Space-time reduction in large-N gauge theories: the view from the lattice

Barak Bringoltz
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Numerical results obtained with S. Sharpe ’08-’09:

“Non-perturbative volume-reduction of large-N QCD with adjoint fermions.”, PRD, arXiv:0906.3538

“Breakdown of large-N quenched reduction in SU(N) lattice gauge theories.”, PRD, arXiv:0805.2146
Outline

I. What is space-time reduction.

II. Space-time reduction with adjoint fermions:
   A) Weak coupling perturbative analysis.
   B) Non-perturbative lattice Monte-Carlo studies with S. Sharpe `09

III. Conclusions + future prospects.
I. What is the space-time reduction?

Given a torus \((L_1 \times L_2 \times L_3 \times L_4)\), with an SU(N) lattice gauge theory \((g^2 N, a m, a \mu, \ldots)\)

Then if:

1. Translation symmetry is intact.
2. \(Z_N\) center symmetry is intact.
3. large-\(N\) factorization holds.

at \(N = \infty\) Wilson loops, Hadron spectra, condensates, etc.

are independent of \(L_1, L_2, L_3, L_4\).

- Reduce cost of large-\(N\) lattice studies.
- Leads to analytic weak-coupling small-volume methods (Unsal `07).
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Conditions derived by:
- Orbifold projections.
  - Neuberger `02,
  - Kovtun-Unsal-Yaffe `06
- Dyson-Schwinger Eqs.
  - Eguchi-Kawai `82
- Perturbation theory.
  - Bhanot-Heller-Neuberger`82,
  - Gross-Kitazawa `82, Parisi-Zhang `82, ...
How to argue it can be valid? can derive Dyson-Schwinger Eqs.

Show that these Eqs’ are independent of $L_{1,2,3,4}$
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Show that these Eqs’ are independent of \( L_{1,2,3,4} \)

\[
A_\mu(x) \rightarrow U_{n,\mu} \in SU(N) \quad @ \quad \text{periodic BC}
\]

\[
S_{\text{gauge}} = Nb \sum_{\substack{n \in \mathbb{Z}_N \cup \{0\} \\ \mu < \nu}} 2\text{Re } \text{Tr} \left( U_{n,\mu} U_{n+\mu,\nu} U_{n+\nu,\mu}^\dagger U_{n,\nu}^\dagger \right)
\]

\[
b = (g^2 N)^{-1}
\]

\[
W_C = \frac{1}{N} \text{tr} \left[ U_{x,\mu} U_{x+\mu,\nu} \cdots U_{x-\hat{\nu}-\hat{\rho},\rho} U_{x-\hat{\nu},\hat{\nu}} \right]
\]

\[
U_{n\mu} \rightarrow \Omega_n U_{n\mu} \Omega_{n+\mu}^\dagger \quad ; \quad \Omega_n \in SU(N)
\]

\[
U_{[(\bar{n},\tau),\mu]} \rightarrow U_{[(\bar{n},\tau),\mu]} z_\mu \quad ; \quad z_\mu \in \mathbb{Z}_N
\]
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Show that these Eqs’ are independent of \(L_{1,2,3,4}\)

<table>
<thead>
<tr>
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How to argue it can be valid? can derive Dyson-Schwinger Eqs.

- Like $\chi_S$ Ward identities:
  \[ U_{n,\mu} \rightarrow U_{n\mu} (1 + i\epsilon t^a) \quad \text{gauge} \]
  \[ U_\mu \rightarrow U_\mu (1 + i\epsilon t^a) \quad \text{reduced} \]

- Crucial difference:
  \[
  \text{gauge} \quad \text{reduced}
  \]
  \[
  \text{Tr (} \cdots U_{n\mu} U_{n+\mu,\nu} \cdots U_{m,\mu} U_{m+\mu,\rho} \cdots \text{) \quad \text{Tr (} \cdots U_\mu U_\nu \cdots U_\mu U_\rho \cdots \text{)}
  \]

- Get extra terms on the reduced side, so for EK reduction to hold:
  \[
  \langle \text{tr (} \cdots \text{) tr (} \cdots \text{)} \rangle \quad \text{reduced}
  \]
  \[ = 0 \]
  
  e.g. \[
  \langle \text{tr (} U_\mu U_\nu^\dagger \text{) tr (} U_\mu^\dagger U_\nu \text{)} \rangle \quad \text{reduced}
  \]
  \[ = 0 \]
• Reduction holds if

\[ \left\langle \text{tr} \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{array} \right\rangle \left\langle \text{tr} \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{array} \right\rangle = 0 \]

1. \( \langle W_{C_1} W_{C_2} \rangle_{\text{reduced}} = \langle W_{C_1} \rangle_{\text{reduced}} \langle W_{C_2} \rangle_{\text{reduced}} + O(1/N^2) \),

2. \( \langle W_{\text{open}} \rangle_{\text{reduced}} = 0. \) or \( U_\mu \rightarrow U_\mu z_\mu ; \ z_\mu \in Z_N \) intact

\[ \langle W_C \rangle_{\text{gauge theory}} = \langle W^\text{reduced}_C \rangle_{\text{reduced}} + O(1/N^2) \].

Lattice \( SU(N) \) on \( L^d \) \( \equiv \infty \) Lattice \( SU(N) \) on \( 1^d \)
I. What is the space-time reduction

Given SU(N) gauge theory on an $L_1 \times L_2 \times L_3 \times L_4$ lattice, defined by $g^2 N, a m, a \mu, \ldots$

Then if:

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at $N=\infty$ Wilson loops, Hadron spectra, condensates, etc.
are independent of $L_{1,2,3,4}$.
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This talk: fate of $\mathbb{Z}_N$. First at weak coupling, then non-perturbatively
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Then if:

- Translation symmetry is intact.
- $\mathbb{Z}_N$ center symmetry is intact.
- Large-N factorization holds.

Not “academic” requirements:

- Breakdown of EK equivalence by formation of a baryon crystal
  BB `08, BB `09

- Wilson loops, Hadron spectra, condensates, etc.
  are independent of $L_{1,2,3,4}$.

This talk: fate of $\mathbb{Z}_N$. First at weak coupling, then non-perturbatively
I. Some History

• Original Eguchi-Kawai. Jan `82. But Bhanot-Heller-Neuberger Feb `82.

• Quenched Eguchi-Kawai. Feb `82 Bhanot Heller Neuberger
  (Also, Parisi, Gross-Kitazawa, Das-Wadia, Migdal-Kazajov) But BB-Sharpe, 2008.

• Twisted Eguchi-Kawai. July `82 Gonzalez-Arroyo Okawa But, Teper-Vairinhos, Hanada-et-al, Bietenholtz et al. 2006.

• Adjoint Eguchi-Kawai. 2007 Kovtun-Unsal-Yaffe Most of this talk.

I. Some History


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  Gonzalez-Arroyo Okawa

Remarks original EK:

- The Bhanot-Heller-Neuberger `82 is suggestive but not conclusive.

- At moderate coupling symmetry may restore.
I. Some History


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Bhanot Heller Neuberger

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\[ 0 \quad Z_N \quad Z_N \quad \infty \text{ (continuum)} \]

\[ 1/g^2N \]
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Remarks original EK:

- The Bhanot-Heller-Neuberger `82 is suggestive but not conclusive.

- At moderate coupling symmetry may restore.

- Indeed, Neuberger-Naraynan-et-al. `02: increase $1^d \rightarrow L^d$. 
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II. Space-time reduction with adjoint fermions

Adjoint fermions are interesting:

- $N_f=1/2$ is softly broken $\mathcal{N}=1$ SUSY.
- $N_f=2$ is in (or close by to) the conformal window.
- Any value of $N_f$ and heavy enough quarks is YM.

Main motivation: study the $N_f=1$ theory

Can study large-$N$ limit of all these with method I will describe.
II. Space-time reduction with adjoint fermions

Study a large-$N$ limit of QCD where quarks are back-reacting on gauge fields

Natural to put quarks in two-antisymmetric:

Armoni-Veneziano-Shifman Planar equivalence `03:

```
\begin{align*}
\text{QCD(AS)} \\
\text{infinite volume} \\
2N_f \text{ fermions}
\end{align*}
```

```
\begin{align*}
\text{QCD(Adj)} \\
\text{infinite volume} \\
N_f \text{ fermions}
\end{align*}
```

```
\begin{align*}
\text{``orientifold equivalence''}
\end{align*}
```

Study 1-flavor adjoint QCD gives physical QCD

Want to study this on a single site: But is the $Z_N$ symmetry intact for this theory?
II.A. Weak coupling analysis, continuum

• Work on $R^3 \times S^1$ with PBC

• Calculate $V_{\text{eff}}$ for Polyakov loops $\Omega$ in continuum perturbation theory.

$$V(\Omega) = V_{\text{Glue}}^{1-\text{loop}}(\Omega) - 2N_f \ V_{\text{Fermi}}^{1-\text{loop}}(\Omega) = (1 - 2N_f) \ V_{\text{Glue}}^{1-\text{loop}}(\Omega) \ ; \ \text{mass} = 0$$

causes eigenvalue attraction $\text{tr}(\Omega) \neq 0$

• $Z_N$ broken if $N_f = 0$ (failure of EK model, or deconfinement)

• $Z_N$ unbroken for $N_f = 1,2$.

• Result is suggestive: not lattice, not single site, only perturbative
II.A. Weak coupling analysis, lattice, $L_{2,3,4} = \infty$, $L_1 = 1$. BB, `09

- Lattice one loop + Wilson fermions and axial gauge ($\Omega_{ab} = e^{i\theta_a} \delta_{ab}$)
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- Lattice one loop + Wilson fermions and axial gauge \( \Omega_{ab} = e^{i\theta_a \delta_{ab}} \)

Get

\[
V(\theta) = \sum_{a \neq b} \int \left( \frac{dp}{2\pi} \right)^3 \left\{ \log \left[ \hat{p}^2 + 4 \sin^2 \left( \frac{\theta^a - \theta^b}{2} \right) \right] - 2N_f \log \left[ \hat{p}^2 + \sin^2 (\theta^a - \theta^b) + m_W(\theta, p) \right] \right\}
\]

\[
\hat{p}^2 = 4 \sum_{i=1}^3 \sin^2 p_i/2 \xrightarrow{a \to 0} \hat{p}^2 \quad \hat{p}^2 = \sum_{i=1}^3 \sin^2 p_i \xrightarrow{a \to 0} \hat{p}^2 \quad m_W = am_0 + \frac{1}{2} \left[ \hat{p}^2 + 4 \sin^2 (\theta^a - \theta^b)/2 \right]
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\]

- What is the value of the one-loop potential at the $Z_N$ invariant g.s.? $\theta^a$:

\[
\sum_{a \neq b} f(\theta_a - \theta_b) \overset{N \rightarrow \infty}{\longrightarrow} N^2 \int \frac{d\omega}{2\pi} f(\omega) \quad \rightarrow \quad V(Z_N) = N^2 \int \frac{d\omega}{2\pi} \int \left( \frac{dp}{2\pi} \right)^3 \left\{ \log \left[ \vec{p}^2 + \omega^2 \right] - 2N_f \log \left[ \vec{p}^2 + \omega^2 + m_W^2(\omega, p) \right] \right\}
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\]

- As if L_1 = \infty: “Embedding of space-time in color space”

perturbation theory: Bhanot-Heller-Neuberger, Gross-Kitazawa, Parisi-Zhang '82, Neuberger '02

beyond perturbation? the soluble 1+1 case: Schon-Thies '01, BB '08.
II. A. Weak coupling analysis, lattice, \( L_{2,3,4} = \infty, L_1 = 1 \). BB, `09

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$$\hat{p}^2 = 4 \sum_{i=1}^{3} \sin^2 p_i/2 \xrightarrow{a \to 0} \overline{p}^2 \quad \hat{p}^2 = \sum_{i=1}^{3} \sin^2 p_i \xrightarrow{a \to 0} \overline{p}^2 \quad m_W = am_0 + \frac{1}{2} \left[ \hat{p}^2 + 4 \sin^2 \left( (\theta^a - \theta^b)/2 \right) \right]$$

- Calculate $V(\theta)$ potential for different $\theta^a$ corresponding $Z_N, Z_N \to \emptyset$, $Z_N \to Z_2$

**lattice units**

$$\kappa = \frac{1}{8 + 2am_0}$$

**massless**

$$\kappa = 1/8$$

**infinite mass**

$$\kappa = 0$$
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These are good news:

Reduction works with \(m=0\)!
II.A. Weak coupling analysis: related developments

- Bedaque et al. `09: Treat $L_1=1$ as 3D theory in a spatial continuum: $\mathbb{Z}_N \rightarrow \mathbb{Z}_2$ at $m=0$
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- Bedaque et al. ’09: Treat $L_1=1$ as 3D theory in a spatial continuum: $\mathbb{Z}_N \rightarrow \mathbb{Z}_2$ at $m=0$

- But can show that this result is UV sensitive BB ’09

$$S_{\text{gauge}} = \frac{2N}{\lambda} \text{Re} \sum_{i<j} \text{Tr} \left( U_{x,i} U_{x+i,j} U_{x+j,i}^\dagger U_{x,j}^\dagger \right)$$

$$+ \frac{2N}{\lambda} \text{Re} \sum_i \text{Tr} \left( U_{x,i} \Omega_{x+i} U_{x,i}^\dagger \Omega_x^\dagger \right)$$

$$S_{\text{one-site}} = \int d^3x \left( \frac{1}{g^2} \text{Tr} \sum_{i<j \in [1,3]} F_{ij}^2 + f^2 \text{Tr} \sum_i |D_i \Omega|^2 \right)$$

\[ \begin{align*}
D_i \Omega(x) &= \partial_i \Omega(x) + i[A_i, \Omega] \\
\Omega &\in SU(N)
\end{align*} \]
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V(\theta) = \sum_{a \neq b} \int \left( \frac{dp}{2\pi} \right)^3 \log \left[ a_i^2 p^2 + \sin^2 \left( \frac{\theta_a - \theta_b}{2} \right) \right] = \Lambda^3 + \sum_{a \neq b} \int \left( \frac{dp}{2\pi} \right)^3 \log \left[ 1 + \frac{1}{a_i^2 p^2} \sin^2 \left( \frac{\theta_a - \theta_b}{2} \right) \right]
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\]

\[
= \Lambda^3 + \Lambda \sum_{a \neq b} \sin^2 \left( \frac{\theta_a - \theta_b}{2} \right) + \ldots
\]

\[
= \Lambda^3 + \Lambda |\text{tr} \Omega_{\text{classical}}|^2 + \ldots, \Omega_{\text{classical}}^{ab} = e^{i\theta^a}
\]

$S_{\text{one-site}}$ is non-renormalizable ... need counter-terms...
II.A. Weak coupling analysis: related developments

What does this teach us?

- Continuum limit in space with $L_1 = 1$ (or for any $D > 2L_1$). Neuberger `02

- need counter-terms $\rightarrow$ new Low Energy Constants (LEC).

- means treating theory as an Effective Field Theory (EFT), and at one-loop:

$$V(\theta) \rightarrow V(\theta) + b_1 |\text{tr} \Omega_{\text{classical}}|^2 + b_2 |\text{tr} \Omega_{\text{classical}}^2|^2$$

Dim-reg hides this and implicitly sets $b_1=b_2=0 \rightarrow b_{1,2} > 0$ fixes $Z_N \rightarrow Z_2$ breaking.
II.A. Weak coupling analysis: related developments

What does this teach us?

- Continuum limit in space with $L_1=1$ (or for any $D > 2L_1$). Neuberger `02

- Need counter-terms $\rightarrow$ new Low Energy Constants (LEC).

- Means treating theory as an Effective Field Theory (EFT), and at one-loop:

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Dim-reg hides this and implicitly sets $b_1=b_2=0 \rightarrow b_{1,2} > 0$ fixes $Z_N \rightarrow Z_2$ breaking.

On $L_1=1$:

But $Z_N$-realization may depend on lattice action ...

Lattice results cannot be anticipated in advance (from Kovtun-Unsal-Yaffe `07)
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Update: Bedaque et al ’09 (postscript): add $b_1, b_2 > 0$, and find $Z_N \rightarrow Z_3$!
II.A. Weak coupling analysis: (very recent) related developments
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- Myers-Hollowood `09 generalized Kovtun-Unsal-Yaffe `07 to nonzero mass
  ** see also earlier Myers-Ogilvie `08

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L = \text{size of } S_1 \quad \text{For } ML > 0 \text{ get } Z_N \to Z_K \quad \text{with} \quad K \sim 1/ML
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V(\theta) = \sum_{a \neq b} \int \left( \frac{dp}{2\pi} \right)^3 \left\{ \log \left[ \hat{p}^2 + 4\sin^2 \left( \frac{\theta_a - \theta_b}{2} \right) \right] - 2N_f \log \left[ \hat{p}^2 + \sin^2 (\theta_a - \theta_b) + m_W^2(\theta, p) \right] \right\}
\equiv \sum_{a \neq b} \sum_r V_r e^{ir(\theta_a - \theta_b)} = \sum_r V_r |\text{tr } \Omega^r|^2 + \text{const.}
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\[ \Omega_{ab} = e^{i\theta_a} \delta_{ab} \]

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V_r = \int \frac{d\omega}{2\pi} e^{-ir\omega} \int \left( \frac{dp}{2\pi} \right)^3 \left\{ \log \left[ \hat{p}^2 + \hat{\omega}^2 \right] - 2N_f \log \left[ \hat{p}^2 + \hat{\omega}^2 + m_W^2(\omega, p) \right] \right\}
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\[ \text{dispersion relations of:} \]

fermions

gluons
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\]

Given mass \( m \), there will be \( r \)'s for which fermions unimportant and \( V_r < 0 \).
But really need a non-perturbative lattice study

- Really interested in $L_{1,2,3,4} = 1$, but IR div’s.
- What happens at $g^2N \simeq 1 - 3$?
- Non-perturbative effect (e.g. QEK and TEK model).

Simulate $N_f = 1$ Wilson adjoint fermions  \( \text{BB+S.Sharpe, 0906.3538} \)

Goal: Map single-site theory in $\kappa$ and $g^2N$

Look for intact $ZN$.

** Lattice`09: contrasting preliminary results  \( \text{Hietanen+Narayanan} \)
II.B. Results of non-perturbative MC lattice simulations

What should we expect? $L_{1,2,3,4}=\infty$:

- Continuum physics
- Strong-to-weak lattice transition
- Strong-coupling/lattice physics

Quarks are light along the line $\kappa_c(b)$.
II.B. Results of non-perturbative MC lattice simulations \text{BB+S.Sharpe, 0906.3538}

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Continuum physics

$(1/g^2 N =) b$

strong-to-weak lattice transition

quarks are light along line

strong-coupling/lattice physics

$\sim 0.04$  
$0.125$

$\kappa_e(b)$

$\sim 0.19$

$0$

$YM$

$K$

$0$

$0.25$

BB+S.Sharpe, 0906.3538

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BB+S.Sharpe, 0906.3538
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II.B. Results of non-perturbative MC lattice simulations  

- Use Metropolis algorithm with weight

\[
P(U) = \exp \left\{ \frac{2}{g^2} \sum_{\mu > \nu} \text{Re} \text{ Tr} \left[ U_{\mu} U_{\nu} U_{\mu}^\dagger U_{\nu}^\dagger \right] \right\} \times \det \left\{ 1 - \kappa \sum_{\mu} \left[ (1 + \gamma_{\mu}) U_{\mu}^G + (1 - \gamma_{\mu}) U_{\mu}^G \right] \right\}
\]

- Determinant is real & positive

- Update N(N-1)/2 SU(2) subgroups in turn on each of the 4 links

- Evaluate determinant explicitly: 50-60% accept.

- Scaling is \((N^2)^3 \times N^2\) \(\rightarrow\) can reach \(N=15\) on PCs

- Measure every 5 sweeps after \(~50\) sweeps thermalizations
  (1 sweep = 5 hits of Metropolis)

- 100-3700 measurements
II.B. Results of non-perturbative MC lattice simulations  BB+S.Sharpe, 0906.3538

Scan no. 1: infinitely massive quarks

X-axis: Real(Polyakov)
Y-axis: Imag(Polyakov)
II.B. Results of non-perturbative MC lattice simulations  

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\[ b = 0 \]
II.B. Results of non-perturbative MC lattice simulations BB+S.Sharpe, 0906.3538

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\[ b = 0 \]

\[ b \approx 0.3 \]
II.B. Results of non-perturbative MC lattice simulations  BB+S.Sharpe, 0906.3538

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X-axis: Real(Polyakov)
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$b = 0$

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$b = 0.5$
II.B. Results of non-perturbative MC lattice simulations  

Scan no. 2: decreasing the quark mass $b=0.5$, SU(10)
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Scan no. 2: decreasing the quark mass $\kappa = 0.5, SU(10)$
II.B. Results of non-perturbative MC lattice simulations \(^{\text{BB+S.Sharpe, 0906.3538}}\)

Scan no. 2: decreasing the quark mass \(b = 0.5\), \(\text{SU}(10)\)

\[
\kappa \approx 0 \quad \kappa = 0.03 \quad \kappa = 0.06
\]

![Graphs showing results for different values of \(\kappa\)]
II.B. Results of non-perturbative MC lattice simulations  BB+S.Sharpe, 0906.3538

Scan no. 2: decreasing the quark mass $b=0.5$, $SU(\tau)$

$k \approx 0$

$k = 0.03$

$k = 0.06$

$k = 0.12$
II.B. Results of non-perturbative MC lattice simulations \cite{BB+S.Sharpe, 0906.3538}

Scan no. 2: decreasing the quark mass $b=0.5$, SU(10)

- $\kappa \approx 0$
- $\kappa = 0.03$
- $\kappa = 0.06$
- $\kappa = 0.12$
- $\kappa = 0.1475$
II.B. Results of non-perturbative MC lattice simulations BB+S.Sharpe, 0906.3538

Scan no. 2: decreasing the quark mass $b=0.5$, $\text{SU}(10)$

<table>
<thead>
<tr>
<th>$\kappa$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$0.12$</td>
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II.B. Results of non-perturbative MC lattice simulations \textbf{BB+S.Sharpe, 0906.3538}

Scan no. 2: decreasing the quark mass $b=0.5$, SU(15)

\[
\kappa = 0 \quad \kappa = 0.06 \quad \kappa = 0.09 \\
\kappa = 0.1275 \quad \kappa = 0.1475 \quad \kappa = 0.155
\]
II.B. Results of non-perturbative MC lattice simulations BB+S.Sharpe, 0906.3538

Scan no. 2: looking for the “critical” line

b=0.35: 1st transition at kappa ~ 0.15
II.B. Results of non-perturbative MC lattice simulations BB+S.Sharpe, 0906.3538

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II.B. Results of non-perturbative MC lattice simulations BB+S.Sharpe, 0906.3538

Scan no. 2: looking for the “critical” line

1st transition structure at all $b$. Extending from $kappa=0.25$ to $0.125$
II.B. Results of non-perturbative MC lattice simulations  BB+S.Sharpe, 0906.3538

“Transition” present for all $N$ studied e.g. $b=0.5$, $N=8$, 10, 11, 13, 15:
II.B. Results of non-perturbative MC lattice simulations  

BB+S.Sharpe, 0906.3538

All results consistent with phase diagram and validity of reduction.

But:

More order parameters for nontrivial breaking of $Z_N$.

For example, $\text{tr} \, U_\mu \, U_\nu \neq 0$ in QEK
II.B. Results of non-perturbative MC lattice simulations  

\[ b = 0.5, \text{SU}(10): \quad \text{tr} \ U_{\mu} \]

\[ \kappa = 0.0001 \quad \kappa = 0.1275 \quad \kappa = 0.245 \]

\[ \kappa = 0.275 \quad \kappa = 0.29 \quad \kappa = 0.495 \]

Indicate \( Z_N \) breaking for \( \kappa \gtrsim 0.28 \).
II.B. Results of non-perturbative MC lattice simulations  

\[ \text{b=0.5, SU}(10): \quad \text{tr} \, U_\mu \, U_\nu \]

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Indicate \( Z_N \) breaking for \( \kappa \gtrsim 0.24 \)
II.B. Results of non-perturbative MC lattice simulations  BB+S.Sharpe, 0906.3538

Perform long runs and measure order parameters of the form

\[ \text{tr} \left[ P_1^{n_1} P_2^{n_2} P_3^{n_3} P_4^{n_4} \right] \text{ with } n_i \in [-5, 5] \rightarrow 14641 \text{ order parameters} ! \]

- For each histogram signal-to-noise for real and imag. part
II.B. Results of non-perturbative MC lattice simulations

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- For each histogram signal-to-noise for real and imag. part

\[ Z_N \text{ unbroken} \]

\[ Z_N \text{ broken} \]
II.B. Results of non-perturbative MC lattice simulations

Results for $K_n$ at $b=1$:

- $N = 10$, $b = 1.0$, $\kappa = 0.09$
- $N = 10$, $b = 1.0$, $\kappa = 0.1275$
- $N = 13$, $b = 1.0$, $\kappa = 0.09$
II.B. Results of non-perturbative MC lattice simulations  BB+S.Sharpe, 0906.3538

What about the bulk transition?
II.B. Results of non-perturbative MC lattice simulations  

**What about the bulk transition?**

\( \kappa = 0 \)

\[ \frac{1}{g^2 N = b} \]

- Signs of bulk.
- Does not involve \((Z_N)^4\) breaking
II.B. Results of non-perturbative MC lattice simulations \( \text{BB+S.Sharpe, 0906.3538} \)

- All our results consistent with conjecture

- Away from critical line, long distance theory is pure-gauge \( \Rightarrow \) original EK ?!?

- Near critical line, obtain adjoint QCD, within \( 1/N \) of physical QCD.
Results of “physical” interest: first pass
II.B. Results of non-perturbative MC lattice simulations \textsuperscript{BB+S.Sharpe, 0906.3538}

**plaquette (action density):**

- Not really physical, but an indication of $1/N^2$ corrections’ size.
- Different from YM even if quarks are heavy
- But nevertheless....

Integrating quarks

Different effective action
II.B. Results of non-perturbative MC lattice simulations \(BB+S.Sharpe, 0906.3538\)

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II.B. Results of non-perturbative MC lattice simulations  BB+S.Sharpe, 0906.3538

**Dirac spectrum:**

Eigenvalue distribution of valence Dirac operator and *quenched* Random Matrix Theory
II.B. Results of non-perturbative MC lattice simulations \textbf{BB+S.Sharpe, 0906.3538}

**Dirac spectrum:**

Eigenvalue distribution of valence Dirac operator and \textit{quenched} Random Matrix Theory

In any case:

Values of $N$ that we used seem sufficient to map the phase-diagram.
III. Conclusions

**Weak coupling with $L_{2,3,4}=\infty, L_1=1$**

- Space-time reduction seems to work for massless Wilson adjoints fermions.

  In progress: analyzing $Z_N \rightarrow Z_K$ for massive fermions.

- Perturbation theory on $L_1=1$ leads to a UV-sensitive free energy.

  Do weak-coupling before MC with a different type of fermion!

**Non-perturbative lattice Monte-Carlo of $N_f=1$ case.**

- Space-time reduction seems to works for YM+Wilson adjoints fermions.

  At couplings where we foresee doing calculations.

  Both heavy and light masses.
III. Future directions

**Can extract physics of QCD(Adj) and QCD(AS)**

- Probably need to develop algorithms that would reduce the $N^8$ scaling of Metropolis!
- Mesons (using the “Gross-Kitazawa trick”)
- Realization of chiral symmetry (of both adjoint sea-quarks and valence fundamental).
- Comparisons with RMT.
- Static potential, string tensions.
- Other theories: is the two-flavor theory (nearly-)conformal?

Currently in progress...