: In addition to the objective, an eyepiece is added with acts like a "magnifier" on the image plane.
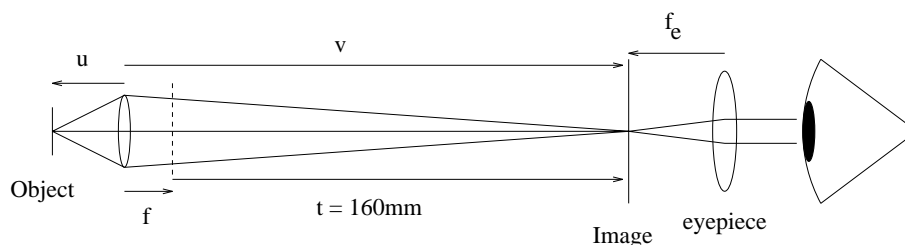
The total magnification of the system is now

$$M = M_{\text{obj}} \times M_{\text{eye}}$$

The M_{eye} is defined with respect to the "Visual Near Point" which is the closest most people can view an object with the naked eye. This is conventionally taken to be 10" (254 mm).

If we view a small object of size h from a distance D then the angular size of this object is

$$\theta_n = \frac{h}{D}$$

If this same object is viewed with an lens of focal length f_e
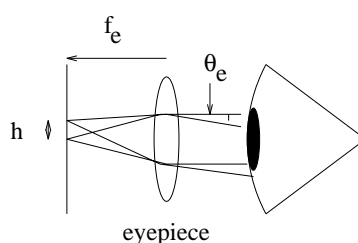
eyepiece

then the angular extent of the object is

$$\theta_e = \frac{h}{f_e}$$

The angular magnification, seen by the eye, is now

$$M_{\text{eye}} = \frac{\theta_e}{\theta_n} = \frac{D}{f_e}$$

where, by convention, we take $D = 254$ mm, and f_e is the focal length of the magnifier (or eyepiece).

So now if we have a objective and we want a $\times 300$ overall magnification we need

$$M_{\text{eye}} = 15 \quad \Rightarrow \quad f_e = 17\text{mm}$$

Usually eyepieces are multi-element systems to allow a wide field of view without the eye being too close to the lens, but the principle is the same.

1.4   Lens Power

Somebody who is myopic (short sighted), tells you that their prescription is -3.75D in the right eye and -4.50D in the left. What are the focal lengths of the their lenses?

Suggest a simple scheme to check that these focal lengths are right. (You may assume that you have a single positive lens of focal length 100 mm that you may wish to use).

Solution

Lens "power" is measured is Diopters, (D) and is defined as

$$D = \frac{1}{f}$$

where f is the focal length *in meters.* So that

$$-3.75\text{D} \Rightarrow f = -26.67\text{cm}$$
$$-4.50\text{D} \Rightarrow f = -22.2\text{cm}$$

by convention, ophthalmic lenses are quoted in Diopters specified to the nearest 0.25 D. This is because 0.25 D is the smallest change that can be perceived by most people.

Aside: many people require a more complex lens that includes a cylinderical component to correct aberrations in the eye.

Both of these lenses are concave, (negative). The typical shape of such a lens is:
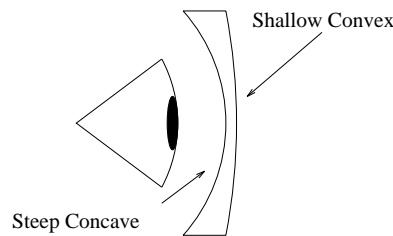
This shape allows the lens to be closer to the eye, so extending the field of view. Also it "looks better" since the thick edge of the lens is hidden behind the frame.

A quick test of a positive lens is to form the image of a distant object, which will be located at the focal length of the lens.
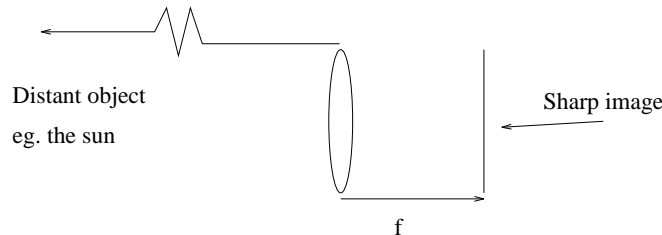
but this will not work with negative lens since you will get a virtual image.

If you have a positive lens of know focal length, in this case $f = 100$ mm, so power is 10 D. If two lenses are placed together then the "powers-add"
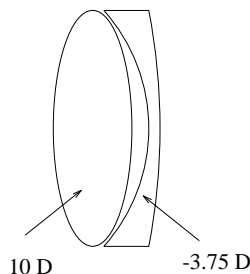
so the effective power of these two lenses is 6.25 D, so the focal length $f = 16$cm.

Similarly with the -4.5D lens, the effect power is 5.50D so the focal length is $f = 18.18$cm.

You can then measure the focal length of the combinations using a distant object as described above.

This is a rather crude method. The power (or focal length), of a lens is usually measured by and imaging instrument called a *phosometer.*

1.5   A Tricky Puzzle

A metal plate with a hole on unknown diameter is placed at one end of an optical bench and a screen at the other. The metal plate is illuminated from behind with a ordinary lightbulb and a lens is used to form a sharp image on the screen. The diameter of the image is 4.5 cm. The lens is then moved towards the screen until a scond sharp image is formed, this time of 2.0 cm in diameter. What is the size of the hole in the metal plate.

Hint: Draw out the two conditions for a sharp image and remember that object and image are interchangable. Yes there *is* enough information to solve this.

Solution

Assume that the radius of the hole in the plate is h_0, so the first condition for imaging is,
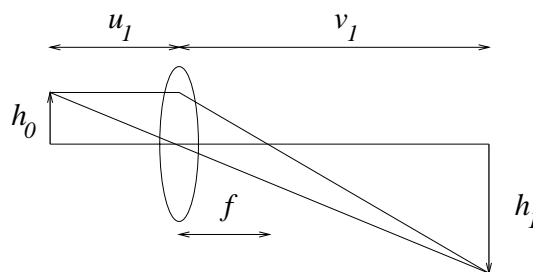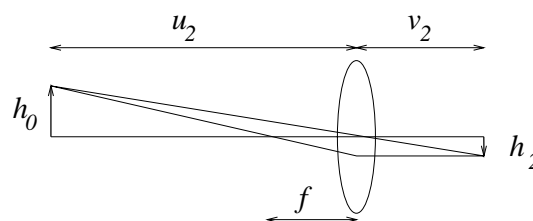

where we have that

$$\frac{1}{u_1} + \frac{1}{v_1} = \frac{1}{f}$$

In this configuration the the object size is:

$$h_0 = \frac{h_1 u_1}{v_1}$$

where $h_1 = 2.25\,\mathrm{cm}$, being the radius of the first sharp image.

We now move the lens towards the screen until we get the second sharp image, which will be at,

where now we have that

$$\frac{1}{u_2} + \frac{1}{v_2} = \frac{1}{f}$$

In this configuration the object size is,

$$h_0 = \frac{h_2 u_2}{v_2}$$

where $h_2 = 1.0\,\mathrm{cm}$, being the radius of the second sharp image.

Now if we interchange object and image in the second configuration (all rays paths are reversible), then we see that,

$$v_2 = u_1 \quad \text{and} \quad u_1 = v_2$$

If we now substitute for u_2 and v_2 is the above expression for h_0 we get the relation that,

$$h_0 = \frac{h_1 u_1}{v_1} = \frac{h_2 v_1}{u_1}$$

so giving that,

$$h_1 u_1^2 = h_2 v_1^2 \quad \text{and thus} \quad \frac{h_1}{h_2} = \frac{v_1^2}{u_1^2}$$

We now take the square root of both sides and invert to get,

$$\frac{u_1}{v_1} = \sqrt{\frac{h_2}{h_1}}$$

which is now just the inverse of the magnification of the first system!

Substitute this into the first expression for h_0 to get

$$h_0 = \sqrt{\frac{h_2}{h_1}} h_1 = \sqrt{h_1 h_2}$$

Simply put in the values of 9/4 and 1 for h_1 and h_2 being the radius of the two images, to get

$$h_0 = 1.5 \text{cm}$$

which is the radius of the object, so the diameter is 3 cm.