Analogue Electronics 6: AC Circuits – transmission lines

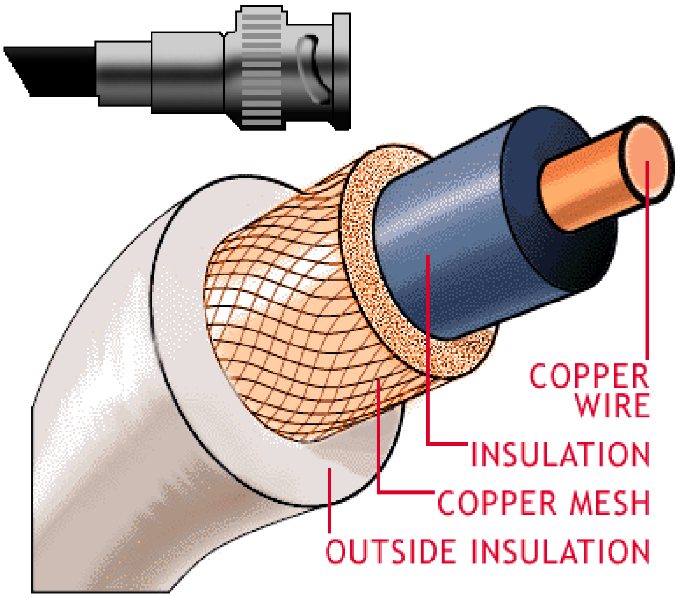
We have talked extensively about *input and output impedances* and what you can expect to happen when you attach two pieces of equipment to each other. With **high frequency signals** we need a new perspective. For slow signals we think about a wire having a *potential difference* (voltage) with respect to ground. For very fast signals we need to start thinking about **waves travelling along a guide**.

The cables commonly used to carry high frequency signals effectively are **transmission lines**, and in electromagnetism they are generally called **wave guides**.

In general a transmission line consists of **two parallel conductors**, one is the signal wire and the other is the ground wire (or shield). Each of these conductors has a *resistance*, a *capacitance* and an *inductance* **per unit length**. An AC voltage at one end causes currents to flow in the conductors, producing fields E and H. The behaviour can be analysed completely in terms of *electromagnetic waves*. We will avoid that here in favour of a more pictorial approach.

Transmission lines:

You have a 30m transmission line as part of your kit in the lab. This is for checkpoint A1 where you investigate the phenomena discussed in this lecture. It is made from RG-58 type **coaxial cable** and fitted with **BNC** connectors (BNC = Bayonet Neill-Concelman), both designed for excellent signal transmission and shielding at affordable cost. The connector plug and inside sections of your transmission line look like this:



http://www.phy.davidson.edu/StuHome/phstewart/IL/speed/Cableinfo.ht

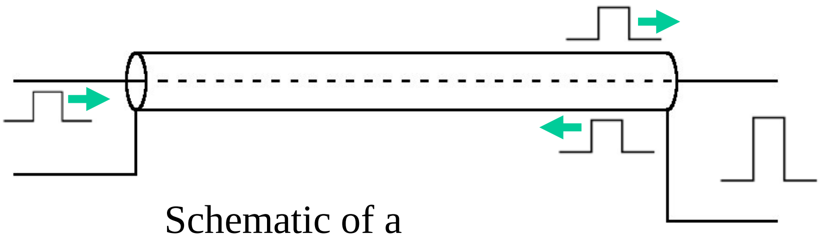
Coaxial, or shot "coax", cables are a common way to connect different pieces of equipment, e.g. used in many of the experiments in the second semester Junior Honours lab course. They are needed where *signal integrity* is important and *small* or *fast signals* need to be shielded against radio frequency noise (RF noise). Bayonet or other locking connectors are used where *frequent* plugging and unplugging operations occur, e.g. the function generator and oscilloscope you use are fitted with BNC sockets.

Lecture 9, 5th October 2012

The aim of this lecture is to demonstrate that:

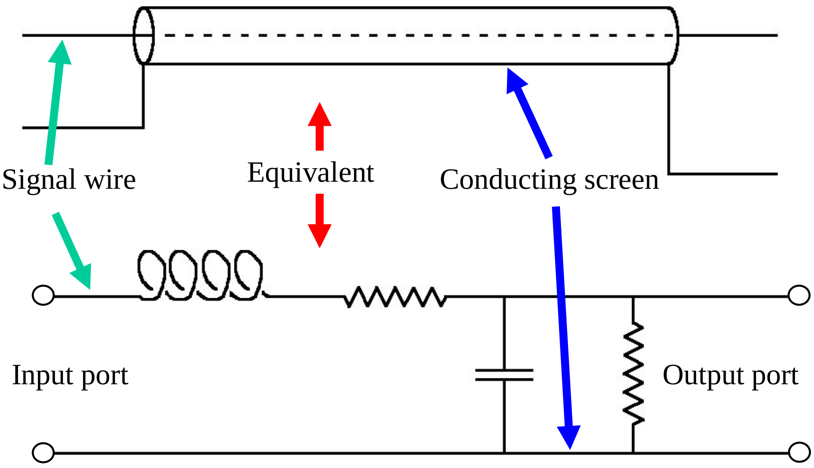
- a coaxial cable has a **characteristic impedance Z_0**,
- you will get **signal reflections** depending on the impedance of the "load" at the end of the cable.

Below is a schematic illustration of a signal, in this case a single square wave, travelling along a transmission line. As stated above the transmission line consists of two wires, which are visible at each end of the sketch. In the middle of the sketch one wire becomes a cylinder with the other wire threaded inside it. The cylinder is a *conducting screen* or *shield* which protects small signals from interference due to external electric fields. The conducting screen is *connected to ground*. The *signal wire* goes straight down the axis of the conducting screen.



Schematic of a transmission line

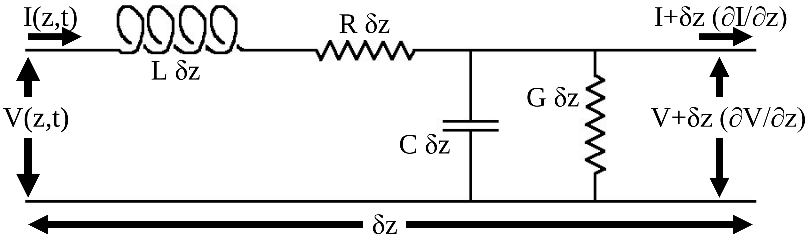
The schematic of the transmission line can be redrawn in terms of an *equivalent arrangement* of components, also referred to as **four terminal** or **two-port network**. This will mean that we can use our existing understanding of circuits to analyse its behaviour. Here we think about the resistance, capacitance and inductance as quantities *per unit length* of cable.



At first you may be surprised to learn that there is some capacitance and induction between the two conductors. But think about it: suppose you send *pulse of charge* down the signal wire. An *electrical field* will form across the insulator to the shield, the same way as in a capacitor. As the charge travels down the wire a *magnetic field* is formed, which in turn causes self inductance. We will try to understand this further in the following.

And note that the resistance between the conductors is large but finite and that the resistance along the wire is small but not zero.

Consider a section of transmission line of *length* δz made from two uniform wires connected between a source and a load, as illustrated below.:



You put in a signal, V(z,t) or I(z,t), and ask how the output looks like after the additional length δz. To answer you need to know the following quantities:

R = resistance / unit length (both lines added together)
G = transverse conductivity / unit length
L = inductance / unit length
C = capacitance / unit length

A *continuous line* then is described by going to the limit δz→0. We can now approach this *equivalent circuit for a transmission* line in the same way we have treated all previous circuits:
Change in voltage (Kirchoff I):

$$V - \left(V + \frac{\partial V}{\partial z}\delta z\right) = RI\delta z + L\frac{\partial I}{\partial t}\delta z$$

Change in current (Kirchoff II):

$$I - \left(I + \frac{\partial I}{\partial z}\delta z\right) = GV\delta z + C\frac{\partial V}{\partial t}\delta z$$

We can simplify these to give the following set of **basic differential equations**:

$$\frac{\partial V}{\partial z} = -\left\{RI + L\frac{\partial I}{\partial t}\right\}$$
$$\frac{\partial I}{\partial z} = -\left\{GV + C\frac{\partial V}{\partial t}\right\}$$

Solutions of the basic equations:

Transmission lines are often used to carry *high frequency signals*, for example a fast pulse associated with the detection of a particle, or a bit in a serial link. As asserted before, any signal form can be considered as a **superposition of sinusoidal signals** (its frequency content is made visible by a Fourier Transformation). Therefore here we can use a simple sine wave to represent our signal in general:

$$V(z,t) = V(z)e^{i\omega t}$$
$$I(z,t) = I(z)e^{i\omega t}$$

If we substitute these into the basic equations above we get the **telegrapher's equations**:

$$\frac{\partial V}{\partial z} = -\{R + i\omega L\}I(z)$$
$$\frac{\partial I}{\partial z} = -\{G + i\omega C\}V(z)$$

So, we have **got rid of the time derivatives** and have replaced them with complex terms.

The next step is to **take the z derivative** of each of these equations, and then to *substitute* the not derived equations for (∂V/∂z) and (∂I/∂z). This gives a pair of equations in terms of V and I alone:

$$\frac{\partial^2 V}{\partial z^2} = \{R + i\omega L\}\{G + i\omega C\}V(z) = p^2 V(z)$$
$$\frac{\partial^2 I}{\partial z^2} = \{R + i\omega L\}\{G + i\omega C\}I(z) = p^2 I(z)$$
$$\text{with} \quad p^2 = \{R + i\omega L\}\{G + i\omega C\}$$

An equation showing that a quantity is *proportional to its own second derivative* has **sinusoidal solutions**. Here we know that we want to deal with complex numbers, so we look for $e^{j\varphi}$ type solutions. In the most general form they look like:

$$V(z) = Ae^{-pz} + Be^{pz}$$
$$I(z) = De^{-pz} + Ee^{pz}$$

Note that p is complex: $= \alpha + i\beta$. So with the time dependence $e^{i\omega t}$ they look like:

$$V(z,t) = Ae^{-\alpha z}e^{i(\omega t - \beta z)} + Be^{\alpha z}e^{i(\omega t + \beta z)}$$
$$I(z,t) = De^{-\alpha z}e^{i(\omega t - \beta z)} + Ee^{\alpha z}e^{i(\omega t + \beta z)}$$

These equations describe waves travelling in the +z (amplitudes A and D) and –z (amplitudes B and E) directions, where α quantifies the attenuation per unit length, ω the frequency and β the phase of the waves.

Impedance Z_0 – characteristic impedance of cable:

Using the time-independent relations we work further towards a relationship between I and V and see how close we can get to Ohm's law. Using:

$$\frac{\partial V}{\partial z} = -\{R + i\omega L\}I(z) \;\Rightarrow\; I(z) = \left(-1/_{\{R + i\omega L\}}\right)\frac{\partial V}{\partial z}$$

and

$$V(z) = Ae^{-pz} + Be^{pz} \quad \Rightarrow \quad \frac{\partial V}{\partial z} = p(-Ae^{-pz} + Be^{pz})$$

we get:

$$I(z) = \sqrt{\frac{G + i\omega C}{R + i\omega L}}(Ae^{-pz} - Be^{pz})$$

and call the impedance term:

$$Z_0 = \sqrt{\frac{R + i\omega L}{G + i\omega C}}$$

to get:

$$I(z) = \left(\frac{1}{Z_0}\right)(Ae^{-pz} - Be^{pz})$$

Our cable, i.e. transmission line, has a complex impedance. This hardly surprises because we represented its behaviour using capacitors and inductors. $\mathbf{Z_0}$ is called the **characteristic impedance** of the cable.

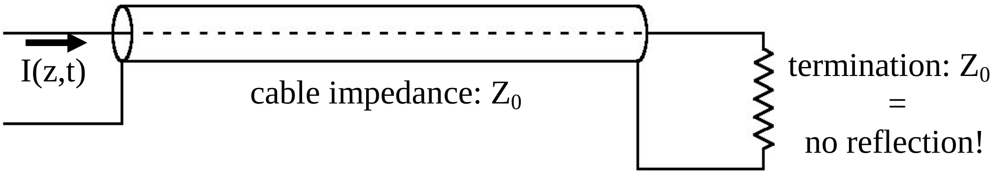
The above already looks close to Ohm's law, but still describes waves going either direction on the cable. Suppose you would have an *infinitely long uniform cable*: sending a pulse down you would never get a reflection in return, i.e. B=0. The above would simplify to:

$$V(z) = Ae^{-pz}$$
$$I(z) = \left(\frac{1}{Z_0}\right)Ae^{-pz}$$

giving:

$$\frac{V(z)}{I(z)} = Z_0$$

i.e. the **impedance seen at any point of an infinite line is Z_0**. One can turn this around and conclude: any line can be made to *behave like an infinite line* if it is **terminated with Z_0** . That is a key point to remember: any signal arriving at the end a line which is terminated with Z_0 **will not be reflected**. This configuration is illustrated below.



Matching impedances as condition to *avoid reflections*: this is the same condition as we discussed for input and output impedances to *maximise power transfer*. Think about it: it is actually the same thing! If no signal is reflected, all power is transmitted to the next element.

Properties of the characteristic impedance Z_0:

The behaviour of the characteristic impedance is not intuitive from the expression:

$$Z_0 = \sqrt{\frac{R + i\omega L}{G + i\omega C}}$$

To get an idea we simplify it down to the terms which have the biggest effect. This way we can see the essential features and the frequency dependence.

In coaxial cables polythene and air often are used as insulators between the central conductor and the outer screen. Both have high resistances, therefore *G usually is very small*: $G \approx 0$.

From your electrodynamics you may remember that in an *ideal conductor* the current only runs at the surface and the centre of the conductor is field-free. In a *real conductor* the penetration of the field actually decreases with the frequency, i.e. for *high frequencies* the resistance in the signal wire and the outer screen actually increases like:

$$R \sim \sqrt{\omega}$$

The term ωL increases faster with the frequency than $R \sim \sqrt{\omega}$, so that for the impedance of a cable the terms for R and G can be neglected from frequencies like $\omega \approx 10^7$, giving:

$$Z_0 \rightarrow \sqrt{L/C}$$

Thus, at *high frequencies* Z_0 becomes *approximately real*, i.e. **purely resistive**.

There are various standards for the impedance of cables, with Z_0 typically 50-100Ω, *depending on the construction* of the cable. You will get to measure this in the lab.

Velocity of signals down a transmission line:

The velocity of the signal and the distance over which it decays can be determined from the details of p in the equations above. Evaluation of α and β:

$$p = \alpha + i\beta = \sqrt{(R + i\omega L)(G + i\omega C)}$$

To simplify take the approximation G=0, then:

$$p = \alpha + i\beta = i\omega\sqrt{LC}\sqrt{\left(1 + {}^{R}/_{i\omega L}\right)}$$

The next step is to assume *large ω* and use the leading term of the Taylor expansion for the root, $(1+x)^n \approx 1+nx$:

$$p = \alpha + i\beta \approx i\omega\sqrt{LC}\left(1 + {}^{R}/_{i2\omega L}\right)$$

re-arrange to:

$$p = \alpha + i\beta \approx {}^{R\sqrt{LC}}/_{2L} + i\omega\sqrt{LC}$$

to find:

$$\alpha \approx {}^{R}/_{2Z_0}$$
$$\beta \approx \omega\sqrt{LC}$$

In this approximation the velocity which the wave crests travel, the phase velocity, becomes:

$$V_{phase} = {}^{\omega}/_{k} = {}^{\omega}/_{\beta} = {}^{1}/_{\sqrt{LC}}$$

i.e. *constant at high frequencies*.

You will notice that we are using concepts that are familiar from courses on waves to analyse the behaviour of a voltage signal. A deep understanding of transmission lines comes from *Maxwell's equations* and the propagation of waves in conducting cavities (also called wave guides). There one can derive the full **dispersion relation ω(k)**, discuss its behaviour in dependence of all four parameters, R, G, L and C, and find the general solutions for the **phase and the group velocity**, $v_{phase}(\omega)$ and $v_{group}(\omega)$.

Examples:

Let's put some numbers into the equations for a couple of the most common types of **copper coaxial cables** (transmission lines), using guessed dimensions.

Insulator: air cored Inner radius a = 0.5 mm Outer radius b = 5 mm

R =	$= 9\times10^{-5}\sqrt{f}\,\Omega m^{-1}$
$C = 2\pi\varepsilon_0 / \ln(b/a)$	$\approx 24\ pFm^{-1}$
$L = (1/2\pi)\,\mu_0 \ln(b/a)$	$\approx 0.46\ \mu Hm^{-1}$

$$V_{phase} = {}^{1}\!/_{\sqrt{LC}} = {}^{1}\!/_{\sqrt{\mu_0\varepsilon_0}} = c$$

$$Z_0 = \sqrt{{}^{L}\!/_{C}} = 138\Omega$$

Insulator: polythene cored Inner radius a = 0.5 mm Outer radius b = 5mm

Dielectric constant $\varepsilon_r = 2.25$ between the inner and outer conductors.
Because the capacitance is proportional to ε_r we can simply modify the values determined for the air cored case:

$$V_{phase} = {}^{c}\!/_{\sqrt{2.25}} = {}^{c}\!/_{1.5} = 0.67c$$
$$Z_0 = \sqrt{{}^{L}\!/_{\varepsilon_r C}} = {}^{138}\!/_{1.5} = 92\Omega$$

We see, that at high frequencies the dimensions of the cable do not matter to first order. Only by choosing the insulator material and its dimensions between signal wire and screen **one can chose the phase velocities and characteristic impedances**. And with plastic materials the refractive index $n = \sqrt{\varepsilon\mu} \approx \sqrt{\varepsilon}$ can be designed, as it depends of the density.

Lets take another look at the **signal integrity** in the cable with air cored insulator. When sending a 100MHz signal through it we are well in the high frequency regime discussed above. So:

$$\alpha \approx {}^{R}\!/_{2Z_0} = \frac{9 * 10^{-5}\sqrt{f}\,\Omega m^{-1}}{2 * 138\Omega} = 3.3\ 10^{-3} m^{-1}$$

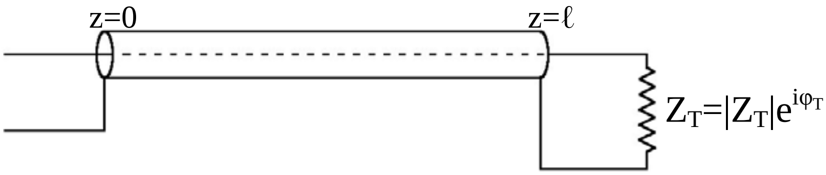
This means that the signal **decreases in amplitude** by the factor *1/e every 303m of cable*. Note, however, that because α actually depends on frequency, the pulse shape will change as the signal will be **distorted** as well as **attenuated**.

If the **integrity of the shape** of the signal is our main goal then we want α to be independent of frequency. This can be achieved! This requires G≠0, as true in real insulators. The condition is:

$$\frac{R}{L} = \frac{G}{C} \quad \Leftrightarrow \quad RC = GL$$

The downside of this is that the *attenuation increases*: $\alpha \approx R/Z_0$. Although the signal is now *undistorted* it is *attenuated twice* the original value. This can be verified by plugging into the equations for Z_0 and *p*.

Terminated lines: (H&H, 13.09, p. 879)
Our previous analysis enabled us to determine the characteristic impedance of a cable and the phase velocity of signals. To evaluate the extent of reflection and transmission for a given type of cable *under various different termination conditions* we can work with **reflection** and **transmission coefficients**. The situation we are working with can be pictured as:



The impedance at some general position z is:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{Ae^{-pz} + Be^{pz}}{Ae^{-pz} - Be^{pz}}$$

where A is the amplitude of the *forward signal* and B the amplitude of the *reflected signal*. Thus, the impedance at the end of the cable, where z=ℓ, is:

$$Z_T = \frac{V(\ell)}{I(\ell)} = Z_0 \frac{Ae^{-p\ell} + Be^{p\ell}}{Ae^{-p\ell} - Be^{p\ell}}$$

which can be rearranged to give the ratio:

$$\frac{\text{Reflected}}{\text{Incident}} = \frac{Be^{p\ell}}{Ae^{-p\ell}} = \frac{Z_T - Z_0}{Z_T + Z_0}$$

This is called the **voltage reflection coefficient, K_R**, and it is complex:

$$K_R = \frac{Z_T - Z_0}{Z_T + Z_0} \quad = |K_R|e^{i\varphi_R}$$

where φ_R is the **phase change** on reflection.

The resultant voltage transmitted to the terminating impedance is the *sum of the forward and backward voltages*. The ratio to the forward voltage is called the **voltage transmission coefficient, K_T**:

$$K_T = 1 + \frac{Be^{p\ell}}{Ae^{-p\ell}} = 1 + \frac{Z_T - Z_0}{Z_T + Z_0}$$

$$K_T = \frac{2Z_T}{Z_T + Z_0}$$

which means:

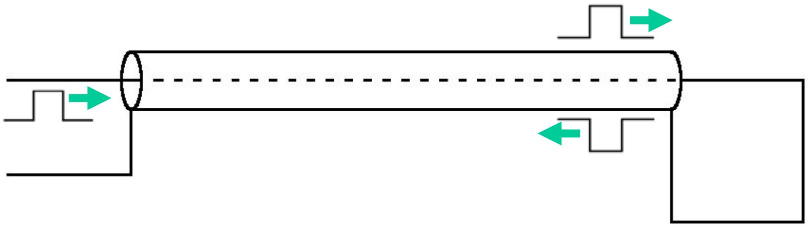
$$K_T - K_R = 1$$

Both of these coefficients apply to *voltages*. There are different formulae for current coefficients, which are derived in a similar way.

We are now in a position to consider the cases that are regularly encountered in the lab.

Special Case 1 : Short circuit termination

As shown in the sketch below, a **short circuit termination** is where the signal wire and the conducting screen are *attached to each other* at the far end of the coaxial cable. The load is zero, hence there cannot be any voltage transmitted to it. In terms of our coefficients this is written $Z_T = 0$:



$$K_R = \frac{Z_T - Z_0}{Z_T + Z_0} = -1 = |K_R|e^{i\varphi_R} \quad \text{and} \quad K_T = \frac{2Z_T}{Z_T + Z_0} = 0$$

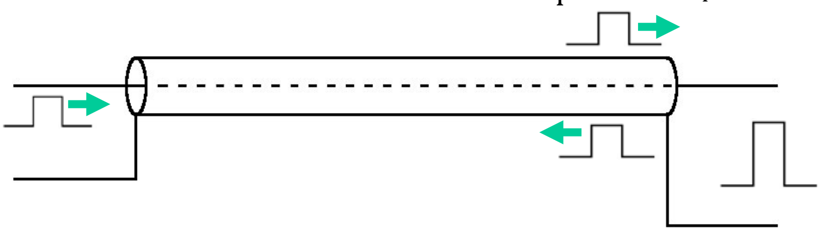
yielding:

$$|K_R| = 1 \quad \text{and} \quad \varphi_R = \pi$$

This means *100% reflection* with a *180° phase change*, as pictured for the pulse in the sketch above. You also can think of this as the charge of the pulse being able to flow at the termination *from the signal wire to the outer screen* and then returning towards the input on the outer screen, hence the inverted polarity of the signal.

Special Case 2: Open circuit termination

In the **open circuit termination** the signal wire and the conducting screen are not attached to each other or to anything else. Hence the load at the end of the cable corresponds to an infinite impedance. In terms of our coefficients this corresponds to $Z_T = \infty$:



$$K_R = \frac{Z_T - Z_0}{Z_T + Z_0} = +1 = |K_R|e^{i\varphi_R} \quad \text{and} \quad K_T = \frac{2Z_T}{Z_T + Z_0} = 2$$

yielding:

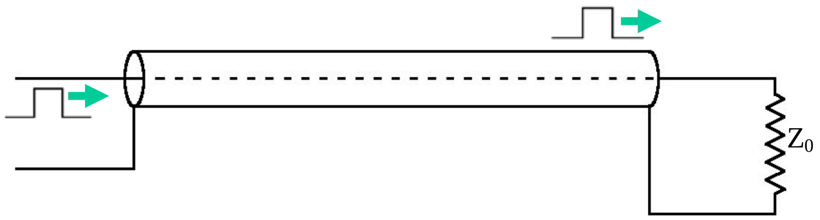
$$|K_R| = 1 \quad \text{and} \quad \varphi_R = 0$$

This again means *100% reflection* but now with *no phase change*. The charge which reaches the end of the signal wire has no other way than to change direction and to *return back on the signal wire*, hence the maintained polarity of the signal.

Note that the double height forward voltage appears across Z_T. From the discussion of input impedances in previous lectures it should no surprise anymore to see that a large input impedance will give rise to a large transmitted signal.

Special Case 3: Termination with Z_0

We already know that for **matched impedances** we do not get any reflections. We found that any line can be made to behave as though it were infinitely long by terminating it with a matching impedance Z_0. In terms of our coefficients we have $Z_T = Z_0$:



$$K_R = \frac{Z_T - Z_0}{Z_T + Z_0} = 0 = |K_R|e^{i\varphi_R} \quad \text{and} \quad K_T = \frac{2Z_T}{Z_T + Z_0} = 1$$

yielding:

$$|K_R| = 0$$

We get no reflections and the full voltage across load.

Reflections will distort the signal on the wire as the components superimpose. In applications which transmit signals this can be rather problematic. It therefore very *often is important to avoid reflections*. Here are a few examples:

- reflections in an analog TV signal between aerial and TV set caused shifted 'ghost images'
- reflections in modern digital TV signals causes the signal quality to degrade, the superposition of signals may cause the receiver to read wrong bits, leading to 'drop outs'
- in a coincidence circuit, where signals e.g. signify the presence of a particle or another type of event, reflections may cause double counting or fake coincidences.

This concludes what is covered in this lecture series on **linear** behavior. After the break of two weeks, where you have time to work on the *first part of your design exercise*, we will continue by looking at the behavior of *non-linear* components.