

Topic 4: Dispersion

4.1 Introduction

Dispersion is where different wavelengths of light take different paths through an optical system usually resulting in polychromatic light being split into its *colours*. There are two physical mechanism for this, being:

1. **Material Dispersion:** where the refractive index of a material depends on the wavelength,
2. **Diffractive Dispersion:** where light is diffracted from grating structure, here the angle of diffraction depends directly on the wavelength of the light.

In this topic, we will consider the first of these, with *Diffractive Dispersion* being considered as part of Topic 7 and 8.

Optical system usually operator over the range:

Colour	Wavelength
Ultra-violet	350 nm
Violet	400 nm
Blue	440 nm
Cyan	490 nm
Green	540 nm
Yellow	580 nm
Red	630 nm
Deep Red	690 nm
Near infra-red	to 1 μm

being the *visible spectrum* (400 \rightarrow 700 nm) plus regions into the long ultra-violet and the near infra-red. This is the regions over what most optical glasses are transparent¹.



We will treat *material dispersion* from a largely qualitative and experimental view. The underlying theory involves the interaction of electromagnetic waves with materials which involves a level of electromagnetic theory that is beyond what is expect in this course.

4.2 Experimental Evidence for Material Dispersion

We have considerable experimental evidence that the refractive index of materials *does* depend on wavelength, for example

1. A prism forms a spectrum with blue light *bend* more than red, as shown in figure 1,
2. The *rainbow*², where water droplet act as spherical lenses and break Sun light into its constituent colours.

¹A wider range of wavelengths in possible, but in involves use of rather exotic crystalline materials

²Full analysis of the rainbow is rather involves, see HYPERPHYSICS web site for a good explanation.



Figure 1: Light being dispersed by a glass prism



Figure 2: A rainbow

This *dispersion* can be useful, for example is a spectrometer to analysis light, or be a severe practical problem, for example is the design of lenses, where, for a simple lens with radii of curvatures R_1 and R_2 we have that

$$\frac{1}{f} = (n - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

so clearly if the refractive index n varies with wavelength, then so does the focal length!



It turns out that everything *except* a vacuum, is dispersive, and in particular the refractive index of *all* optical materials (glass and plastic) varies with wavelength.

4.3 Abbe Number

The simplest characterisation of dispersion is the Abbe, or V_d number of a material. This is experimentally measure by determining the refractive index at three standard wavelengths³, these being:

³The use of these three wavelengths is historical, in practice the Helium d-line is often replaced by the Sodium D-line at 589.3 nm.

Line	Wavelength	Colour	Index
Helium d line	587.6 nm	Yellow	n_d
Hydrogen F Line	486.1 nm	Blue-Green	n_F
Hydrogen C Line	656.3 nm	Red	n_C

The dispersion is then characterised by

$$V_d = \frac{n_d - 1}{n_F - n_C}$$



For all practical optical materials $n_F > n_C$ so V_d is always positive, and for typical glasses with, typically,

Index	Abbe Number
$n_d < 1.6$	$V_d > 50$
$n_d > 1.6$	$V_d < 50$

Note the *larger* V_d is the *less* dispersive the material is, so the smaller is the change in refractive index with wavelength.

There are actually many hundreds⁴ of glasses with

$$\begin{aligned} n_d &= 1.49 \rightarrow 2.0 \\ V_d &= 85 \rightarrow 20 \end{aligned}$$

with the general rule that low refractive index glasses have *high* Abbe numbers, so being having *low* dispersion, while high refractive index glasses have *high* dispersion, so *low* Abbe numbers. A plot of class types is shown in figure 3 showing the distribution of refractive index against Abbe number.

Glass type can be roughly classified into *crown* (low index) and *flint* with the two most common being.

Type	Name	n_d	V_d
Borosilicate Crown	BK7	1.51680	64.29
Dense Flint	F4	1.61659	36.63

The rather old, an highly empirical scheme has lasted to the present day with glasses catalogued by a 6 figure number being

$$nnnVVV \quad \text{or} \quad nnn : VVV$$

where

$$\begin{aligned} nnn &\rightarrow 3 \text{ significant digits of } n_d - 1 \\ VVV &\rightarrow 3 \text{ significant digits of } V_d\text{-number} \end{aligned}$$

so BK7 is also known as type 517643 and F4 as type 617366.

⁴The Schott's glass catalogue contains more than 100 glass types, other manufacturers offer similar numbers.

image showing this effect. This effect occurs because the *Chromatic Aberration* in the eye results in the *red* and *blue* light being focused in different planes so they appear to be at different distances.

The effect is most often seen as *colour fringing* in imaging system where the object and image distance are fixed, as shown in figure 4. A bright spot in the object is imaged as a *spectrum* in the image plane due to the focal length of the imaging lens being dependent on wavelength. This effect is known as *Transverse Chromatic Aberration* since the rays are moved laterally in the imaging plane. This effect is most obvious is a poorly aligned and setup overhead projector!

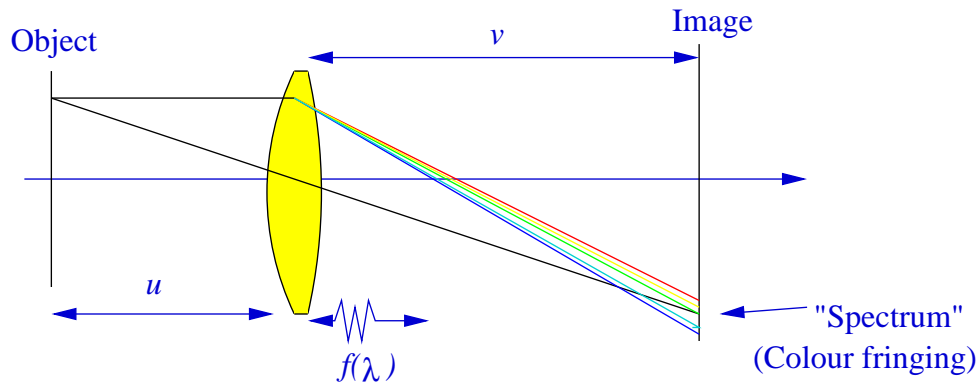


Figure 4: Transverse Chromatic Aberration in an imaging system.

Chromatic aberration is largest of the optical aberrations, so we will look at dealing with this in some detail.

4.5 The Achromatic Doublet



The key to dealing with all aberration in optical systems is to use multiple components with one *canceling* the aberrations of the other while still producing the required imaging properties. Here consider the simplest case to two thin lenses of power p_1 and p_2 in contact with each other at the d-line wavelength, then the total power is,

$$p = p_1 + p_2$$

If the power change between F-line and C-line light for each lens is δp_1 and δp_2 respectively, the power change of the pair is just,

$$\delta p = \delta p_1 + \delta p_2$$

which, from above, is given by

$$\delta p = \frac{p_1}{V_1} + \frac{p_2}{V_2}$$

where V_1 and V_2 are the V_d numbers for the glass from which each lens is made.

So if V_1 and V_2 are different, we can use the dispersion of the second lens to *cancel* the dispersion of the first, by setting

$$p_1 V_2 = -p_2 V_1 \quad \text{so that} \quad p_2 = -\frac{p_1 V_2}{V_1}$$

Note that p_2 is *negative*, so we need a

Low Dispersion Positive Lens + High Dispersion Negative Lens



combination to forms the zero dispersion doublet.

So if we want a lens of power p , then if the V_d numbers of the glasses are V_1 and V_2 the power of the two lenses to give $\delta p = 0$ are,

$$p_1 = \frac{V_1}{V_1 - V_2} p \quad \text{and} \quad p_2 = \frac{V_2}{V_2 - V_1} p$$

Note the same formula apply if we want $p < 0$, so the double has a negative focal length.

Example: if we want a $p = 10\text{D}$ ($f = 100\text{mm}$) and we use BK7 crown glass and F4 flint glass, we get that

V_d	Power (D)	Focal Length
64.29	23.24	43.03 mm
36.63	-13.24	-75.53 mm

There are many schemes for making a doublet, but by far the most common is the cemented doublet shown in figure 5. Here the positive and negative lenses are cemented together and share a common central radius of curvature.

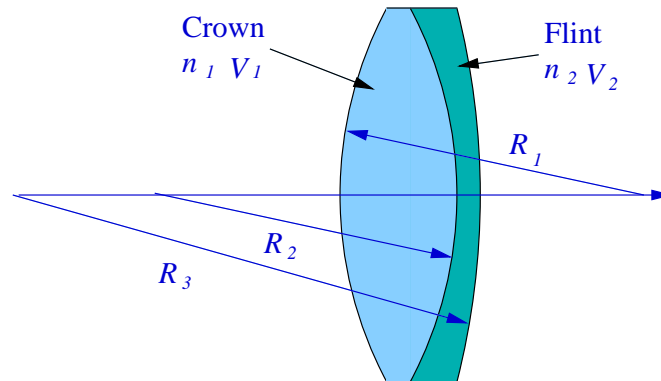


Figure 5: The cemented doublet of positive crown glass lens cemented to negative flint glass lens

This scheme is mechanically stable with the gap between the lenses being filled with an optical glue with the same refractive index as one of the lenses⁵, and so easy to fabricate. The simplest of all doublets is the Fraunhofer Doublet, where the crown glass positive lens is symmetric, so $R_1 = -R_2$, which for a given glasses, you can calculate all the required radii, see question 4.2

This analysis is correct for *thin lenses* where we have ignored the thickness of each component of the doublet. Doing any further analytic analysis results in very messy expressions that do not extract any more physics, such analysis is much better done numerically, for example by the matrix ray techniques discussed in the last section where the system is analysed for different wavelengths.

⁵There is a corresponding matched glue available for almost every available type of glass.

4.6 Cauchy Formula for Dispersion


What is really needed for analysis of dispersion is an analytic expression for refractive index against wavelength. The simplest, and surprisingly successful, is the empirical Cauchy's relationship, that

$$n(\lambda) = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \frac{D}{\lambda^6} + \dots$$

where A, B, C, D, \dots are experimentally determined constants.

In practice, Cauchy's relationship is usually truncated to three terms, giving

$$n(\lambda) = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4}$$

 which can be fitted by accurately measuring the refractive index at three wavelengths. For useful optical materials, this gives an accuracy of typically ± 1 in the fourth decimal place, over the visible spectrum of $400 \rightarrow 700$ nm which is good enough for all but the highest quality optical systems.

The previous measures of n_d and V_d constitute *two* measures and can be used to form the reduced form of Cauchy's relationship, that


$$n(\lambda) = A + \frac{B}{\lambda^2}$$

where A and B can be expressed in terms of n_d and V_d , see question 4.3. Even this reduced form will typically give an accuracy of ± 1 in the third decimal place, so good enough for most system.

4.7 Beyond Cauchy's Formula

Cauchy's Formula is an empirical fit to the refractive index within the visible region and while very useful, does not explain any of the underlying physics. A much better model comes from considering the interaction with electro-magnetic waves with a material where the material is modeled as atoms surrounded by an electron cloud which is displaced by the presence of the electric field associated with the wave. If we assume that the material is *lossless*, then from this model *it can be shown*⁶ that the refractive index of such a media varies as

$$n^2(\omega) = 1 + \frac{Ne^2}{m_e(\omega_0^2 - \omega^2)}$$

 where ω is the angular frequency of the electric field, ω_0 is the natural angular frequency of the electron cloud, m_e is the effective mass of the electron in the cloud, N is the number of atoms per unit volume, and e is the charge on the electron.

We can express this in terms of λ , by substitution of

$$\lambda = \frac{c}{2\pi\omega} \quad \text{and} \quad \lambda_0 = \frac{c}{2\pi\omega_0}$$

which, after a bit or re-arrangement, gives that

$$n^2(\lambda) = 1 + \frac{a\lambda^2}{\lambda^2 - \lambda_0^2}$$

⁶The derivation is included in the notes, but is not part of this course. The background electromagnetism used in this derivation will be covered towards the end of the concurrent Electromagnetism course.

where

$$a = \frac{4\pi^2 N e^2 \lambda_0^2}{c^2 m_e}$$

This is known as Sellmeier's relation⁷.

The model treats the electron cloud as a lightly damped simple harmonic oscillator with natural frequency ω_0 . The resonant frequency will therefore be ω_0 , so such as oscillator will strongly absorb radiation with frequency ω_0 . This identifies λ_0 is the as the wavelength at which light is absorbed. Figure 6 shows a least square fit $n(\lambda)$ to the measured refractive index values for BK7 glass at the λ_F , λ_d and λ_C lines which gives the value of $a = 1.2646$ and $\lambda_0 = 0.097621 \mu\text{m}$. This suggest an absorption about 100 nm, which is far into the ultra-violet region of the spec-

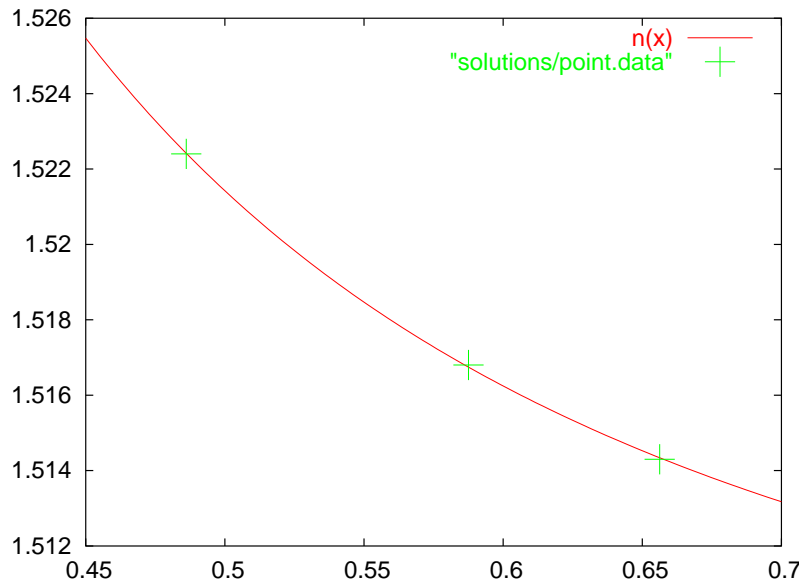


Figure 6: Least squares of Sellmeier relations to the measured refractive index values for BK7

trum. Experimentally this is exactly what you get with different optical materials having strong absorption in the ultra-violet, with the higher index, and higher dispersion material having absorptions at longer λ_0 , in the nearer ultra-violet.

This model predicts that

1. Optical materials will absorb in the ultra-violet.
2. High refractive index materials tend to have high dispersion.
3. Very high refractive index materials tend to absorb at in the blue/ultra-violet part of the spectrum. Such materials tend to be yellow in colour due to absorption of blue light.
4. Absorption limits the maximum practical refractive index, usually to ≈ 2.0 , diamond is the highest optical material with $n \approx 2.3$.

All these observations fit optical materials very well.

⁷This was postulated empirically by Sellmeier, the electromagnetic argument was formulated by Lorentz.

This basic theory assumes that there is only one absorption wavelength, but can easily be extended to include multiple absorption wavelengths at λ_0 , λ_1 , etc. In this case the refractive index become,

$$n^2(\lambda) = 1 + \sum_{i=0} \frac{a_i \lambda^2}{\lambda^2 - \lambda_i^2}$$

This model extends the useful range of the dispersion formula well into the near ultra-violet and near infra-red where Cauchy's relation breaks down.

This is the physical basis for the dispersion formula used by many optical glass manufacturer, which is,

$$n^2(\lambda) = 1 + \frac{B_1 \lambda^2}{\lambda^2 - C_1} + \frac{B_2 \lambda^2}{\lambda^2 - C_2} + \frac{B_3 \lambda^2}{\lambda^2 - C_3}$$

where B_i and C_i are experimentally determined constants. Such a fit typically gives six significant figures of accuracy in the refractive index over the region of $200 \text{ nm} \rightarrow 2.3 \mu\text{m}$ ⁸. See question 4.5 for the standard method of accurately measuring refractive index.

The manufactures of optical materials will typically characterised their glasses by quoting accurately fitted values for:

1. n_d and V_d refractive index and Abbe number measure at the helium d-line, 578.56 nm.
2. n_e and V_e refractive index and Abbe number measure at the mercury e-line, 546.07 nm.
3. $B_1 \rightarrow B_3$ and $C_1 \rightarrow C_3$ for the above dispersion formula where λ is in microns.
4. Many other parameters including transmission at various wavelengths, thermal characteristics and mechanical characteristics.

This allow you to calculate the refractive index over the region which the material is transparent. The transmission for 25 mm thick of three glasses, N-BK7 ($n_d = 1.51$), F4 ($n_d = 1.61$) and LASFN9 ($n_d = 1.85$), are shown in figure 7 for the range $300 \rightarrow 1000 \text{ nm}$. N-BK7 and F4 shown high transmission over the visible range, but LASFN9 has considerable absorption for blue light and would appear strongly *yellow* so would not be suitable for system with light shorted than about 580 nm. All three of these glasses shown good transmission well into the infra-red. .

4.8 Ray Calculation for Different Wavelength

Now that we have an numerical expression for the variation of refractive index with wavelength we can apply the *ray tracing* methods seen in the last topic as a range of wavelengths.

The *matrix methods* allow us to analyse focal lengths and principal plane location, see question 4.7 for a numerical example. We will consider this again in the next topic when we look at optical systems.

When combined with full *ray tracing* using the vector form of Snell's Law we get a fully geometric optical analysis of a system at the required wavelength. For *white light* systems this usually means tracing rays at a range of wavelengths and combining the results. For systems where the output is designed to be viewed by the eye, the usual wavelength are *red* (650 nm), *green* (550 nm) and *blue* (450 nm), covering the whole visible spectrum.

⁸For example Schott's glass catalogue.

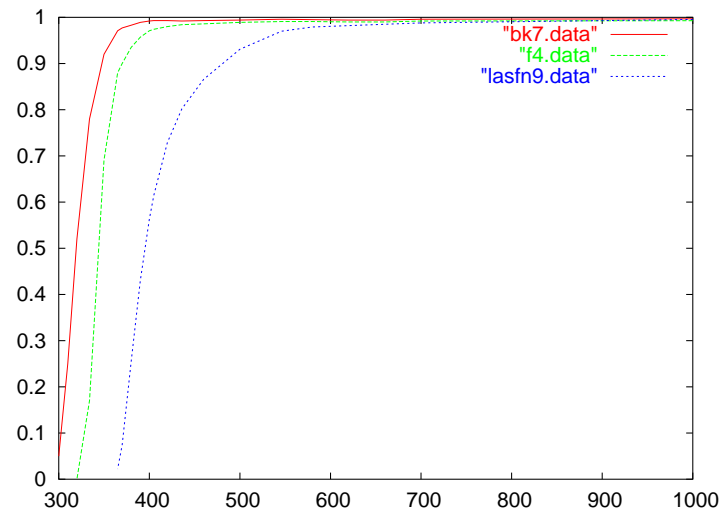


Figure 7: Transmission against wavelength plot for 25 mm thick sample for Schott's N-BK7, F4 and LASFN9 glass.

4.9 Optical Materials

The traditional optical material has been glass having the properties that:

Advantages of Glass:

- available is a huge range of refractive index and dispersion combinations, many hundreds,
- high transmission across wide range of wavelength,
- easy to polish into spherical surfaces so easy to *mass produce*,
- refractive index changes with temperature are *very* small, can almost always be ignored,
- mechanically stable, very low thermal expansion coefficients, different glasses can be glued together to form *doublets*,
- can stand very high temperature processing, for example to apply surface coatings,
- surfaces are *hard* being both scratch and chemical resistant.

Problems with Glass:

- very high mass, especially high index glasses which are doped with lead,
- difficult to make *aspherical* surfaces, expensive to polish and difficult to machine to a good surface,
- relatively easily breakable and tends to *shatter*.

Given these characteristics glass is used in 95% of high quality optical systems and instruments. There are a large number of pre-made components and a huge number of *standard designs* based on glass components.

The alternative material is plastic, either perspex or polycarbonate, which looks an excellent alternative.

Advantages of Plastic:

- very low mass, especially for high refractive index materials,
- can be *injection molded* to give very cheap high volume low quality components,
- can be easily machined to aspherical surfaces,
- almost *unbreakable*

however when we look further, plastics are less attractive.

Disadvantages of Plastic

- limited range of refractive index and dispersion combinations,
- high refractive index material, ($n > 1.7$) are *very* expensive,
- lower light transmission and limited wavelength transmission range,
- significant refractive index changes with temperature, factor of 100 more than with glass,
- not mechanically stable, large thermal expansions make *doublets* impossible to make,
- no high temperature processing possible, makes applying surface coating much more complex and expensive,
- surfaces *soft*, easily damaged by abrasion and chemicals,
- finite lifetime especially if exposed to ultra-violet light; *goes yellow with age*.

Given these characteristics, plastics are only used low quality optical system that use very cheap injection molded plastic components, such as *disposable camera*, or in systems where weight and safety is critical, for example spectacle lenses, hand held magnifier, head mounted viewers. In these systems optical quality comparable to glass is possible, but at a very much greater cost.

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Other Optical Materials

Glass or plastic are the obvious choices in the *visible* and *near infra-red* regions where these materials are transparent.

1. **Ultra-violet** no glasses or plastics are transparent below about 350 nm. Need to use synthetic fused silica to 290 nm and synthetic sapphire to about 200 nm. Shorter than this all components must be reflective. Very restrictive range of materials all with low refractive index. Very expensive.
2. **Infra-Red** most glasses highly transparent to beyond $\approx 2\mu\text{m}$. For longer wavelengths we need more exotic materials, mainly semi-conductors that are *not* transparent in the visible region. There are two standard wavelength regions
 - (a) $3 \rightarrow 5\mu\text{m}$ covered by Barium Fluoride and Silicon

(b) $8 \rightarrow 12\mu\text{m}$ covered by Zinc Selenide, Zinc Sulphide and Germanium.

all these materials are very expensive and difficult to manufacture. They also have *very* high refractive indexes ≈ 4 for materials in the useful $8 \rightarrow 12\mu\text{m}$ thermal infrared region.

4.10 Summary

In this section we have:

1. Looked at the experimental evidents for dispersion,
2. Discussed the empirical measure of Abbe Number, and how this this leads to a simple analysis of a thin singlet lens, resulting in *chromatic aberration*,
3. Used Abbe Number to design a simple *doublet* lens,
4. Discussed the empirical Cauchy formula for dispersion,
5. Outlined the physical model that lies behind dispersion, and how this leads to Sellmeier's Relation, and then on to the numerical relations used to characterise glass,
6. compared and contrasted the optical and mechanical properties of glass and plastic as practical optical materials.

Workshop Questions

4.1 Imaging with a Flint Singlet

A single lens made of dense flint type F4, with $V_d = 36.63$ and focal length 100mm at the d-line, is used to image an object containing red and blue regions placed 120mm in-front of the lens. Calculate distance between the planes where the red and blue part of the object are imaged and comment on you answer. Assume the blue and the red objects are at the F and C line respectively.

Hint You will need to make some approximations to solve this.



4.2 Fraunhofer Doublet

A Fraunhofer Doublet consists of a cemented double of crown and flint glass with the crown element being symmetric. If such a doublet is make from a combination of BK7 crown and F4 flint glass, calculate the curvatures and sketch the shape of the doublet with a focal length of 500 mm.



4.3 Cauchy Formula in practice

For the two parameters Cauchy's formula for refractive index given by

$$n(\lambda) = A + \frac{B}{\lambda^2}$$

derive expressions for A and B in terms of n_d and V_d .

Use the values for BK7 quoted in Guenther table 7A.2 of

$$\begin{aligned} n_d &= 1.51680 & V_d &= 64.17 \\ n_C &= 1.51432 & n_F - n_C &= 0.008054 \end{aligned}$$

to check,graphically, how good the Cauchy's formula fits these points.



4.4 Simplification of Sellmeier's Relation

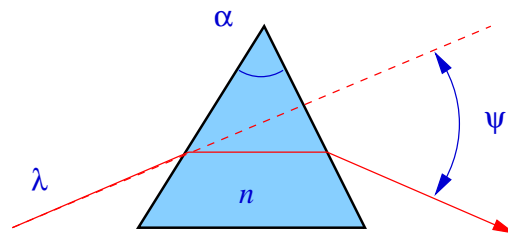
Show that the two parameter Cauchy's Formula for dispersion given by

$$n(\lambda) = A + \frac{B}{\lambda^2}$$

is an simplification of Sellmeier's relation, and hence derive an expression for λ_0 , the absorption wavelength for the material in terms of the two Cauchy constants A and B .

4.5 Measuring Refractive Index

The most accurate scheme for measuring refractive index is to make an accurate isosceles prism from the material and is measure the *angle of minimum deviation* against wavelength. The condition for *minimum deviation* occurs in the geometry as shown below,



where the ray path through the prism is parallel with the base.

Derive an expression for the refractive index n in terms of the angle of minimum deviation ψ and α the prism angle.

If you have spectrometer that can measure angles to an accuracy of ± 1 minute of arc, to what accuracy can you determine n for a given wavelength using an equilateral prism of glass with refractive index ≈ 1.6 . How can this accuracy be improved?

4.6 Imaging with a Flint Singlet II

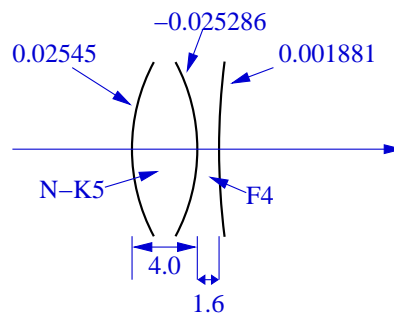
Repeat question 4.1 numerically using Cauchy's Formula for the refractive index relation and the values of n_d and V_d for F4 dense flint glass given in the notes.

You should either use the supplied JAVA classes and Ray Matrix methods or a spreadsheet.



4.7 Focal Length of a Doublet

The details of an achromatic doublet are shown below,



where the curvatures and distances are in mm and the glass parameters are given in the SCHOTT'S GLASS TABLES linked from the web page.

Write a JAVA program, using the Ray Matrix methods and the REFRACTIVE INDEX methods, to plot the focal length of this doublet over the range $0.38 \rightarrow 780 \mu\text{m}$ and comment on the result.

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Forced Oscillator Dispersion

The section is non-examinable and is included for completeness.

If we consider the outer electron cloud round an atom in the presence of an electric field \mathbf{E} as shown in figure 8 The electron clouds will be displaced by an amount \mathbf{l} which is equivalent to a induced dipole of strength

$$\mathbf{p} = q\mathbf{l}$$

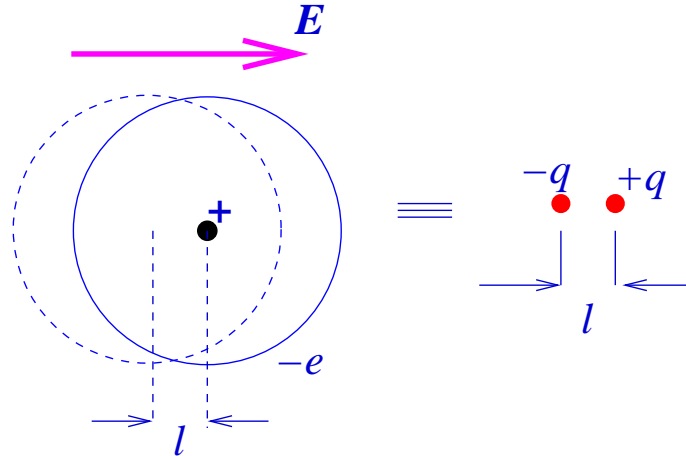


Figure 8: Electron cloud displaced by electric field.

If in the material there are N identical atoms per unit volume, then the total induced dipole (per unit volume) is

$$\mathbf{P} = N\mathbf{p}$$

The material is *linear* if this induced dipole is linearly proportional to the electric field. If we can assume that we can write,

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E}$$

where χ is called the Polarisability of the material, and \mathbf{P} is known as the *Dielectric Polarisation*.

Consider the x direction only; we apply an electric field of angular frequency ω , then

$$E_x(t) = E_0 \sin(\omega t)$$

so the equation of motion of the electron (cloud) can be described by a forced, damped oscillator, where $E_x(t)$ is the forcing term. We get that

$$m_e \frac{d^2 x}{dt^2} = -eE_x - m_e \omega_0^2 x - 2m_e \gamma \frac{dx}{dt}$$

where,

- m_e = effective mass of electron
- ω_0 = Natural Frequency
- γ = Damping coefficient

This has a standard solution (See Physics 2A), that

$$x(t) = \frac{-eE_x(t)}{m_e(\omega_0^2 - \omega^2 + 2i\gamma\omega)}$$

This is for one electron, so for N electrons per unit volume,

$$P_x = -exN = \epsilon_0\chi E_x$$

so the polarisability is given by

$$\chi = \frac{Ne^2}{m_e(\omega_0^2 - \omega^2 + 2i\gamma\omega)}$$

For optically useful materials the γ is very small (low loss) and $\omega - \omega_0$ is large, so χ is (almost) real and a slow varying function of ω . (For all optical materials $\omega < \omega_0$).

Now with Maxwell's Equations in a media,

$$\begin{aligned}\nabla \cdot \mathbf{D} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \wedge \mathbf{B} &= \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \\ \nabla \wedge \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}\end{aligned}$$

but in a non-magnetic material, the current \mathbf{J} is the rate of change of the polarisation \mathbf{P} , so that

$$\mathbf{J} = \frac{\partial \mathbf{P}}{\partial t}$$

which gives us the wave equation in a media as

$$\nabla^2 \mathbf{E} - \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} = 0$$

Take plane in the z direction then we have the \mathbf{E} is constant in both x and y so we get,

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} - \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} = 0$$

so considering a single frequency ω , we can write

$$\mathbf{E} = E_z \exp(i\omega t) \quad \text{and} \quad \mathbf{P} = P_z \exp(i\omega t)$$

which for each component gives

$$\frac{\partial^2 E_z}{\partial z^2} + \mu_0 \epsilon_0 \omega^2 E_z + \mu_0 \omega^2 P_z = 0$$

So if we now substitute for P_z the polarisation, we get that

$$P_z = \epsilon_0 \chi E_z$$

giving

$$\frac{\partial^2 E_z}{\partial z^2} + \frac{\omega^2}{c^2} (1 + \chi) E_z = 0$$

Then if we try a solution of the form (a plane wave),

$$E_z = A(z) \exp(-i\kappa z)$$

then, if we can assume that χ is real only (sensible for optical material), then we get that

$$\kappa^2 = \frac{\omega^2}{c^2} (1 + \chi)$$

so we identify

$$(1 + \chi) = n^2$$

the refractive index of the material, so giving that

$$n(\omega) = \sqrt{1 + \chi(\omega)}$$

where for optical materials, γ is small and $\omega_0^2 - \omega^2$ is large, we get,

$$\chi(\omega) \approx \frac{Ne^2}{m_e (\omega_0^2 - \omega^2)}$$

where ω_0 is the natural (linear) frequency of the material dipole. This gives an expression for the refractive index

$$n^2(\omega) = 1 + \frac{Ne^2}{m_e (\omega_0^2 - \omega^2)}$$

For all practical optical materials $\omega < \omega_0$, so

$$n > 1 \quad \text{and} \quad n \text{ increases with } \omega$$

with typical shape of n is shown in figure 9 where $\omega_0 = 1000$ and other factors are chosen at random.

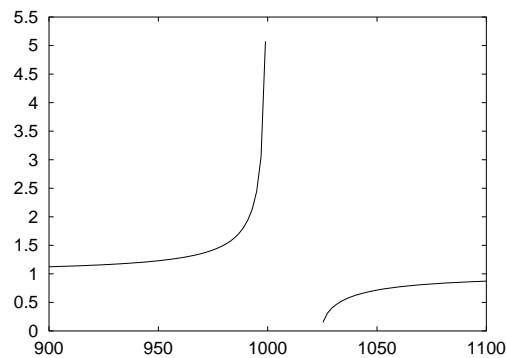


Figure 9: Plot of dispersion against frequency for $\omega_0 = 1000$