Lecture 7, 1st October 2012

Analogue Electronics 4: AC Circuits – capacitors and complex numbers

Electrical engineers work with complex numbers earlier in their degree programmes than physicists do – in this lecture you are going to begin to discover why.

Road to the frequency domain:

We are now going to *go back and repeat the previous lecture* – except this time we will use **complex numbers**. Although this does not make a huge difference for the differentiators and integrators described previously it does become increasingly important as the circuits become more complicated.

Starting with lecture 4 you have come into contact with the term **impedance**, mainly in contexts like *input impedance* and *output impedance*. It was mentioned that ohmic resistance is only one contribution to the impedance of a circuit. And it was mentioned that impedance describes the opposition of a circuit to the flow of a current for the **full frequency spectrum**. This is what we will deal with now for its ohmic, capacitive and inductive contributions.

Impedance & Reactance: (H&H, p. 29)

When an ohmic resistor is introduced into a circuit it has no effect on the time dependence of the voltages and currents. This *time-independent* behaviour is called **resistance**, and we have already discussed its properties. Components which *alter the input waveform*, such as capacitors and inductors, show a reactive behaviour. The *time-dependant* reaction to an input waveform is therefore called **reactance**. It turns out that *Ohm's law can then be generalized* as:

impedance = resistance + reactance

with the resistance being *frequency independent* and the reactance being *frequency dependent*.

Linear signal transformation:

Recall that you can *compose any time-dependant signal*, y(t), from a *superposition of sine waves*, characterised by their frequency, ω_i, their relative phase shift, φ_i, and their Amplitude, A_i, by which they contribute:

$$y(t) = \sum_{i=0}^{\infty} \mathrm{A_i} \sin(\omega_i t + \varphi_i)$$

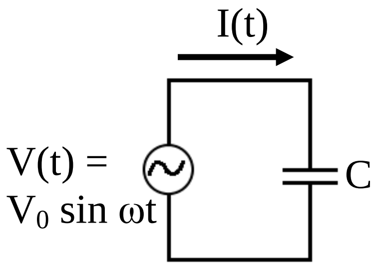
Thus, the discussion of complex waveforms always can be reduced to the discussion of sine (and cosine) waves. As a rule of thumb: *fast signal changes need high frequency content*. This you should have come across when studying the **Fourier Transformation**.

Capacitors and inductors are still *passive components*. When they are subjected to sine waves they react only by altering the *amplitude* and *phase* of the sine wave, dependent on its frequency, but the **output is still a sine wave**. This behaviour is called **linear**. Thus, a complex input signal gets transformed depending on its frequency content in terms of individual sine waves. As the transformation is frequency dependent, in general the output signal wave has a *distorted shape* compared to the input signal, since the frequency components have been altered differently.

Note that resistors are passive and linear components as well. Only they do not alter the phase and the alteration of the amplitude is universal, i.e. not depending on the frequency.

We will come to **non-linear** behaviour, usually found in *active components*, at later lectures.

Reactance of a capacitor: (H&H, 1.18, p. 30)
In order to characterize this component it is useful to look at the response for a single frequency.



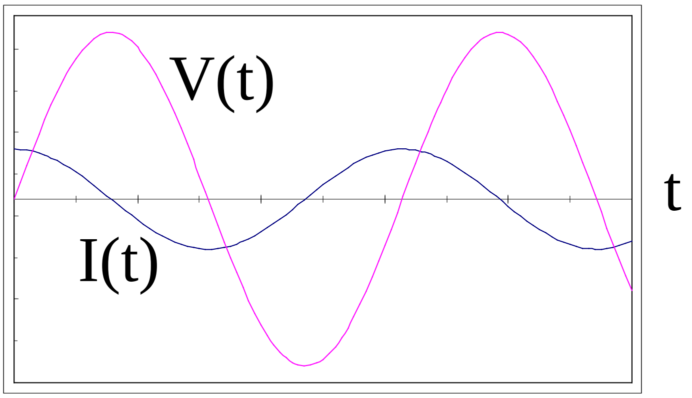
If a capacitor is driven by a voltage:

$$V(t) = V_0 \sin \omega t$$

then the current is

$$I(t) = C\frac{dV}{dt} = C\omega V_0 \cos \omega t$$

i.e. the current **leads** the input voltage by 90°, see graph below.



Disregarding the 90° phase shift for a moment, the current can be written in the form:

$$I = \frac{V}{{}^{1}/_{\omega C}}$$

Compared with Ohm's law, *I=V/R*, one can see that the effective resistance of a capacity is:

$$X_C = \frac{1}{\omega C}$$

This effective resistance combined with the phase shift is the **reactance** of the capacity.

Note that for low frequencies (slowly varying signals) a capacitor has a high reactance, *it blocks DC currents*. For high frequency (quickly varying signals) a capacitor has a low reactance –*it almost acts like a conducting wire*.

Voltages and currents as complex numbers:

The *amplitude* and *phase* information of wave phenomena can be represented and manipulated in a compact way using **complex numbers**. Note that quantities with a *physical representation* in the real world, like electrical currents which are generated by a flow of electrons, only will be represented by the *real part* or the complex numbers.

Using the complex representation the input voltage becomes:

$$V(t) = V_0 e^{i(\omega t + \varphi)} = V_0(\cos(\omega t + \varphi) + i\sin(\omega t + \varphi))$$

with the amplitude V_0, frequency ω and phase shift φ.

In the following the reactance of a capacitor is presented using complex numbers. We still refer to same oscillating circuit pictured above.

If a capacitor is driven by a voltage: $V(t) = V_0 e^{i\omega t}$

Then the current is: $I(t) = C\frac{dV}{dt} = i\omega C V_0 e^{i\omega t}$

In Ohm's law form: $I(t) = \frac{V(t)}{{}^{-i}/_{\omega C}}$

And the reactance of a capacitor becomes: $X_c = \frac{-i}{\omega C} = \frac{1}{i\omega C}$

Again the reactance becomes very large (small) at very low (high) frequencies.

Ohm's law generalized:

It is time to generalise Ohm's law to:

$$V = IZ \quad \text{with } Z = R + X$$

where Z is the **impedance**, R the *resistance* and X the *reactance*. Here V, I and Z are *complex numbers*.

The impedance Z is a *direct extension* of the concept of resistance. Other rules to Ohm's Law can be applied in the same manner, e.g. impedances for components arranged in series or in parallel can be *combined in the same way*:

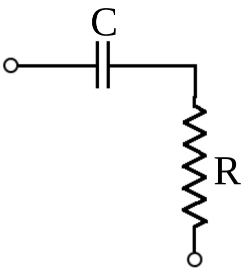
In series: $Z = Z_1 + Z_2 + Z_3 + \cdots$
In parallel: ${}^{1}/_{Z} = {}^{1}/_{Z_1} + {}^{1}/_{Z_2} + {}^{1}/_{Z_3} + \cdots$

For the linear components that we have encountered so far the impedances are:

For resistors: $Z_R = R$
For capacities: $Z_C = {}^{-i}/_{\omega C} = {}^{1}/_{i\omega C}$
For inductivities: $Z_L = i\omega L$

Resistors only contribute resistance while capacitors and inductors only contribute reactance. Most circuits have both – and consequently the impedance has real and imaginary parts.

Phasor diagrams: (H&H, 1.20, p. 39)
The impedance of a circuit can most easily be thought about as *a point in the complex plane*. For example, in the following circuit fragment we have two components in series. As a result we can add the two impedances together. This gives us a complex impedance, Z, that can be considered as an amplitude, |Z|, and a phase, φ, in the complex plane:

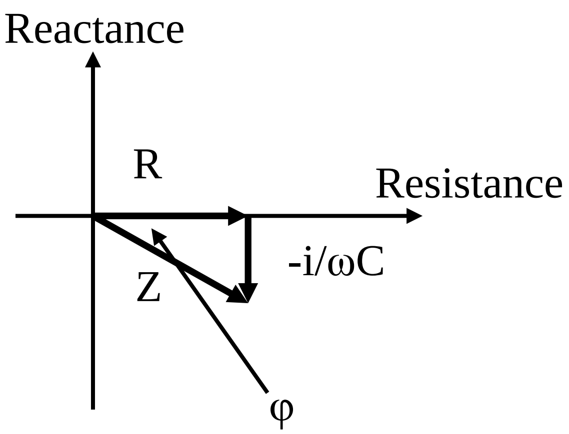


Combined impedance: $Z = R - {}^{i}/_{\omega C}$

Amplitude: $|Z| = \sqrt{R^2 + {}^{1}/_{\omega^2 C^2}}$

Phase: $\varphi = \tan^{-1} \frac{-{}^{1}/_{\omega C}}{R}$

Here the frequency response of the circuit fragment is shown in the complex plain. The real axis is the resistance and the imaginary axis is the reactance. The impedance is the combination of the two.



Power in reactive circuits: (H&H, p. 33)
Resistive components *dissipate energy* while reactive components do not. A capacitor stores the energy in the electric field and an inductor stores it in the magnetic field. Hence the consumption of power in a circuit with resistive and reactive components can be quite involved. In particular, the *instantaneous power can change sign* over a single period of an AC-circuit. This corresponds to, for example, the charging and discharging of a capacitor.

The power averaged over one oscillation period, T = 2π/ω, is:

$$P = \frac{1}{T}\int_0^T V(t)\, I(t) dt$$

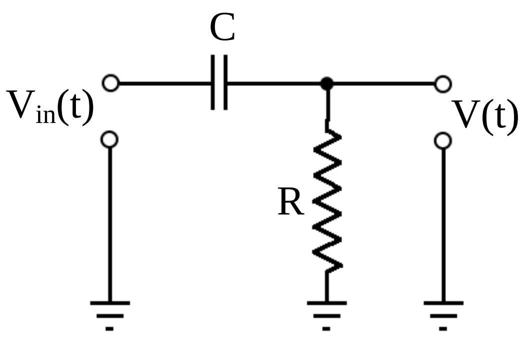
Using complex numbers the average power is the real part of the product of the complex rms amplitudes V and I*:

$$P = Real(VI^*)$$

Here I* denotes the complex conjugate of I. This formalism is often more convenient.

Frequency domain: (H&H, 1.19, p. 35)
The simplest **frequency filters** are *frequency dependent voltage dividers* (go back to lecture 4, if you can't remember what a voltage divider is). The purpose of filters is to let signals in a *particular frequency range* pass without attenuation, while the signals outside this range are subject to attenuation, in dependence to the frequency.

Depending on the arrangement *RC elements* can act as *high-pass* and as *low-pass* filters.
Let's first look at the **high-pass** configuration of a RC unit as shown below. We already know that the capacitor in the upper branch will block DC currents but is rather transparent for fast changing currents.



For a voltage divider with two resistors we determined in lecture 4:

$$V = \frac{V_{IN} R_2}{R_1 + R_2}$$

where R_1 is the resistance in the upper branch and R_2 is the current sink to ground. Using the generalisation of Ohm's law we get:

$$V(t) = \frac{V_{in}(t) Z_2}{Z_1 + Z_2}$$

For the high pass filter this becomes:

$$V(t) = V_{in}(t) \frac{R}{-{}^{i}/_{\omega C} + R}$$

and after a few calculations:

$$V(t) = V_{in}(t) \frac{R\left(R + {}^{i}/_{\omega C}\right)}{R^2 + {}^{1}/_{\omega^2 C^2}} = V_{in}(t) \frac{\left(1 + {}^{i}/_{\omega RC}\right)}{1 + {}^{1}/_{\omega^2 R^2 C^2}}$$

At low frequencies the second term in the denominator becomes exceedingly large. This shows that at low frequencies the high pass filter strongly attenuates V(t).

Now we ask ourselves, what is the amplitude of V(t) as a function of frequency, ω? It is:

$$|V| = \sqrt{VV^*} = \frac{|V_{in}|}{\sqrt{1 + {}^{1}/_{\omega^2 R^2 C^2}}}$$

It turns out that $1/RC$ is the **characteristic frequency scale** in this problem.

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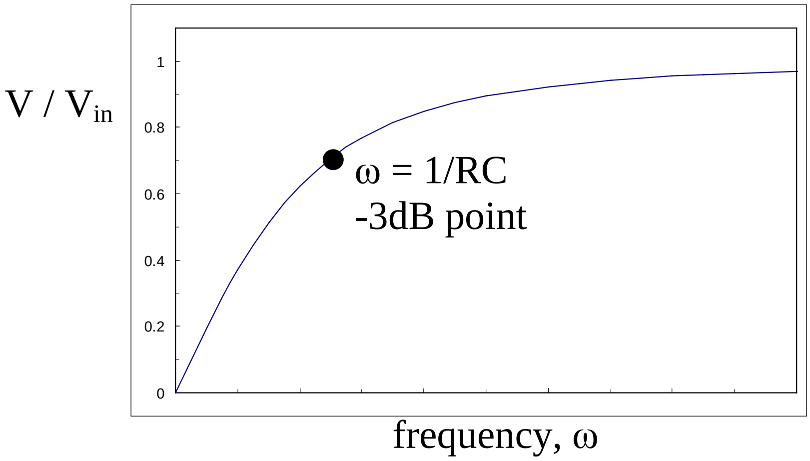
The function V/Vin(ω) is drawn here:



It behaves as expected: for frequencies near zero the attenuation is very strong and for high frequencies it becomes negligible. Because the *shape of this curve is universal* for this kind of filter the behaviour of the filter can be described by pointing out just one *characteristic point*:

$$\omega = \frac{1}{RC}$$

At this frequency the attenuation will be such that:

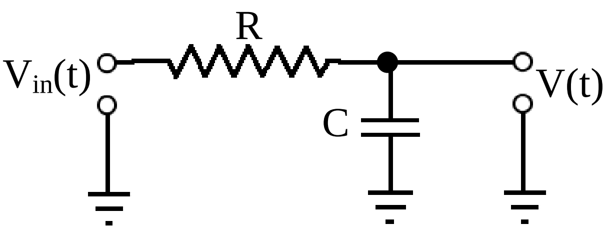
$$\frac{V}{V_{in}} = \frac{1}{\sqrt{2}} \quad \text{i.e. the power changes by a factor of 2:} \quad \frac{V^2}{V_{in}^2} = \frac{1}{2}$$

A reduction in power by a factor of two corresponds to an approximate -3dB attenuation on the Dezibel range (we will come back to it in a later lecture):

$$gain_{dB} = 20\ log_{10}\left(\frac{V}{V_{in}}\right)$$

This point sometimes also is referred to as **turn-on point**.

Next we look at the **low-pass** configuration of a *RC unit,* shown below. These are voltage dividers with the capacitor in the lower branch and we know already that it will pass DC currents but will smooth out fast alternating charges.



As before this can be analysed as a voltage divider using impedance:

$$V(t) = \frac{V_{in}(t)Z_2}{Z_1 + Z_2} = V_{in}(t)\frac{{-i}/{\omega C}}{R - {i}/{\omega C}}$$

and after a few calculations:

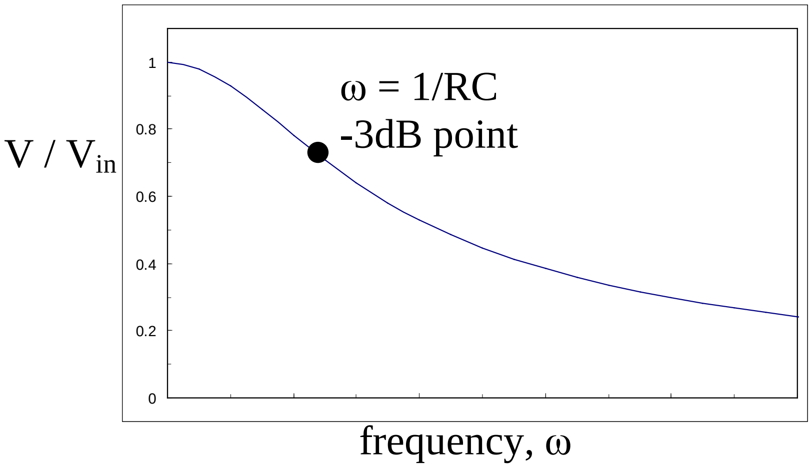
$$V(t) = V_{in}(t)\frac{1/_{\omega^2C^2} - {}^{iR}/_{\omega C}}{R^2 + 1/_{\omega^2C^2}} = V_{in}(t)\frac{1 - i\omega RC}{1 + \omega^2R^2C^2}$$

This circuit attenuates at high frequencies as the second term in the nominator becomes large.

Again we ask ourselves, what is the amplitude of V(t) as a function of frequency, ω? It is:

$$|V| = \sqrt{VV^*} = \frac{|V_{in}|}{\sqrt{1 + \omega^2R^2C^2}}$$

The function $V/V_{in}(\omega)$ is drawn here:



By this point we find our expectation confirmed that *RC* is the **characteristic frequency scale** in this problem as well. The -3dB attenuation point, where the power is reduced by a factor of two, here sometimes is referred to as **turn-off point**.

Notes:

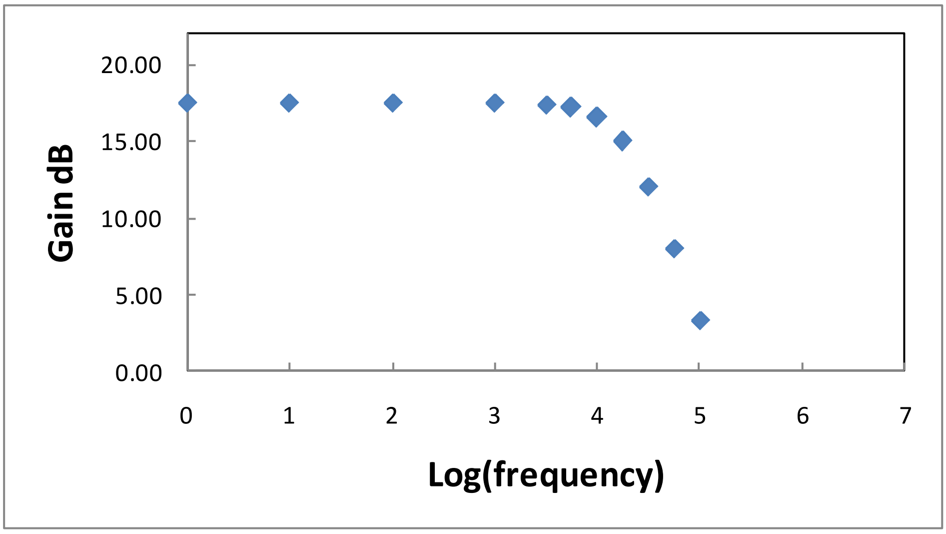
In the previous lecture we looked at the output voltage of a RC unit in the **time domain** and found RC to the **characteristic time constant** of its behaviour. Here we look at the output signal of a RC unit in the **frequency domain** and find 1/RC to be **characteristic frequency** of the found behaviour.

Here we only have discussed the response of the RC units to a signal with a *single frequency*. Arbitrary signals will be a **superposition** of some or many frequency components: however, we can consider the response of the circuit to **each individual frequency in turn**.

Bode plot:

Here both the *amplitude* and the *phase* information are presented together for a RC unit in low-pass configuration. This set of plots gives a complete characterization of the filter circuit.

The term **Bode plot** specifically refers to a plot of the *gain* of a unit versus the *frequency*, with both axes in *logarithmic representation*. In this log-log display wide parameter ranges can be covered and deviations from exponential behaviour, plotted linearly in this representation, are easy to spot.



The phase information is plotted on a linear scale on the y-axis with a log scale for the frequency axis.