Introduction to the Standard Model

Lecture 11

III. Spontaneous Symmetry Breaking and the Higgs Mechanism

Outline: We now study gauge invariant models with a nontrivial vaccuum structure. This means the action is gauge invariant but the ground state (or vaccuum) is not:

$$
\begin{aligned}
S\big[\phi, \psi, F^{\mu\nu}\big] &= S\big[U\phi, U\psi, UF^{\mu\nu}U^{\dagger}\big] \\
S\big[\langle\phi\rangle, \langle\psi\rangle, \langle F^{\mu\nu}\rangle\big] &\neq S\big[U\langle\phi\rangle, U\langle\psi\rangle, U\langle F^{\mu\nu}\rangle U^{\dagger}\big]
\end{aligned}
$$

where $\langle\cdots\rangle$ denotes the *vaccuum expectation value* (or vev).

A vev for fermions ($\rightarrow$ spin) or gauge fields ($\rightarrow \langle\vec{E}\rangle, \langle\vec{B}\rangle$) would be incompatible with the observed isotropy of space.

$$\Rightarrow \langle\psi\rangle, \langle F^{\mu\nu}\rangle = 0$$

The vev for a scalar is allowed

$$\langle\phi\rangle \neq 0$$

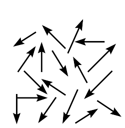
We will see that the Goldstone Theorem creates a massless mode for each generator T_a which does not leave the vaccuum invariant, $T_a\langle\phi\rangle \neq \langle\phi\rangle$.

Recall that the explicit mass term in the Lagrangian is not invariant. In combination with a gauge theory, the massless *Goldstone boson* will lead to a massive vector boson ($\rightarrow$ *Higgs mechanism*). A massless gauge boson together with a Goldstone boson leads to a massive gauge field. The gauge field acquires a longitudinal component through interaction with a nontrivial vaccuum.

Spontaneou Symmetry Breaking (SSB)

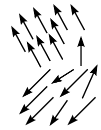
SSB is not specific to particle physics

i. Ferromagnets if $T > T_{\text{Currie}}$ then all spins are randomly aligned and uncorrelated resulting in a zero-mean magnetisation:



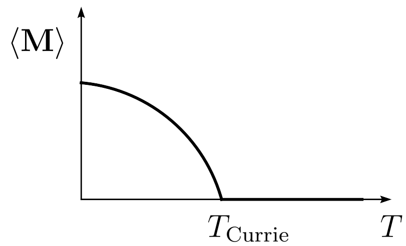
$$\Rightarrow \langle\mathbf{M}\rangle = 0$$

as $T \rightarrow T < T_{\text{Currie}}$, the spins align and become correlated giving rise to a mean magnetisation:



$\Rightarrow \langle \mathbf{M} \rangle \neq 0$

We can see this graphically by plotting the order parameter, $\langle \mathbf{M} \rangle$, against the temperature, T:



ii. Superconductors $\mathcal{L}$ has $U(1)$ gauge symmetry which corresponds to charge conservation.
if $T < T_{\text{crit}}$, the electrons will form pairs ("Cooper pairs") with spin alignment:

$$e^- \uparrow e^- \downarrow \qquad \text{with charge } Q = 2 \text{ and spin } \sigma = 0$$

The order parameter in this case is the average of the Cooper pair density: $\langle \phi(e^- \uparrow e^- \downarrow) \rangle$
The system gains energy if the electrons are not independent but act in pairs.

If the vaccuum symmetry is broken then a massless mode is created.

Example: SSB in $U(1)$ gauge theory

Note: As $U(1)$ is isomorphic to $SO(2)$ the models are equivalent.

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad \left(\text{or } \widetilde{\phi} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \text{ for } SO(2) \right)$$

$$\begin{aligned} \mathcal{L} \rightarrow \mathcal{L}_{\text{KG}} =& \partial_\mu \phi^\dagger \partial^\mu \phi - \mu^2 \phi^\dagger \phi - \lambda \left(\phi^\dagger \phi \right)^2 \qquad \text{with } \lambda > 0 \\ =& \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{\mu^2}{2} \left(\phi_1^2 + \phi_2^2 \right) - \frac{\lambda}{4} \left(\phi_1^2 + \phi_2^2 \right)^2 \end{aligned}$$

$\mathcal{L}$ is invariant under a global transformation

$$\phi \rightarrow e^{i\theta} \phi \qquad \left(\text{or } \widetilde{\phi} \rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \widetilde{\phi} \right)$$

The ground state is where the energy is minimised

$$
\begin{aligned}
\mathcal{H} =& \vec{\pi}\dot{\vec{\phi}} - \mathcal{L} \\
=& \frac{\partial \mathcal{L}}{\partial \dot{\phi}_1}\dot{\phi}_1 + \frac{\partial \mathcal{L}}{\partial \dot{\phi}_2}\dot{\phi}_2 - \mathcal{L} \\
=& \frac{1}{2}\underbrace{\left(\pi_1^2 + \pi_2^2\right)}_{\geq 0} + \frac{1}{2}\underbrace{\left(\nabla\phi_1 \cdot \nabla\phi_1 + \nabla\phi_2 \cdot \nabla\phi_2\right)}_{\geq 0} + V(\phi_1, \phi_2)
\end{aligned}
$$

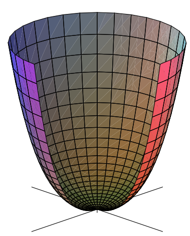
Hence, we see that to minimise $\mathcal{H}$ we need the minimum of the potential

$$
V(\phi_1, \phi_2) = \frac{\mu^2}{2}\left(\phi_1^2 + \phi_2^2\right) + \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2\right)^2
$$

The minimum condition reads

$$
\frac{\partial V}{\partial \phi_1} = \frac{\partial V}{\partial \phi_2} = 0 \quad \Leftrightarrow \quad \left.\begin{aligned} \phi_1\left(\mu^2 + \lambda(\phi_1^2 + \phi_2^2)\right) &\stackrel{!}{=} 0 \\ \phi_2\left(\mu^2 + \lambda(\phi_1^2 + \phi_2^2)\right) &\stackrel{!}{=} 0 \end{aligned}\right\} (*)
$$

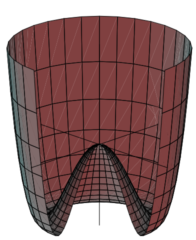
case 1. $\mu^2 > 0$**:** $\phi_1 = \phi_2 = 0$ is the ground state or vaccuum solution.



ϕ_1, ϕ_2 are real scalar fields with mass μ

The vaccuum state $\langle\phi_1\rangle = \langle\phi_2\rangle = 0$ is trivially invariant under rotations in the ϕ_1,ϕ_2-plane.

case 2. $\mu^2 < 0$**:** $(*)$ has a nontrivial solution



$$
2\phi^*\phi = \phi_1^2 + \phi_2^2 = \frac{-\mu^2}{\lambda} = v^2 > 0
$$

The ground state along the circle which gives infinite possibilities for the ground state. This is called the "champagne bottle" or "Mexican hat" potential.

As the theory is $U(1), SO(2)$ invariant we may choose

$$
\langle\phi_1\rangle = 0, \quad \langle\phi_2\rangle = v \qquad \left(\text{ or } \langle\widetilde{\phi}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \right)
$$

Applying a phase transform to the vaccuum we find

$$e^{i\Lambda\theta}\langle\phi\rangle \neq \langle\phi\rangle \qquad U(1) \text{ transform}$$

The physical spectrum is obtained after expanding around the vev of the theory

$$\phi_1 = \pi, \ \ \phi_2 = \sigma = H + \langle\phi_2\rangle = H + v$$

where the new fields have

$$\langle\pi\rangle = 0, \ \ \langle H\rangle = 0$$

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\partial_\mu\pi\partial^\mu\pi + \frac{1}{2}\partial_\mu H\partial^\mu H - \frac{\mu^2}{2}\left[\pi^2 + (v+H)^2\right] - \frac{\lambda}{4}\left[\pi^2 + (v+H)^2\right]^2 \\ =& \frac{1}{2}\partial_\mu\pi\partial^\mu\pi + \frac{1}{2}\partial_\mu H\partial^\mu H - \frac{\mu^2}{\lambda}\left[\pi^2 + v^2 + 2vH + H^2\right] - \frac{\lambda}{4}\left[\pi^2 + v^2 + 2vH + H^2\right]^2\end{aligned}$$

Collect the terms with different powers of π, H:

$\sim$	π^0, H^0 :	$-\frac{\mu^2}{\lambda}v^2 - \frac{\lambda}{4}v^4$	irrelevant constant; can't affect the E.o.M
$\sim$	π^0, H^1 :	$-\mu^2 v - \lambda v^3 = 0$	linear terms give rise to tadpoles, nonexistent in the theory
$\sim$	π^2, H^0 :	$-\frac{\mu^2}{2}v^2 - \frac{\lambda}{2}v^2 = 0$	π's are massless
$\sim$	π^0, H^2 :	$-\frac{\mu^2}{\lambda}v^2 - \frac{\lambda}{2}v^2 - \lambda v^2$	gives mass to the H term $MH^2 - 2\lambda v^2 = -2\mu^2 > 0$

The other terms define the interactions between π, H.
We conclude that

- π is a massless spin-zero boson, the *Goldstone boson*

- H is a massive spin-zero boson, the *Higgs boson*

Generalisation to $SO(N)$

$$\vec{\phi} = \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_{N-1} \\ \sigma \end{pmatrix} \in \mathbb{R}^N$$

This is in the fundamental representation of $SO(N)$ where the generators $U \in SO(N)$ are such that $UU^{-1} = 1$ and $detU = 1$. There are $N(N-1)/2$ generators, all of which are antisymmetric matrices.

$$U = \exp\big(i \sum_{i<j}^{N} \Lambda_{ij} T^{(ij)}\big) \quad \in SO(N)$$

where

$$\big(T^{(ij)}\big)_{kl} = -i\big(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\big)$$

The Lagrangian

$$\mathcal{L} \;= \frac{1}{2}\partial_\mu \vec{\phi}^{\,\mathrm{T}} \partial^\mu \vec{\phi} - \frac{\mu^2}{2}\big(\vec{\phi}^{\,\mathrm{T}}\vec{\phi}\big) - \frac{\lambda}{4}\big(\vec{\phi}^{\,\mathrm{T}}\vec{\phi}\big)^2$$

is invariant under global $SO(N)$ transformations.
For $\mu^2 < 0$, $V(\vec{\phi})$ is minimal if $\phi_i\phi_i = -\frac{\mu^2}{\lambda} = v^2 > 0$

hence we choose

$$\langle\vec{\phi}\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix}$$

$\langle\vec{\phi}\rangle$ is invariant under $SO(N-1)$ transformations (generators $T^{(ij)}$ with $i < j < k$ which are defined by the $T^{(ij)}\langle\vec{\phi}\rangle = \langle\vec{\phi}\rangle$). The remaining $N-1$ generators $T^{(ik)}$ *break* the vacuum as $T^{(ik)}\langle\vec{\phi}\rangle \neq \langle\vec{\phi}\rangle$. There are $\frac{1}{2}(N(N-1)) - \frac{1}{2}(N-1)(N-2) = N-1$ broken generators
We see that the vev breaks $SO(N)$ spontaneously to $SO(N-1)$.
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Looking at the spectrum, we find

- $\pi_{j=1,\ldots,N-1}$ massless Goldstone bosons

- $\sigma = H + v \rightarrow H$ is a massive state with mass $M_H^2 = 2\lambda V^2$ known as the Higgs boson mass