Analogue Electronics 5: AC Circuits – oscillating LCR circuits

In the last lecture we began to look at using complex numbers to represent the behaviour of reactive circuits. Here we apply this to the case of circuits which oscillate. You hopefully will find in this electrical example some aspects to be familiar from your mechanics courses, and, hence, that it will be a useful way to practice using complex numbers. We will use complex numbers in AC circuits to generate resonances and for tuning.

Example: Graphic Equalizer

You will be familiar with the idea that arbitrarily complicated sounds can be broken down into bands of frequencies that can be amplified independently: this is what happens in a **graphic equalizer**. The one shown below can be used on a stage to modify the sound of an electric guitar.

The oscillations of the strings of the guitar are picked up in a sensor and translated into **oscillations of a current** in a conductor. **Filter circuits** in the graphics equalizer, using the same principles we discussed in the last lecture, *split* the **frequency spectrum** into several bands, which in turn are routed through individually regulated amplifier stages, before they are *mixed* (merged) again into one signal.

These procedures, of course, are not limited to the range of acoustic signals. In electronics you generally just deal with higher frequencies, and you do not talk about the *pitch of a sound* but about the **frequency of a sine wave**.

And remember: **any signal shape can be produced by a superposition of sine waves**.

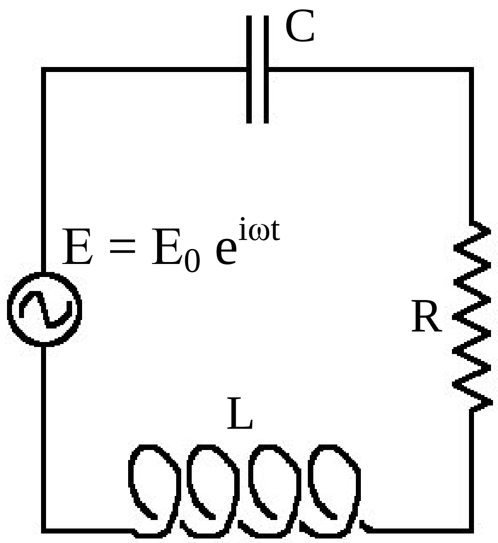
To recap:

- The output of a **linear circuit**, driven by a sine wave at some frequency ω, is itself a *sine wave at the same frequency* (only the *amplitude* and *phase* may have changed).
- We *analyze* linear circuits (those with resistors, capacitors and inductors) by asking how the output voltage (amplitude and phase) depends on the input voltage, for a sine wave input, in dependence of the *frequency*.
- We use the **generalised Ohm's law** to specify the current corresponding to a particular voltage.
- Write signals (voltages) in the form $V = V_0 e^{i(\omega t + \varphi)}$, which makes arithmetic much simpler than using sines and cosines directly.
- To find actual voltages – after a calculation – look just at the **real part** of the result.

Series LCR circuit:

This is a simple example which is analogous to a familiar mechanical situation. This hopefully helps to understand it.

Take a series arrangement of a capacity, a resistor and an inductor. By attaching an alternating power supply, ε, we force the voltage across the components to oscillate with angular frequency, ω. The resistor *dissipates energy* while the capacitor and inductor sometimes *store* and sometimes *release energy*.



An **analogous mechanical setup** to this AC circuit is the following: a weight is suspended on a horizontal rail, on which it can move with friction. A spring is attached horizontally, which on the other side is forced by a sinoidal movement.

The capacitor stores **potential energy** in an *electric field* in a similar manner to the spring storing energy in its *deformation*. The inductor stores energy in a *magnetic field* in a similar manner to the weight storing **kinetic energy** via its *motion*. Therefore:

- the *inverted capacity*, 1/C, relates to the *spring constant*, K, and
- the *inductance*, L, relates to the *inert mass*, m.

See further relations in the following table:

Driven LCR circuit (analogue computer)		Driven weight on a spring	
Charge on capacitor	Q	Displacement	x
Current	I	Velocity	v
Energy stored in capacitor	$\frac{1}{2}Q^2/C$	Energy stored in spring	$\frac{1}{2}Kx^2$
Energy stored in inductor	$\frac{1}{2}LI^2$	Kinetic energy	$\frac{1}{2}mv^2$

Previously we would have analysed the circuit with Kirchoff's voltage law:

$$V = V_R + V_C + V_L$$
$$V = IR + \frac{Q}{C} + L\frac{dI}{dt}$$

Now we will switch to use the *complex equivalents*. To highlight this we use the following notation: complex voltage, **E**, complex current, **Y**, and complex impedance, **Z.**

We would like to see a simple relationship between the voltage and the current and therefore express all quantities in terms of impedance:

$$E = YZ_R + YZ_C + YZ_L$$
$$E = YR + \frac{Y}{i\omega C} + Yi\omega L$$
$$E = Y\left\{R + i\left(\omega L - \frac{1}{\omega C}\right)\right\}$$

The content of {} is the total impedance of the circuit, Z_T, which, no surprise, is a complex number:

$$Z_T = R + i\left(\omega L - \frac{1}{\omega C}\right)$$

This can be written in polar representation:

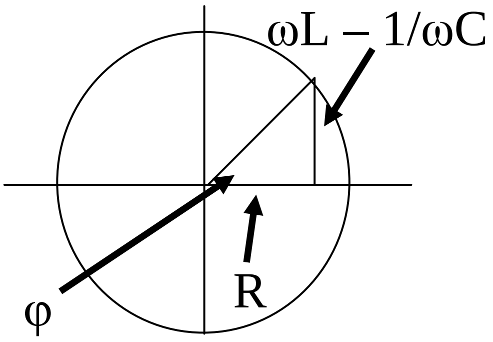
$$Z_T = |Z_T|e^{i\varphi}$$

where $|Z_T|$ is the magnitude and φ is the phase of the complex impedance.

The magnitude of Z_T is found by multiplying Z_T by its complex conjugate and then taking the square root:

$$|Z_T| = \sqrt{Z_T {Z_T}^*} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

The phase of Z_T is derived in the complex plane considering the triangle formed by the *real*



part (adjacent), *the imaginary part* (opposite) and the *magnitude* (hypotenuse). See below:
The phase, φ, therefore is:

$$\tan\varphi = \frac{\omega L - \frac{1}{\omega C}}{R}$$

We can state our results using the **complex form of Ohm's Law**:

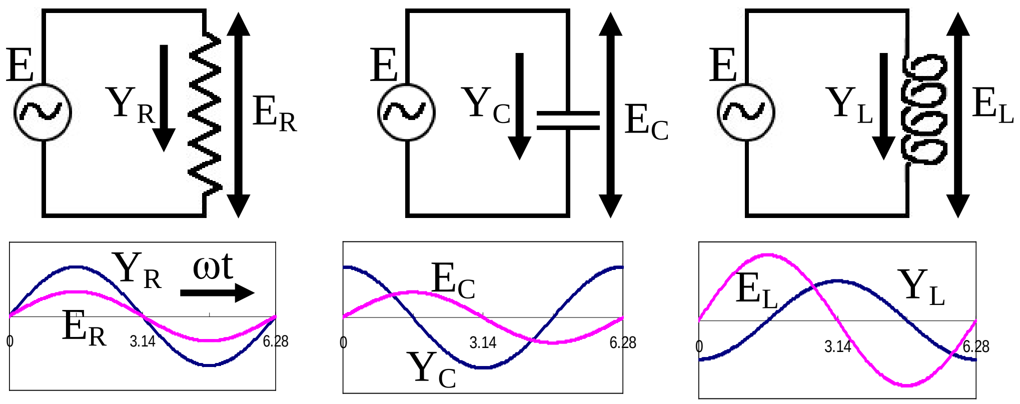
$$Y = \frac{E}{Z_T} = \frac{E_0 e^{i\omega t}}{|Z_T| e^{i\varphi}} = \frac{E_0 e^{i(\omega t - \varphi)}}{|Z_T|}$$

Significance of the phase φ:

A phase shift appears between the voltage across and the current through the circuit. If φ is the phase associated with the complex impedance Z_T then the phase of the current Y is different from that of the voltage E by –φ.

In a capacitive circuit: the current leads the voltage: $\varphi < 0$.
In an inductive circuit: the voltage leads the current: $\varphi > 0$.

This is illustrated below:

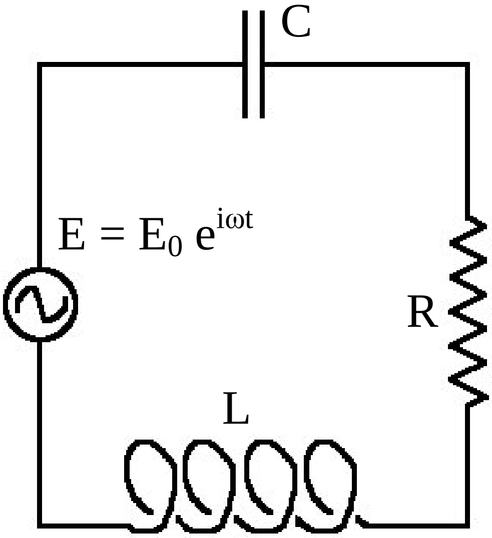


Resistive circuit: voltage and current in phase

Capacitive circuit: voltage lags current

Inductive circuit: voltage leads current

We have now completed the most **basic analysis of the LCR circuit** driven at an angular frequency, ω, summarized below.



Instantaneous analysis:

$$E = Y\left\{R + i\left(\omega L - \frac{1}{\omega C}\right)\right\}$$

The magnitude of the complex impedance, Z_T:

$$|Z_T| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

Phase φ:

$$\tan\varphi = \frac{\omega L - \frac{1}{\omega C}}{R}$$

Resonant nature of series LCR circuit: (H&H, 1.22, p. 41)
What now will be of interest is the way this circuit's oscillations become **resonant** if the driving frequency is *appropriately tuned*. The same is true for mechanical systems as well. Below we will again use the complex number representation in the discussion of resonance.

The **resonant point** of the circuit is the frequency at which the current flowing around the circuit will become *maximal*. First we want to find the appropriate *angular frequency*, the *maximal current* and the *phase shift* at the resonant point.

The magnitude of current, Y, is given by the ratio of the amplitude, E_0, to the driving voltage to the amplitude of the complex impedance, $|Z_T|$:

$$Y = \frac{E_0}{|Z_T|} = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

If the angular frequency, ω, is varied the maximum current is found when:

$$\omega L - \frac{1}{\omega C} = 0$$

This occurs when:

$$\omega L = \frac{1}{\omega C} \quad \text{hence} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

This angular frequency is the **resonant frequency, ω_0**, for the LCR circuit.

At this angular frequency the circuit resonates with a **maximum current**:

$$Y_{max} = \frac{E_0}{R}$$

The **resonant phase** of voltage with respect to current is:

$$\tan\varphi = \frac{\omega L - \frac{1}{\omega C}}{R} = 0 \quad \text{hence} \quad \varphi = 0$$

The circuit transits from capacitive lead to inductive lead as the frequency passes through the resonance.

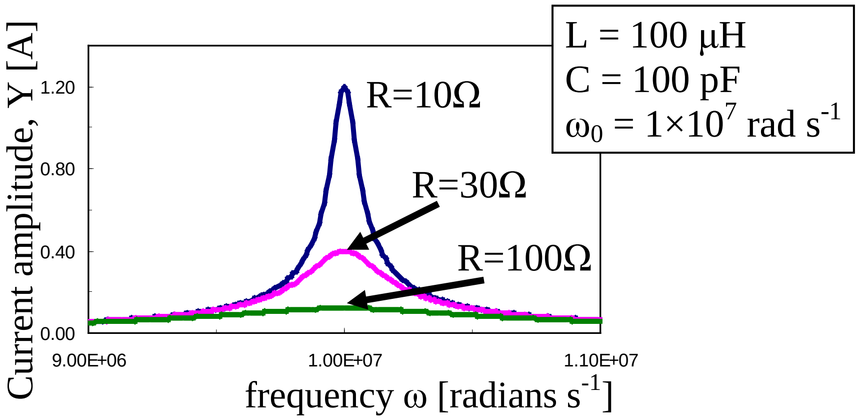
Our original expression for the current in the circuit:

$$Y = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

can be rewritten in terms of the resonant frequency using $\omega_0^2 = 1/LC$:

$$Y = \frac{E_0 \, {}^{\omega}/_{L}}{\sqrt{\left({}^{R\omega}/_{L}\right)^2 + (\omega^2 - \omega_0{}^2)^2}} = \frac{E_0 \omega C \omega_0^2}{\sqrt{(R\omega C\omega_0^2)^2 + (\omega^2 - \omega_0{}^2)^2}}$$

Below this function is plotted for example component values. The symmetric behaviour around the resonance frequency, ω_0, is evident, as is the dependency of the peak size on the energy dissipation in the circuit, $Y_{max} = E_0 / R$:



Next we can determine the width of the peak by finding the **full width at half maximum (FWHM)**, i.e. the width between the two points at which the current is half of its resonant value.

$$\text{FWHM} = 2\Delta\omega = 2\left|\left(\omega_{1/2} - \omega_0\right)\right|$$

determined from:

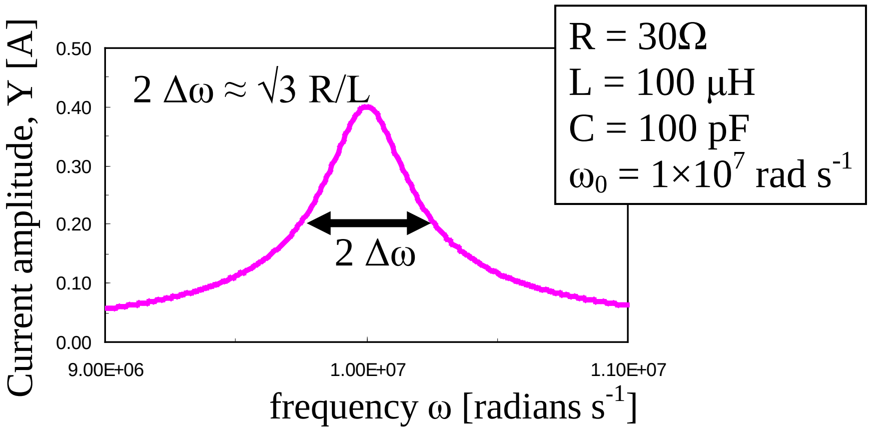
$$\frac{E_0}{2R} = \frac{E_0 \, {}^{\omega}/_{L}}{\sqrt{\left({}^{R\omega}/_{L}\right)^2 + (\omega^2 - \omega_0{}^2)^2}}$$

After solving this quadratic equation in ω one finds as *leading term*:

$$\text{FWHM} \approx \sqrt{3}\frac{R}{L}$$

In the leading term the width of the peak is *proportional to the resistance*, i.e. the peak becomes more pronounced as the power dissipation decreases. Thus, superconducting (R=0) resonant circuits could be considered to be of great interest.

Here the width of the peak is indicated for one of the above examples:

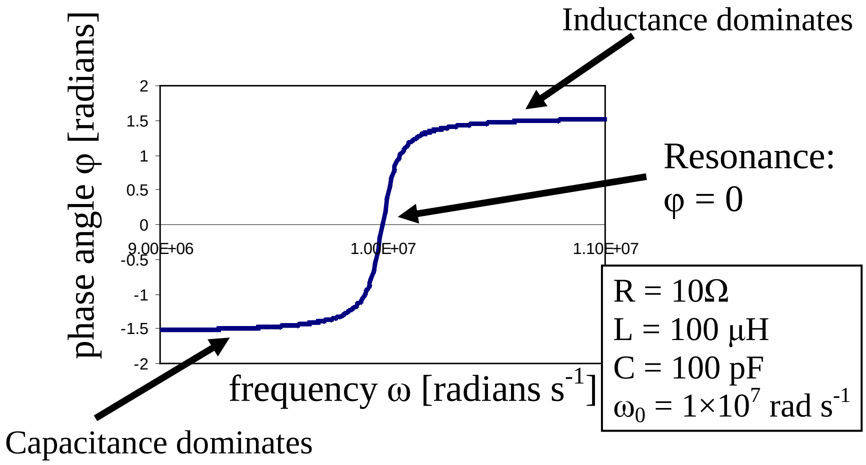


We have already seen that the *phase shift changes as we pass through resonance*.

$$\tan\varphi = \frac{\omega L - \frac{1}{\omega C}}{R}$$

Using $\omega_0^2 = 1/LC$ this can be rewritten as:

$$\tan\varphi = \frac{(\omega^2 - \omega_0^2)}{R\omega/L}$$



<u>In summary:</u>

Useful signatures of resonance are:

- There is a peak in the amplitude of the current.
- The phase shift is equal to zero.
- The phase shift switches signs either side of resonance.

Power in AC circuits:

In the LCR circuit above the energy is *supplied* by the function generator, as **electrical energy**. This sets up a flow between energy stored in capacitor and energy stored in inductor. All the while energy *dissipated* in resistor in the form of **heat**.

The **power dissipated** in the resistor is the energy dissipated per second:

$$P = Y^2 R$$

using the complex current:

$$Y = Ie^{i(\omega t - \varphi)}$$

However care is required when squaring Y, as the actual current is only the real part:

$$\begin{aligned} Y &= \Re\left(I\, e^{i(\omega t - \varphi)}\right) \\ &= I \cos(\omega t - \varphi) \end{aligned}$$

Note that I is the amplitude of the real AC current. And only this current needs to be substituted into the expression for the power dissipated in the resistor:

$$P = Y^2 R = I^2 R \cos^2(\omega t - \varphi)$$

This is an expression for the **instantaneous power** at time t. It describes the dissipation at any moment in the circuit. Notice the change from *instantaneous current*, Y, to the *magnitude of the current*, I.

Often we are interested in the **average behaviour** of the circuit over longer periods of time. Then we need to integrate the instantaneous properties over *multiples of the AC period*. So:

$$P_{av} = I^2 R \langle \cos^2(\omega t - \varphi) \rangle$$

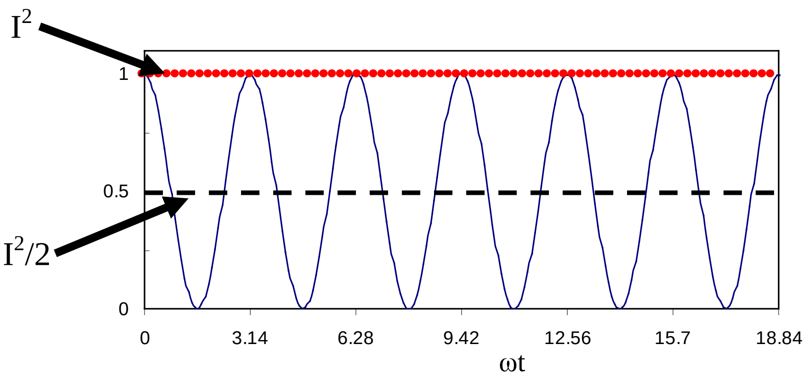
where

$$\langle \cos^2(\omega t - \varphi) \rangle = \frac{1}{2}$$

Thus the **average power dissipated** is:

$$P_{av} = \frac{1}{2} I^2 R$$

This is half of the maximum power dissipated instantaneously each cycle, as illustrated below:



Root mean square (RMS) quantities:

We can write the average power as:

$$P_{av} = \frac{1}{2} I^2 R = \left(\frac{I}{\sqrt{2}}\right)^2 R$$

The quantity in brackets is given the name **root mean square** of *I*:

$$I_{rms} = \frac{I}{\sqrt{2}}$$

The formula for the *power dissipated* then returns to the form familiar from DC circuits:

$$P_{av} = I_{rms}{}^2 R$$

This notation can also be used for voltages, so that:

$$V_{rms} = \frac{V}{\sqrt{2}}$$

Thus we have **Ohm's law for RMS quantities**:

$$I_{rms} = \frac{V_{rms}}{Z}$$

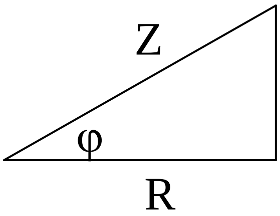
where Z is the total impedance:

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

Phase and the dissipated power:

The power delivered to the resistor can be written in terms of the generator voltage, V_{rms}, and the phase of the impedance, φ:

$$P_{av} = I_{rms}\left(\frac{V_{rms}}{Z}\right) R = I_{rms} V_{rms}\left(\frac{R}{Z}\right) = I_{rms} V_{rms} \cos\varphi$$



with:

$$\cos\varphi = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

We can see that:

- The maximum power is delivered when: *φ=0* giving *cos(φ)=1,* i.e. at resonance
- In purely capacitive or inductive circuits: *cos(φ)=0,* i.e. no power is delivered

Take home message:

By using *complex quantities* we have been able to analyse the behaviour of a resonant circuit via familiar concepts such as Ohm's Law and the power dissipated in a resistor.