On comparing the phase diagram of SU(N) gauge theories at weak and strong coupling

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How do we compare phase diagrams at weak and strong coupling?

- The phase diagram of $SU(N)$ gauge theories at weak coupling can be obtained by perturbation theory on small volume manifolds such as the sphere $[S^1 \times S^3]$.
- This can be compared to the phase diagram at strong coupling obtained using lattice gauge theory simulations on large volumes such as the torus with one shorter dimension $[S^1 \times \mathbb{R}^3]$.

Example: adjoint QCD

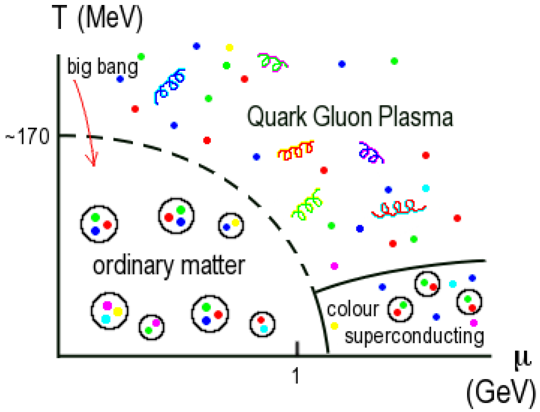
- Why useful?: Learn about QCD at large N from adjoint QCD.

orientifold planar equivalence: (Armoni, Shifman, Veneziano)	The large N equivalence of QCD(AS/S) and adjoint QCD
large N reduction: (Kovtun, Unsal, Yaffe)	Volume independence within adjoint QCD

- How?: Formulating adjoint QCD on the sphere.

- Compare: Results for the phase diagram of adjoint QCD as a function of volume and fermion mass. Comparison of perturbation theory and lattice results.

Why not just study QCD directly?



QCD phase diagram. The natural state of ordinary terrestrial matter is one in which quarks and gluons are confined.

- Confined phase

  Quarks and gluons are bound into hadrons.

- Deconfined phase

  Quarks and gluons are free in the QGP.

- Superconducting phases

  Cooper pairing of quarks occurs at neutron star densities.

The coupling strength in QCD

Euclidean spacetime QCD partition function:

$$Z_{QCD} = \mathrm{Tr} e^{-\beta H} = \int \mathscr{D}A \mathscr{D}\bar{\psi} \mathscr{D}\psi e^{-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x} \mathscr{L}_{QCD}}$$

Lagrangian:

$$\mathscr{L}_{QCD} = \frac{1}{4g^2} \mathrm{Tr}_F \left(F_{\mu\nu} F_{\mu\nu} \right) + \bar{\psi} \left(\not{D} + M - \gamma_4 \mu \right) \psi$$

running coupling strength:

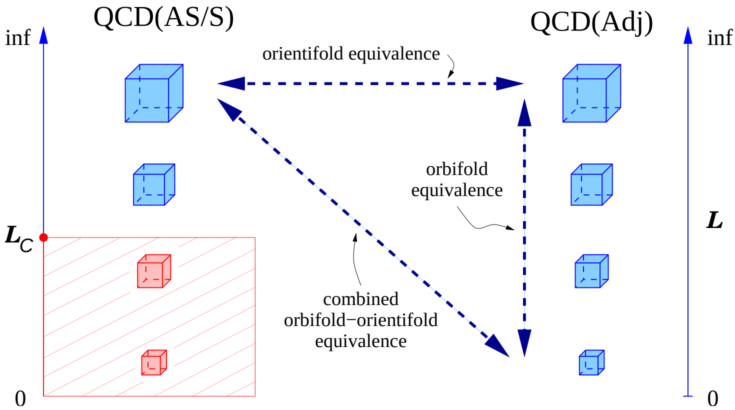
$$g^2(k) = \frac{g^2(k_0)}{1 + \frac{g^2(k_0)}{(4\pi)^2} (\frac{11}{3} N - \frac{2}{3} N_f) ln(k^2/k_0^2)}$$

In QCD $N = 3$ and $N_f = 6$:

- $g(k) \uparrow$ as $k \downarrow$
  - strong coupling at low energies (large distances)
  - **Perturbation theory is not valid in the low energy confined phase of QCD**

What to do?

- Lattice gauge theory works regardless of the coupling strength.
- However, simulations can be computationally expensive.
- Implement shortcut:



Kovtun, Unsal and Yaffe: "Volume independence in large Nc QCD-like gauge theories" (hep-th/0702021).

Conditions for equivalence

Orientifold equivalence:	Charge conjugation symmetry cannot be broken in QCD(AS/S).
Orbifold equivalence: (volume independence)	Adjoint QCD, formulated on both volumes, must have $Z(N)$ symmetry intact.

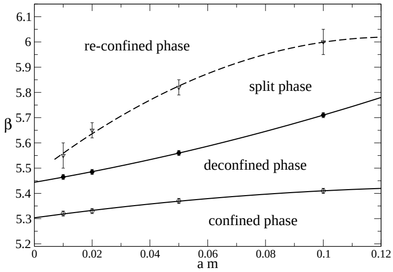
- $Z(N)$ symmetry breaking is realized in terms of the Polyakov loop order parameter $P = e^{\beta A_0}$:

In the confined phase $\mathrm{Tr} P = \mathrm{Tr}(zP) = 0, \quad \text{where } z \in Z(N).$

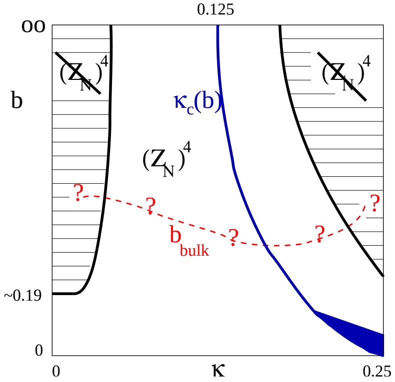
In the deconfined phase $\mathrm{Tr} P \neq \mathrm{Tr}(zP) \neq 0.$

Relevant checks from the lattice

- Orientifold planar equivalence: DeGrand, Hoffmann, Schaefer, and Liu (2006) calculated the quark condensate in one-flavour QCD and found it to agree with the SYM prediction.
- Cossu and D'Elia (2009), Bringoltz and Sharpe (2009) performed lattice simulations of adjoint QCD, but found potentially contradictory results.



Cossu and D'Elia 0904.1353



Bringoltz and Sharpe 0906.3538

Perturbative check?

- Can formulate adjoint QCD on a finite volume to access the Yang-Mills deconfinement transition in the large N limit (Aharony et al (hep-th/0310285)).
- adding massive fermions should allow for comparison with lattice results, using the knowledge that $\beta \sim 1/L_{S^1}$.

Region of validity of 1-loop calculations

Properties of $SU(N)$ gauge theories on $S^1 \times S^3$

- Valid for $\min[R_{S^1}, R_{S^3}] \ll \Lambda_{QCD}^{-1}$
  - Small S^1:
    - Good: Allows study at any N and in the limit of large 3-volume.
      $\mathbb{R}^3 \times S^1$: YM/QCD: $m=0, \mu=0$: Gross, Pisarski, Yaffe (Rev.Mod.Phys.53:43,1981), $\mu \neq 0$: Korthals Altes, Pisarski and Sinkovics (hep-ph/9904305), $m \neq 0$: Meisinger and Ogilvie (hep-ph/0108026), QCD(Adj/AS/S): $m=0, \mu=0$: Unsal and Yaffe (hep-th/0608180), $m \neq 0$: Myers and Ogilvie (arXiv:0903.4638)
    - Bad: Have to be in the limit of high temperatures (or small S^1)
  - Small S^3:
    - Good: Allows study at any temperature (or any S^1).
      $S^3 \times S^1$: YM: Aharony et al (hep-th/0310285), QCD(Adj/AS/S): $m=0$: Hollowood and Naqvi (hep-th/0609203), Unsal (hep-th/0703025), $m \neq 0$: Hollowood and Myers (arXiv:0907.3665)
    - Bad: Must be in small 3-volume. Finite N studies are more complicated.

1-loop Lagrangian

$$\mathscr{L}_E = -\frac{1}{2}\bar{A}_0^a(\tilde{D}_0^2(a)+\Delta^{(s)})\bar{A}_0^a - \frac{1}{2}B_i^a(\tilde{D}_0^2(a)+\Delta^{(v,T)})B_i^a$$
$$-\frac{1}{2}C_i^a(\tilde{D}_0^2(a)+\Delta^{(v,L)})C_i^a - \bar{c}(\tilde{D}_0^2(a)+\Delta^{(s)})c + \bar{\psi}(\not{D}_A(a)+m)\psi$$

where

$$A_i = B_i + C_i.$$

- B_i = transverse: $\nabla_i B_i = 0$
- C_i = longitudinal: $C_i = \nabla_i f$

and

$$\tilde{D}_0 \equiv \partial_0 + \alpha^a T_A^a$$

where α is the only zero mode

$$\alpha \equiv \frac{1}{Vol(S^1\times S^3)}\int_{S^1\times S^3} \mathrm{d}\tau\, \mathrm{d}^3x\, A_0(\mathbf{x})$$

1-loop partition function

The partition function, at one loop:

$$Z = \det_{l=0}^{1/2}(-\tilde{D}_0^2(a) - \Delta^{(s)})\det^{-1}(-\tilde{D}_0^2(a) - \Delta^{(v,T)})\det^{N_f/4}(-\not{D}^2(a) - \Delta^{(f)})$$

Eigenvalues and degeneracies of Laplacians on S^3:

$$\Delta^{(type)}\Omega_{j,l,m_1,m_2}(\theta_1,...,\theta_3) = -\varepsilon_l^{(type)2}\Omega_{j,l,m_1,m_2}(\theta_1,...,\theta_3)$$

Example: scalars

$$\varepsilon_l^{(s)2} = l(l+2)/R^2$$

$$d_l^{(s)} = (l+1)^2$$

for scalars and spinors $l = 0, 1, ...,$
for vectors $l = 1, 2, ...$

1-loop partition function: S^1 contribution

The eigenvalues of the Dirac operator can be computed in frequency space in terms of the Matsubara frequencies:

$$\tilde{D}_0 \rightarrow i\omega_n^+ + \alpha$$

where the Matsubara frequencies are

$$\omega_n^+ = 2n\pi/L$$

We define the Polyakov loop:

$$P(\vec{x}) = \mathscr{P} e^{\int_0^L \mathrm{d}\tau\, A_0(x)} = e^{L\alpha} = \mathrm{diag}\{e^{i\theta_1}, ..., e^{i\theta_N}\}$$

one-loop effective action

To get the effective action we need the $\ln Z$. Calculating the sum over Matsubara frequencies and simplifying one can show that

$$\mathrm{Tr}_A \sum_{l=0}^{\infty} \ln\left(-\tilde{D}_0^2 + \varepsilon_l^2\right) = \sum_{l=0}^{\infty} d_l \left[L\varepsilon_l - 2\sum_{n=1}^{\infty} \frac{1}{n} e^{-nL\varepsilon_l} \mathrm{Tr}_A P^n\right]$$

for the fermion contribution:

$$\varepsilon_l^{(f)} \to \omega_l = \sqrt{\varepsilon_l^{(f)2} + m^2}$$

Effective action is:

$$S_{1-loop} = -\ln Z$$
$$= \sum_{n=1}^{\infty} \frac{1}{n} \left[1 - \sum_{l=0}^{\infty} d_l^{(v,T)} e^{-nL\varepsilon_l^{(v,T)}} + 2N_f \sum_{l=0}^{\infty} d_l^{(f)} e^{-nL\omega_l^{(f)}}\right] \mathrm{Tr}_A(P^n)$$

1-loop effective action in Yang-Mills theory [Aharony et al (hep-th/0310285)]

In terms of the Polyakov loop $P = \mathrm{diag}\{e^{i\theta_1}, \ldots, e^{i\theta_N}\}$ the effective action is

$$
\begin{aligned}
S(P) &= \sum_{n=1}^{\infty} \frac{1}{n} (1 - z_v(nL/R))\, \mathrm{Tr}_A P^n \\
&= \sum_{n=1}^{\infty} \frac{1}{n} (1 - z_v(nL/R)) \sum_{i,j=1}^{N} \cos[n(\theta_i - \theta_j)]
\end{aligned}
$$

where

- L is the length of S^1
- R is the radius of S^3

$$
\begin{aligned}
z_v(nL/R) &= \sum_{l=0}^{\infty} d_l^{(v,T)} e^{-nL\varepsilon_l^{(v,T)}} \\
&= \sum_{l=0}^{\infty} 2l(l+2) e^{-nL(l+1)/R}
\end{aligned}
$$

The weak-coupling analogue of the deconfinement transition temperature can be calculated in the large N limit. It is $T_d R \simeq 0.759$ or $L_d/R \simeq 1.317$.

1-loop effective action: $SU(N)$ gauge theories + fermions

The effective action for N_f Dirac flavours of fermions in representation R with mass m and chemical potential μ is

$$S(P) = \sum_{n=1}^{\infty} \frac{1}{n} (1 - z_v(nL/R)) \mathrm{Tr}_A P^n + 2 \sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n} N_f z_f(nL/R, mR) \left[e^{nL\mu} \mathrm{Tr}_R(P^{\dagger n}) + e^{-nL\mu} \mathrm{Tr}_R(P^n) \right]$$

where

$$z_f(nL/R, mR) = 2 \sum_{l=1}^{\infty} l(l+1) e^{-nL\sqrt{(l+1/2)^2 + m^2 R^2}/R}$$

The effective action for N_f Majorana flavours of adjoint fermions with mass m and $\mu = 0$ simplifies to

$$S(P) = \sum_{n=1}^{\infty} \frac{1}{n} (1 - z_v(nL/R) + N_f z_f(nL/R, mR)) \sum_{i,j=1}^{N} \cos[n(\theta_i - \theta_j)]$$

Large N limit

In the limit of large N it is helpful to consider the distribution of the Polyakov loop eigenvalues around the circle (following Aharony et al hep-th/0502149):

$$Z(L/R) = \int [\mathrm{d}\theta]\, e^{-\sum_{n=1}^{\infty} \frac{1}{n}(1 - z_v(nL/R) + N_f z_f(nL/R, mR))|\mathrm{Tr} P^n|^2}$$

Take:

$$\rho_n \equiv \int e^{in\theta} \rho(\theta) \mathrm{d}\theta = \frac{1}{N} \mathrm{Tr}(P^n), \qquad \rho(\theta) = \frac{1}{N} \sum_{i=1}^{N} \delta(\theta - \theta_i),$$

$$f(nL/R, mR) \equiv (1 - z_v(nL/R) + N_f z_f(nL/R, mR)).$$

Then

$$Z(L/R) = \int \mathrm{d}\rho_n \mathrm{d}\bar{\rho}_n e^{-N^2 \sum_{n=1}^{\infty} \frac{1}{n} f_n |\rho_n|^2}.$$

When N is large the path integral can be solved using the saddle point approximation.

Phases of large N

$$Z(L/R) = \int \mathrm{d}\rho_n \mathrm{d}\bar{\rho}_n e^{-N^2 \sum_{n=1}^{\infty} \frac{1}{n} f_n |\rho_n|^2}.$$

Fourier analyze the density:

$$\rho(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \rho_n e^{in\theta}$$

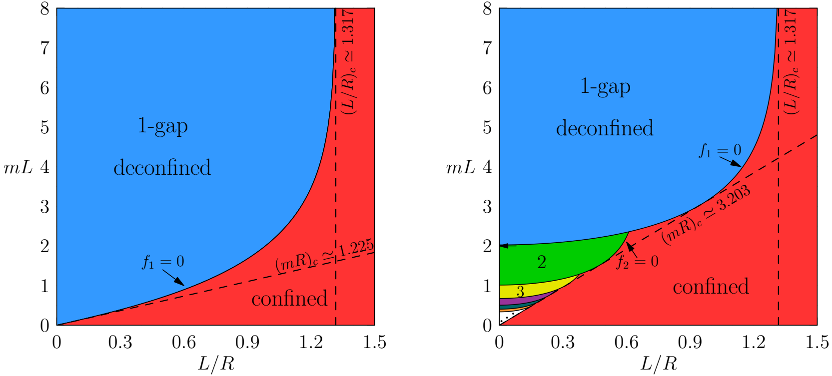
where $\rho_0 = 1$, and $\rho_n^* = \rho_{-n}$.

Phases:

Confined: $f_n > 0$ for all n, $\rho_n = \frac{1}{N}\mathrm{Tr}P^n = 0$.

k-gap: $f_k < 0$, $\rho_k = \frac{1}{N}\mathrm{Tr}P^k \neq 0$, but $\mathrm{Tr}P^l = 0$ for $mod[l,k] \neq 0$.

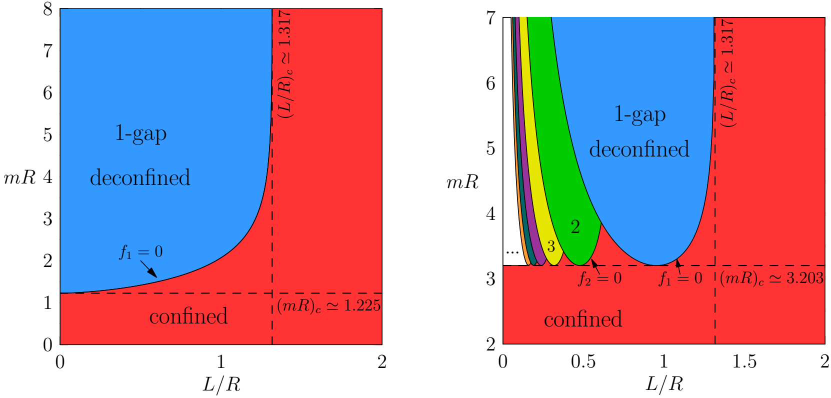
Large N: Phase diagram in $(L/R, mL)$ plane



$N_f^M = 1$ $\qquad$ $N_f^M = 2$

- The confined phase persists without phase transition for all L/R below a critical mass $(mR)_c$ which increases with N_f.
- For $N_f \geq 2$ partially confined phases become possible.

Large N: Phase diagram in $(L/R, mR)$ plane



$N_f^M = 1$ $\quad$ $N_f^M = 2$

- The confined phase is only transition-free below $(mR)_c$, and for $L/R > 1.317$.
- For $N_f \geq 2$ the confined phase at small L/R and $mR > (mR)_c$ the confined phase is pushed all the way to the (mR)-axis.

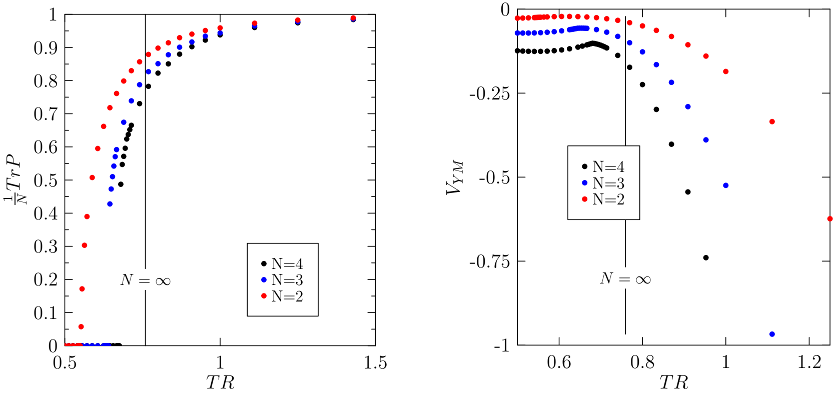
Finite N

- The phase diagram can also be calculated at finite N by numerically minimizing the effective action directly (the saddle-point approximation is only valid for large enough N).
- Expectation values of observables can also be computed exactly by performing the integrals over the gauge fields. This serves as a useful check, but not used as the primary technique since there is no suitable observable to distinguish the k-gap phases: $\langle \mathrm{Tr} P \rangle = 0$ in all phases.

The finite N phase diagram is useful for several reasons

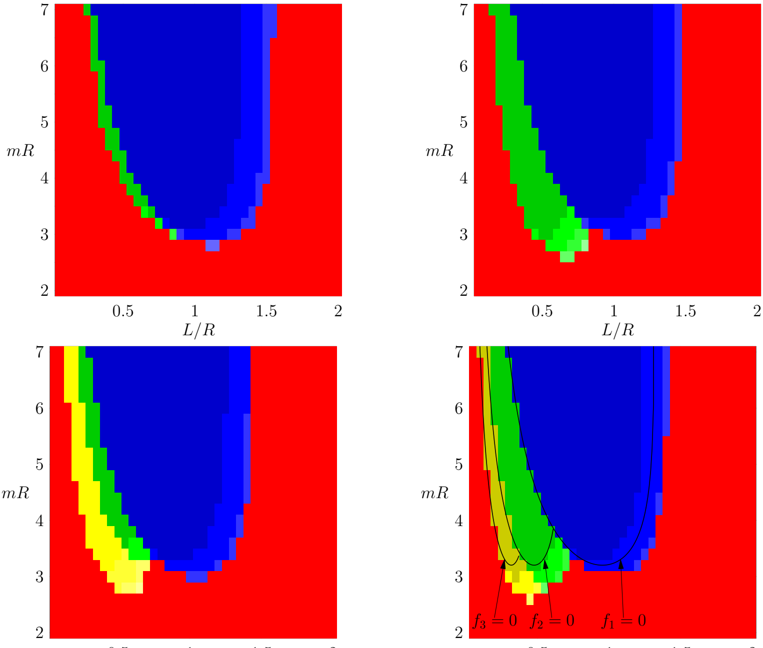
1. We can determine how the large N limit is approached and perhaps show when the saddle point approximation becomes valid.
2. It is possible to compare with finite N results on $\mathbb{R}^3 \times S^1$ by taking the limit $L/R \to 0$.
3. The phase diagram at finite N and finite volume can be compared more easily with lattice results.

Finite N: Yang-Mills theory

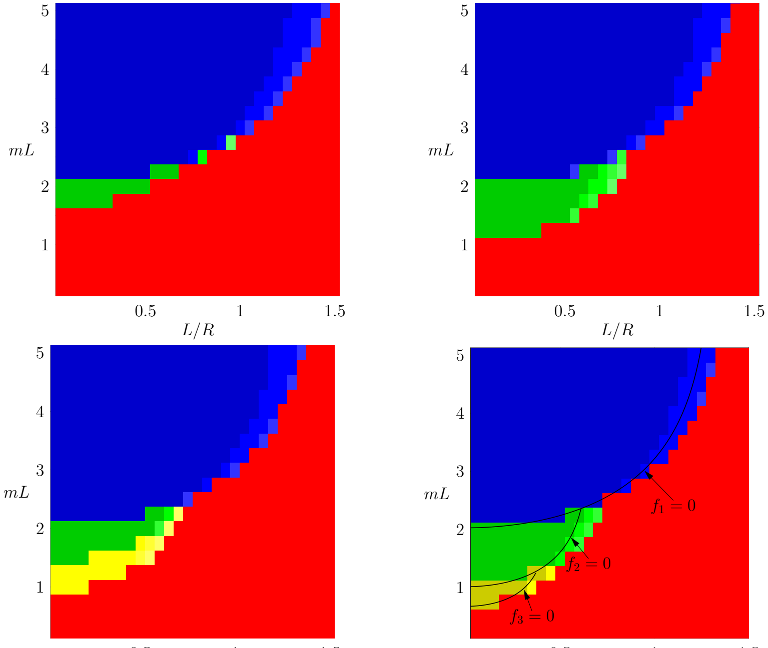


- Even for $N = 2$, the weak-coupling deconfinement transition is clearly indicated by the discontinuity in $\mathrm{Tr}P$.
- The transition increases in sharpness as N is increased. The formation of a discontinuity in the free energy that is sharpened with increasing N is consistent with a first order phase transition in the large N limit.

$N = 3, 4, 5, 6$: Phase diagram in $(L/R, mR)$ plane

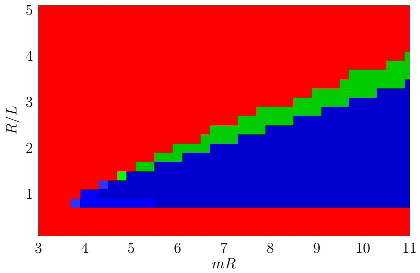


$N = 3, 4, 5, 6$: Phase diagram in $(L/R, mL)$ plane

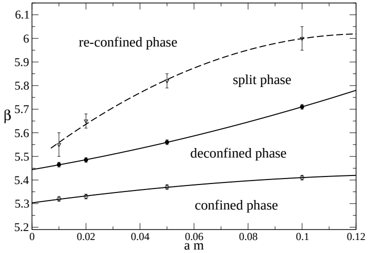


Comparison with $N = 3$ lattice results of Cossu and D'Elia (arXiv:0904.1353)

Thanks go to Cossu and D'Elia for allowing us to show their figures.



$S^3 \times S^1$

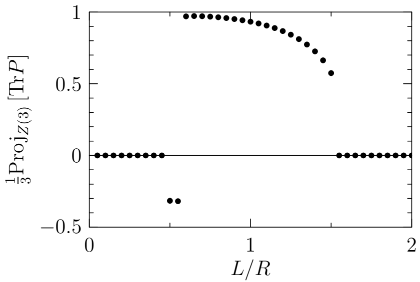


$16^3 \times 4$ lattice

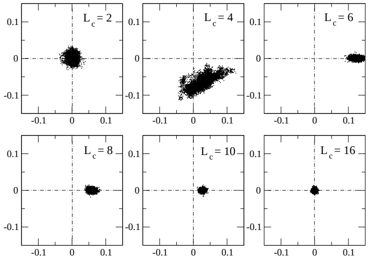
In the $S^3 \times S^1$ result the confined phase passes through for small enough mR as R/L is increased. This is unclear for the lattice results. It might be that the deconfined phase drops down to lower values of mR for larger coupling, or it could be that the chiral limit has not been reached in the simulations.

Comparison of the Polyakov loop order parameter

For a fixed value of the fermion mass, we can also compare the behaviour of the Polyakov loop with Cossu and D'Elia (arXiv:0904.1353):



$S^3 \times S^1$, $mR = 6$



$16^3 \times L_c$ lattice, $ma = 0.10$

- At a qualitative level the behaviour of the Polyakov loop appears to agree for all L/R, even within the deconfined phase.

Conclusions

- The phase diagram of adjoint QCD on $S^3 \times S^1$ has a rich phase structure.
- This phase diagram can be calculated at large N limit using the saddle point approximation. The confined phase is transition free for all L/R if mR is below a critical value which increases with N_f.
- At finite N the partition function can be evaluated numerically to get a phase diagram which agrees with the $\mathbb{R}^3 \times S^1$ result in the $L/R \to 0$ limit. It also approaches the large N result, and closely resembles the lattice result of Cossu and D'Elia.

Future work

A similar analysis can be performed for other gauge theories. One can consider:

- different fermion representation
- different gauge field representation
- adding scalar fields
- different boundary conditions
- incorporating finite chemical potential
- other manifolds (eg. torus)