Constructing One-Loop Amplitudes

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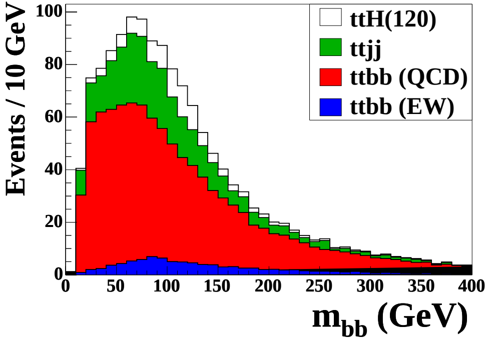
Outline

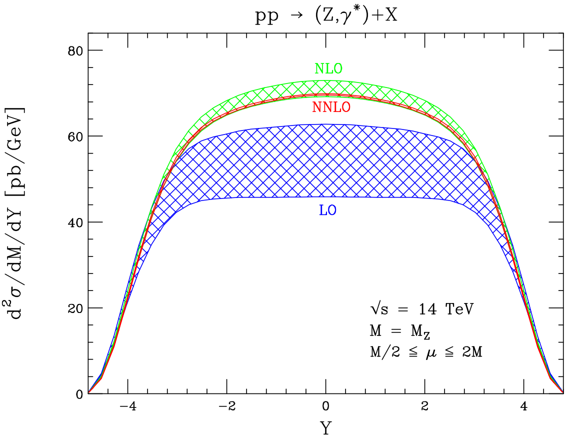
1. One-loop amplitudes and the unitarity method
2. Tadpole coefficients from exotic cuts
   [with B. Feng]
3. Single cut integration
   [with E. Mirabella]

QCD background at LHC



Higgs Signal With Background (top quark associated production)

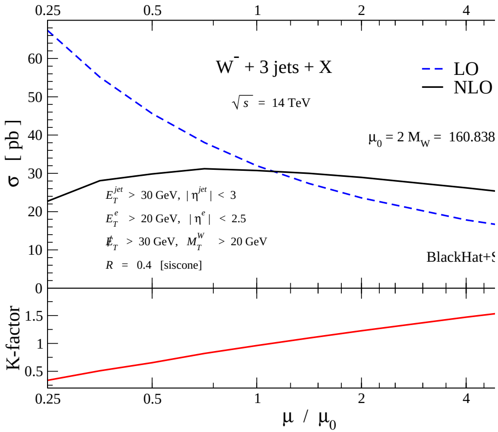
Next-to-Leading-Order Effects Are Large



Rapidity distribution of a Z boson at LHC. $\alpha_s = 0.121$ at M_Z.

[Anastasiou, Dixon, Melnikov, Petriello]

NLO reduces scale dependence



Scale dependence of cross sections for W + 3 jets.

[BlackHat collaboration (Berger et al.)]

Prioritized Wish List, Next-to-Leading Order

[Les Houches Physics at TeV colliders 2007, NLO multileg working group: Summary report; updated 2009]

Done [a]	$p\ p \to t\ \bar{t}\ b\ \bar{b}$	background for $t\bar{t}H$
Done [b]	$p\ p \to t\ \bar{t}\ + 2$ jets	relevant for $t\bar{t}H$
	$p\ p \to V\ V\ b\ \bar{b}$	relevant for benchmark processes
	$p\ p \to V\ V + 2$ jets	VBF $\to H \to VV$
Done [c]	$p\ p \to V + 3$ jets	new physics
	$p\ p \to b\ \bar{b}\ b\ \bar{b}$	Higgs and new physics
	$p\ p \to t\ \bar{t}\ t\ \bar{t}$	new physics
	$p\ p \to W\ b\ \bar{b}\ j$	new physics

[a]: Bredenstein, Denner, Dittmaier, Pozzorini; Bevilacqua, Czakon, Papadopoulos, Pittau, Worek

[b]: Bevilacqua, Czakon, Papadopoulos, Worek

[c]: Berger, Bern, Dixon, Febres Cordero, Forde, Gleisberg, Ita, Kosower, Maître; Ellis, Melnikov, Zanderighi

One-loop amplitudes: analytic results

2006: 6 gluons. Complexity of $2 \to 4$ scattering in QCD
[Ellis, Giele, Zanderighi; Bedford, Berger, Bern, Bidder, Bjerrum-Bohr, Brandhuber, RB, Buchbinder, Cachazo, Dixon, Dunbar, Feng, Forde, Ita, Kosower, Mastrolia, Perkins, Spence, Travaglini, Xiao, Yang, Zhu]

Completed recently:
pp $\to$ Higgs + 2 jets. [Badger, Berger, Campbell, Del Duca, Dixon, Ellis, Giele, Glover, Mastrolia, Risager, Sofianatos, Williams, Zanderighi]
pp $\to$ $t\bar{t}$ [Badger, Sattler, Yudin]
$0 \to d\bar{u}Q\bar{Q}\bar{\ell}\ell$, W-mediated. [Badger, Campbell, Ellis]

Analytic techniques

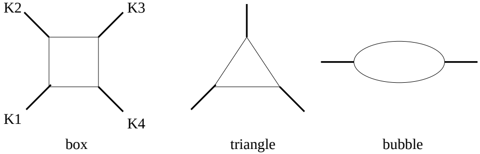
- may have advantages in stability and speed,
- extend readily to larger numbers of external particles,
- and have numerical counterparts.

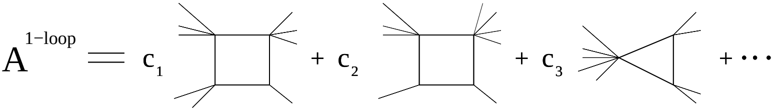
One-Loop Amplitudes

In 4-dimensional massless theories, reduction of Feynman integrals brings the one-loop amplitude to the form

$$A = \sum_i d_i \text{ (box)} + \sum_i c_i \text{ (triangle)} + \sum_i b_i \text{ (bubble)} + \text{rational}$$

where the master integrals have scalar structure and are known explicitly. [in dim. reg.: Bern, Dixon, Kosower]

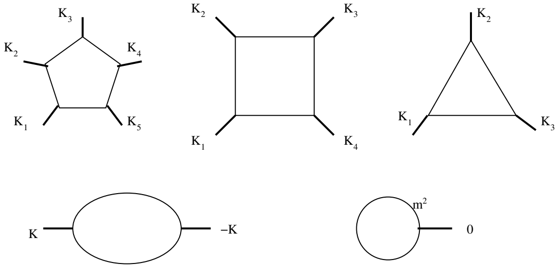




One-Loop Amplitudes

In $D = 4 - 2\epsilon$ dimensions, and allowing for internal masses, the result of reduction is

$$A = \sum_i e_i \text{ (pentagon)} + \sum_i d_i \text{ (box)} + \sum_i c_i \text{ (triangle)} + \sum_i b_i \text{ (bubble)} + \sum_i a_i \text{ (tadpole)}$$

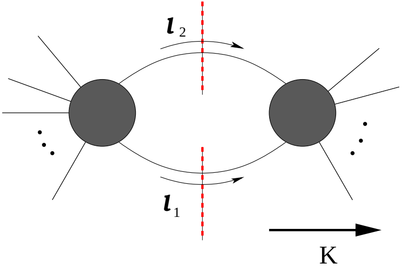


Unitarity Cuts: Loops from Trees

$$\Delta A^{\text{1-loop}} = \int d\mu \;\; A^{\text{tree}}_{\text{Left}} \;\times\; A^{\text{tree}}_{\text{Right}}$$

where

$$d\mu = d^4\ell_1 \; d^4\ell_2 \; \delta^{(4)}(\ell_1 + \ell_2 - K) \; \delta(\ell_1^2) \; \delta(\ell_2^2)$$

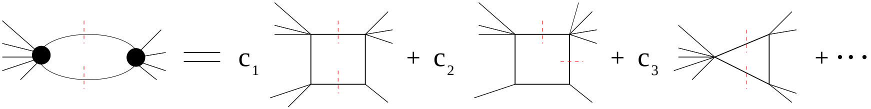


By unitarity, this is the discontinuity of the amplitude across a branch cut. [Cutkosky]

Amplitudes from unitarity cuts

$$\Delta A^{1-\mathrm{loop}} = \sum c_i \Delta I_i$$

Tree level input.



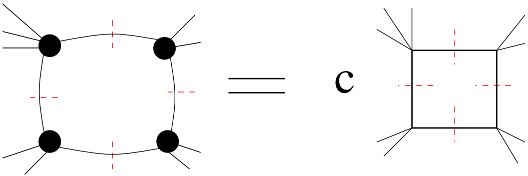
Matching 4-dimensional cuts can suffice to determine reduction coefficients! Logarithms with unique arguments.
"CUT-CONSTRUCTIBILITY"
[Bern, Dixon, Dunbar, Kosower]

But: we get several coefficients together in the same equation.

How do we evaluate a unitarity cut?

Box Coefficients from Quadruple Cuts

[RB, Cachazo, Feng]



Generalized Unitarity: Try replacing all four propagators by delta functions.

This operation isolates any given box.

In four dimensions, these four delta functions localize the integral completely. This computation is very easy!

Box Coefficients from Quadruple Cuts

The box coefficients computed from quadruple cuts are given by

$$\frac{1}{2}\sum_{\mathcal{S}} A_1^{\rm tree} A_2^{\rm tree} A_3^{\rm tree} A_4^{\rm tree}$$

$\mathcal{S}$ is the set of all solutions of the on-shell conditions for the internal lines.

$$\mathcal{S} = \{\ell |\ \ell^2 = 0, \quad (\ell - K_1)^2 = 0, \quad (\ell - K_1 - K_2)^2 = 0, \quad (\ell + K_4)^2 = 0\}$$

Can these equations always be solved?

In complexified momentum space, there are exactly 2 solutions.

From momentum vectors to spinors

Change from Lorentz 4-vector to spinor indices with Pauli matrices:

$$p_{a\dot{a}} = \sigma^{\mu}_{a\dot{a}}\; p_{\mu} \qquad\qquad a, \dot{a} = 1, 2$$

For a null vector (massless particle):

$$0 = p^2 = \det(p_{a\dot{a}}) \implies p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}.$$

Lorentz-invariant spinor products:

$$\begin{aligned} \langle \lambda\ \lambda' \rangle &\equiv \epsilon_{ab}\, \lambda^a\, \lambda'^b \\ [\tilde{\lambda}\ \tilde{\lambda}'] &\equiv \epsilon_{\dot{a}\dot{b}}\, \tilde{\lambda}^{\dot{a}}\, \tilde{\lambda}'^{\dot{b}} \end{aligned}$$

Spinor formulas are elegant

Gluons, tree level:

$$A(p_1^-, p_2^+, p_3^-, p_4^+, \ldots, p_n^+) = \frac{\langle\lambda_1 \ \lambda_3\rangle^4}{\langle\lambda_1 \ \lambda_2\rangle\langle\lambda_2 \ \lambda_3\rangle \ldots \langle\lambda_{n-1} \ \lambda_n\rangle\langle\lambda_n \ \lambda_1\rangle}$$

[Parke, Taylor; Berends, Giele]

Cut integrals

A closer look at the cut integral:

$$\Delta A^{1-\mathrm{loop}} = \int d\mu \, A^{\mathrm{tree}}(-\ell, i, \ldots, j, \ell - K) \, A^{\mathrm{tree}}(K - \ell, j+1, \ldots, i-1, \ell)$$

$$d\mu = d^4\ell \;\; \delta^+(\ell^2) \; \delta^+((\ell - K)^2)$$

Change to homogeneous (CP^1) spinor variables with

$$\ell_{a\dot{a}} = t \; \lambda_a \tilde{\lambda}_{\dot{a}}.$$

Integration measure:

$$\int d^4\ell \; \delta^+(\ell^2) \; (\bullet) = \int_0^\infty dt \;\; t \int_{\tilde{\lambda}=\bar{\lambda}} \langle \lambda \; d\lambda \rangle \; [\tilde{\lambda} \; d\tilde{\lambda}] \; (\bullet)$$

[Cachazo, Svrček, Witten]

Spinor integration

[Anastasiou, RB, Buchbinder, Cachazo, Feng, Kunszt, Mastrolia]

- Change variables, $\ell = t\lambda\tilde{\lambda}$, and use the spinor measure,

$$\int d^4\ell \; \delta(\ell^2)\delta((\ell - K)^2) = \int dt \; t \int \langle\lambda \; d\lambda\rangle[\tilde{\lambda} \; d\tilde{\lambda}]\delta((t\lambda\tilde{\lambda} - K)^2)$$

- Use 2nd delta function to perform t-integral.
- Evaluate with residue theorem.
- Identify cuts of basis integrals and read off coefficients.
- We have given formulas for the resulting coefficients.

Cuts of Master Integrals

$$\Delta I_2 = \int \langle\lambda\ d\lambda\rangle[\tilde\lambda\ d\tilde\lambda]\frac{K^2}{\langle\lambda|K|\tilde\lambda]^2}$$

$$\Delta I_3 = \int \langle\lambda\ d\lambda\rangle[\tilde\lambda\ d\tilde\lambda]\frac{1}{\langle\lambda|K|\tilde\lambda]\langle\lambda|Q_1|\tilde\lambda]}$$

$$\Delta I_4 = \int \langle\lambda\ d\lambda\rangle[\tilde\lambda\ d\tilde\lambda]\frac{1}{K^2}\frac{1}{\langle\lambda|Q_1|\tilde\lambda]\langle\lambda|Q_2|\tilde\lambda]}$$

$$Q_j \equiv -K_j + \frac{K_j^2}{K^2}K$$

Cutting the Amplitude

Starting point is the product of tree amplitudes:

$$C = c \int d^4\ell \, \frac{\prod_{i=1}^{k+n}(2\ell \cdot P_i)}{\prod_{j=1}^{k}(\ell - K_j)^2} \;\; \delta(\ell^2) \;\; \delta((\ell - K)^2)$$

We define the following vectors:

$$\begin{aligned} Q_j &= -K_j + \frac{K_j{}^2}{K^2} K, \\ R_i &= P_i. \end{aligned}$$

Then the cut integral can be rewritten:

$$C = c \int \langle \lambda \; d\lambda \rangle [\tilde{\lambda} \; d\tilde{\lambda}] \frac{(K^2)^{n+1}}{\langle \lambda | K | \tilde{\lambda}]^{n+2}} \frac{\prod_{i=1}^{k+n} \langle \lambda | R_i | \tilde{\lambda}]}{\prod_{j=1}^{k} \langle \lambda | Q_j | \tilde{\lambda}]}$$

Reduction by partial fractions

Split the factors in the denominator with partial fractions.

$$\frac{\prod_{j=1}^{k-1} \langle\lambda|R_j|\tilde{\lambda}]}{\prod_{i=1}^{k} \langle\lambda|Q_i|\tilde{\lambda}]} = \sum_{i=1}^{k} \frac{1}{\langle\lambda|Q_i|\tilde{\lambda}]} \frac{\prod_{j=1}^{k-1} \langle\lambda|R_j Q_i|\lambda\rangle}{\prod_{m=1, m\neq i}^{k} \langle\lambda|Q_m Q_i|\lambda\rangle}$$

$$\frac{\prod_{j=1}^{n-1} \langle\lambda|R_j|\tilde{\lambda}]}{\langle\lambda|K|\tilde{\lambda}]^n \langle\lambda|Q|\tilde{\lambda}]} = \frac{\prod_{j=1}^{n-1} \langle\lambda|R_j Q|\lambda\rangle}{\langle\lambda|KQ|\lambda\rangle^{n-1}} \frac{1}{\langle\lambda|K|\tilde{\lambda}]\langle\lambda|Q|\tilde{\lambda}]}$$
$$+ \sum_{p=0}^{n-2} (-1)^{n-p} \frac{\prod_{j=1}^{n-p-2} \langle\lambda|R_j Q|\lambda\rangle \langle\lambda|R_{n-p-1} K|\lambda\rangle \prod_{t=n-p}^{n-1} \langle\lambda|R_t|\lambda]}{\langle\lambda|K|\tilde{\lambda}]^{p+2} \langle\lambda|QK|\lambda\rangle^{n-p-1}}$$

Box coefficients

$$C[K_r, K_s, K] = \frac{1}{2}\left(\mathcal{T}^{(N)}(\ell) D_r(\ell) D_s(\ell)\right)\Big|_{\lambda \to P_{sr,1}, \tilde{\lambda} \to P_{sr,2}} + \{P_{sr,1} \leftrightarrow P_{sr,2}\}$$

$$\begin{aligned} P_{sr,1} &= Q_s + \left(\frac{-Q_s \cdot Q_r + \sqrt{\Delta_{sr}}}{Q_r^2}\right) Q_r, \\ P_{sr,2} &= Q_s + \left(\frac{-Q_s \cdot Q_r - \sqrt{\Delta_{sr}}}{Q_r^2}\right) Q_r, \\ \Delta_{sr} &= (Q_s \cdot Q_r)^2 - Q_s^2 Q_r^2. \end{aligned}$$

Triangle coefficients

$$
\begin{aligned}
C[K_s, K] &= \frac{1}{2(N+1)!\sqrt{\Delta_s}^{N+1}\langle P_{s,1}\ P_{s,2}\rangle^{N+1}} \\
&\times \frac{d^{N+1}}{d\tau^{N+1}} \left(\mathcal{T}^{(N)}(\ell) D_s(\ell) \langle\lambda|K|\tilde{\lambda}]^{N+1} \Big|_{\tilde{\lambda}\to Q_s\lambda, \lambda\to P_{s,1}-\tau P_{s,2}} \right) \Bigg|_{\tau\to 0} \\
&+ \{P_{s,1} \leftrightarrow P_{s,2}\}
\end{aligned}
$$

$$
\begin{aligned}
P_{s,1} &= Q_s + \left(\frac{-Q_s \cdot K + \sqrt{\Delta_s}}{K^2} \right) K, \\
P_{s,2} &= Q_s + \left(\frac{-Q_s \cdot K - \sqrt{\Delta_s}}{K^2} \right) K, \\
\Delta_s &= (Q_s \cdot K)^2 - Q_s^2 K^2.
\end{aligned}
$$

Bubble coefficient

$$C[K] = K^2 \sum_{q=0}^{N} \frac{(-1)^q}{q!} \frac{d^q}{ds^q} \left(\mathcal{B}^{(0)}_{N,N-q}(s) + \sum_{r=1}^{k} \sum_{a=q}^{N} \left(\mathcal{B}^{(r;a-q;1)}_{N,N-a}(s) - \mathcal{B}^{(r;a-q;2)}_{N,N-a}(s) \right) \right) \Bigg|_{s=0},$$

$$\mathcal{B}^{(0)}_{N,m}(s) \equiv \frac{d^N}{d\tau^N} \left(\frac{(2\eta \cdot K)^{m+1} \langle \lambda|K|\tilde{\lambda}]^N}{N![\eta|\eta' K|\eta]^N (m+1)(K^2)^{m+1} \langle \lambda\ \eta \rangle^{N+1}} \mathcal{T}^{(N)}(\ell) \Bigg|_{\substack{\tilde{\lambda} \to (K+s\eta)\cdot\lambda \\ \lambda \to (K-\tau\eta')\cdot\eta}} \right) \Bigg|_{\tau \to 0},$$

$$\mathcal{B}^{(r;b;1)}_{n,m}(s) \equiv \frac{(-1)^{b+1}}{b!(m+1)\sqrt{\Delta_r}^{b+1} \langle P_{r,1}\ P_{r,2} \rangle^b} \times$$
$$\frac{d^b}{d\tau^b} \left(\frac{\langle \lambda|\eta|P_{r,1}]^{m+1} \langle \lambda|Q_r \eta|\lambda\rangle^b \langle \lambda|K|\tilde{\lambda}]^{N+1}}{\langle \lambda|K|P_{r,1}]^{m+1} \langle \lambda|\eta K|\lambda \rangle^{n+1}} \mathcal{T}^{(N)}(\ell) D_r(\ell) \right) \Bigg|_{\substack{\tilde{\lambda} \to (K+s\eta)\lambda,\ \lambda \to P_{r,1} - \tau P_{r,2} \\ \tau = 0}}$$

$$\mathcal{B}^{(r;b;2)}_{n,m}(s) \equiv \frac{(-1)^{b+1}}{b!(m+1)\sqrt{\Delta_r}^{b+1} \langle P_{r,1}\ P_{r,2} \rangle^b} \times$$
$$\frac{d^b}{d\tau^b} \left(\frac{\langle \lambda|\eta|P_{r,2}]^{m+1} \langle \lambda|Q_r \eta|\lambda\rangle^b \langle \lambda|K|\tilde{\lambda}]^{N+1}}{\langle \lambda|K|P_{r,2}]^{m+1} \langle \lambda|\eta K|\lambda \rangle^{n+1}} \mathcal{T}^{(N)}(\ell) D_r(\ell) \right) \Bigg|_{\substack{\tilde{\lambda} \to (K+s\eta)\lambda,\ \lambda \to P_{r,2} - \tau P_{r,1} \\ \tau = 0}}$$

D-dimensional cuts

We have been doing the unitarity cut in four dimensions, very convenient for the spinor formalism.

We missed the “rational” terms.

Various ways to get rational terms of 4-d amplitudes:

- on-shell recursions at one loop [Bern, Dixon, Kosower; Berger, Dixon, Forde, Kosower]
- Feynman diagrams with targeted reduction [Xiao, Yang, Zhu; Binoth, Guillet, Heinrich]
- Special set of Feynman rules [Ossola, Papadopoulos, Pittau]

Or, compute D-dimensional cuts. [Ellis, Giele, Kunszt, Melnikov, Zanderighi; Ossola, Papadopoulos, Pittau; Anastasiou, RB, Feng, Kunszt, Mastrolia; Badger]

Unitarity in $D = 4 - 2\epsilon$ dimensions

Orthogonal decomposition, keeping external momenta in 4 dimensions. [Bern, Chalmers, Mahlon, Morgan]

$$\int d^{4-2\epsilon}\ell_{4-2\epsilon} \quad = \quad \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)} \int_0^1 du\ u^{-1-\epsilon} \int d^4\ell_4.$$

where $\ell_{4-2\epsilon} = \ell_4 + \ell_{-2\epsilon}$ and $\ell_{-2\epsilon}^2 = \frac{K^2}{4}u$.

Relate the ℓ_4 to a null 4-momentum ℓ via the cut momentum K.

$$\ell_4 = \ell + \xi K, \qquad \ell^2 = 0$$

$$\int d^4\ell_4 = \int d\xi \ \ d^4\ell \ \ (2\ell \cdot K)\ \delta^+(\ell^2).$$

This is the delta function we need to start spinor integration; two more delta functions from the original cut. Perform the 4-d cut integral as before.

(Cf. methods by Ossola, Papadopoulos, Pittau; Forde; Ellis, Giele, Kunszt; Kilgore; Giele, Kunszt, Melnikov; Badger)

Cuts of D-dimensional Master Integrals

$$\Delta I_2 = \int_0^1 du\ u^{-1-\epsilon} \int \langle\lambda\ d\lambda\rangle[\tilde\lambda\ d\tilde\lambda] \quad K^2\sqrt{1-u}\frac{1}{\langle\lambda|K|\tilde\lambda]^2}$$

$$\Delta I_3 = \int_0^1 du\ u^{-1-\epsilon} \int \langle\lambda\ d\lambda\rangle[\tilde\lambda\ d\tilde\lambda] \quad \sqrt{1-u}\frac{1}{\langle\lambda|K|\tilde\lambda]\langle\ell|Q|\ell]}$$

$$\Delta I_4 = \int_0^1 du\ u^{-1-\epsilon} \int \langle\lambda\ d\lambda\rangle[\tilde\lambda\ d\tilde\lambda] \quad \frac{\sqrt{1-u}}{K^2}\frac{1}{\langle\ell|Q_1|\ell]\langle\ell|Q_2|\ell]}$$

D-dimensional unitarity algorithm

$$\Delta A \;=\; \int_0^1 du\; u^{-1-\epsilon} \int d^4\ell \;\; \delta(\ell^2)\; \delta(\sqrt{1-u}\; K^2 - 2K \cdot \ell)$$

1. 4d cut: get u-dependent coefficients of master integrals. u-dependence is polynomial. [RB, Feng, Yang; RB, Feng, Mastrolia]
2. Treat polynomial u-dependence of integrand. Two choices:
   2.1 For each term in the polynomial, use shift identities to get coefficients of 4d master integrals.
   2.2 Use dimensionally shifted master integrals.

The u-integral is not done explicitly.

Massive particles

Cut amplitude:

$$\int_0^1 du\ u^{-1-\epsilon} \int \langle\lambda\ d\lambda\rangle[\tilde{\lambda}\ d\tilde{\lambda}] \left(\frac{\sqrt{\Delta[K^2, M_1^2, M_2^2]}}{K^2} \right) \frac{(K^2)^{n+1}}{\langle\lambda|K|\tilde{\lambda}]^{n+2}} \frac{\prod_{j=1}^{n+k} \langle\lambda|R_j|\tilde{\lambda}]}{\prod_{i=1}^{k} \langle\lambda|Q_i|\tilde{\lambda}]}$$

- For scalar particles, the formalism/formulas for integral coefficients will look the same.
- Integral coefficients are polynomials in u.
- New element: tadpole and massless bubble integrals.
- Self-energy and mass renormalization contributions may require gauge fixing. [Ellis, Giele, Kunszt, Melnikov].

Tadpole approaches

1. Universal divergent behavior
2. Add an "auxiliary propagator"
3. Generalized unitarity: single cuts

Tadpoles from IR + UV divergences

- Tadpoles and null bubble integrals are cut-free but diverge as $1/\epsilon$.
- The coefficients can sometimes be fixed by the known universal UV/IR divergences.
- Examples:
  4-gluon amplitude with a massive fermion loop [Bern, Morgan].
  $t\bar{t}gg$, using Mitov-Moch small-mass factorization result [Badger].

Tadpoles from double cut with auxiliary propagator

[RB, Feng]

Unitarity cuts are a powerful tool.

If we just had two propagators instead of one, we could apply our formalism.

Let's try inserting an extra propagator to the integrand.

We make use of the integrand classification offered for the numerical algorithm of Ossola, Papadopoulos, Pittau (OPP).

The integrand is decomposable as a sum of master integrands and "spurious terms" that integrate to zero.

The OPP Decomposition: work at the integrand level

[Ossola, Papadopoulos, Pittau]

Decompose at the integrand level.

$$A^{1-\mathrm{loop}} = \int d^{4-2\epsilon}\ell \; \frac{N(\ell)}{D_0 D_1 \cdots D_{m-1}}$$

$$D_i = (\ell - K_i)^2 - M_i^2 - \mu^2$$

Expand the integrand in terms of the scalar master integrands,

$$I^{(i)} = \frac{1}{D_i}, \qquad I^{(i,j)} = \frac{1}{D_i D_j}, \qquad I^{(i,j,r)} = \frac{1}{D_i D_j D_r}, \qquad \ldots$$

along with "spurious" terms that integrate to zero.

The OPP Decomposition: work at the integrand level

$$\frac{N}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} = \sum_i [a(i) + \widetilde{a}(\ell; i)] I^{(i)} + \sum_{i<j} [b(i,j) + \widetilde{b}(\ell; i,j)] I^{(i,j)}$$

$$+ \sum_{i<j<r} [c(i,j,r) + \widetilde{c}(\ell; i,j,r)] I^{(i,j,r)}$$

$$+ \sum_{i<j<r<s} [d(i,j,r,s) + \widetilde{d}(\ell; i,j,r,s)] I^{(i,j,r,s)}.$$

The OPP algorithm (in 4d) is to solve $\bar{D}_i(\ell) = 0$ numerically, multiply through, and substitute the solutions...

We will use this expansion differently.

Tadpole coefficients via Auxiliary Propagator

Recall:

$$D_i = (\ell - K_i)^2 - M_i^2 - \mu^2$$

Introduce an auxiliary denominator factor D_K:

$$D_K = (\ell - K)^2 - M_K^2 - \mu^2$$

For now, K and M_K^2 are just variables.

Plan: Unitarity cut of the propagators D_0 and D_K.

Then, decouple effects of D_K. Somewhat like single cut.

Double cut with auxiliary propagator

Structure of the integrand and its cuts implies

$$b_K(K,0) = a(0) + [\text{contributions from } \widetilde{a},\ \widetilde{b},\ \widetilde{c},\ \widetilde{d}]$$
$$c_K(K,0,i) = b(0,i) + [\text{contributions from } \widetilde{b},\ \widetilde{c},\ \widetilde{d}]$$

We'd like to decouple the "spurious contributions" as much as possible.

Use double cuts to get $c_K(K,0,i)$, $b(0,i)$, $b_K(K,0)$. Then solve the equations for $a(0)$.

$$\begin{aligned} a &= \text{tadpole coefficient} \\ b &= \text{bubble coefficient} \\ c &= \text{triangle coefficient} \end{aligned}$$

Conditions for decoupling (most) spurious terms

$$K \cdot K_i = 0, \qquad \forall i \tag{1}$$

$$M_K^2 = M_0^2 + K^2. \tag{2}$$

Condition (2) can be taken as a definition of M_K^2.
Condition (1) is treated formally.

Then there is only one spurious term that survives, and we can solve the equations for $a(0)$.

Procedure

1. Construct the true integrand $I = A^{\text{tree}}_{1-cut}/D_0$ and the auxiliary integrand $I_K = A^{\text{tree}}_{1-cut}/(D_K D_0)$.
2. From I_K, find bubble and triangle coefficients $b_K(K,0)$ and $c_K(K,0,i)$.
3. From I, find bubble coefficients $b(0,i)$.
4. The tadpole coefficient is given by imposing the conditions (1) and (2) in

$$a(0) = b_K(K,0) + \sum_i \frac{K_i^2 - M_i^2 + M_0^2}{4K_i^2} \left[c_K(K,0,i) - b(0,i)\right]\Big|_{\mu^2}.$$

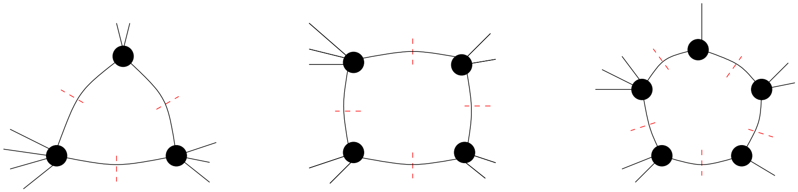
Requires all $K_i^2 \neq 0$.

Generalized Cuts

The discontinuity across a branch cut was given by putting two propagators on shell with delta functions.

This notion can be generalized.

It only helps to add more delta functions:



Generalized Cuts

Triple cuts had found some applications. [Bern, Dixon, Kosower] But using complex momenta in generalized cuts opened the door wide.

Applications of generalized cuts – with complex momenta:

- One-loop box coefficients [RB, Cachazo, Feng]
- Cut all propagators of multi-loop amplitudes
  [Buchbinder, Cachazo; Bern, Carrasco, Johansson, Kosower]

- One-loop: sequential approach to box, triangle, bubble, tadpole coefficients by quadruple, triple, double, single cuts
  [Ossola, Papadopoulos, Pittau; Mastrolia; Forde; Kilgore; Ellis, Giele, Kunszt; Giele, Kunszt, Melnikov]

Generalized Unitarity: the Single Cut

[RB, Mirabella]

We find that we can extract coefficients after doing *some* of the integration and then matching integrands.

The single cut has a single delta function and no momentum channel

$$\int d^4\ell_4 \; \delta(\ell_4^2)$$

The delta function integral is done by spinor techniques, a bit like D-dimensional unitarity.

Again we introduce an extra momentum vector K.

Generalized Unitarity: the Single Cut

See also the D-dimensional single cut of Glover, Williams used for rational parts of gluon amplitudes.

Single-cut techniques based on the Feynman Tree Theorem need different propagator prescriptions. [Catani, Gleisberg, Krauss, Rodrigo, Winter; Caron-Huot]

Applications for planar multiloop supersymmetric integrands. [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka]

Single Cut Technique

Replace one propagator by a delta function:

$$\Delta_{D_i}[I] \equiv \int d^4k\ \delta^{(+)}(D_i) \left(\frac{N(k)}{D_0 \cdots D_{i-1} D_{i+1} \cdots D_k} \right) .$$

Decompose loop momentum to get a null variable:

$$k = \ell + \xi K, \qquad \ell^2 = 0$$

Now K is free.

Expand K in two null momenta, $K = p + q$, to replace null ℓ by three independent variables $t, z, \bar{z}$. [Mastrolia]

$$\ell = t\lambda\tilde{\lambda}; \qquad \lambda = |p\rangle + z\,|q\rangle\,, \qquad \tilde{\lambda} = |p] + \bar{z}\,|q]\,.$$

Single Cut Technique

The single cut measure is

$$\int_0^\infty \frac{dt}{4} \int (idz \wedge d\bar{z}) \frac{K^2 t^2 (1 + z\bar{z})}{\sqrt{t^2(1 + z\bar{z})^2 + u}},$$

where

$$u \equiv \frac{4m^2}{K^2}.$$

For convenience, we set $u = 0$, equivalent to $K^2 \to \infty$.

Single Cut Technique

Perform the $\int dz\ d\bar{z}$ integral by the Generalized Cauchy Formula:

If $G(z,\bar{z})$ is a primitive of $F(z,\bar{z})$, then

$$\int_D dz\ d\bar{z}\ F(z,\bar{z}) = \oint_{\partial D} dz\ G(z,\bar{z}) - 2\pi i \sum_{\text{poles } z_j} \mathrm{Res}\{G(z,\bar{z}), z_j\}.$$

The line integral dominates in the large K^2 limit. There $z = \Lambda e^{i\alpha}$ and Λ is large.

In double cuts, the t integral was done by the second delta function. Here, it diverges and we leave it *undone*.

Single Cuts of Master Integrands

Primitives for master integrals:

$$\begin{aligned} \text{tadpole} &\simeq \Lambda^2 t, \\ \text{bubble} &\simeq \log(\Lambda^2), \\ \text{triangle} &\simeq \frac{1}{\Lambda}\log(\Lambda^2), \\ \text{box} &\simeq \frac{1}{\Lambda^2}\log(\Lambda^2). \end{aligned}$$

Discard terms subleading in Λ, then logs.
Single cut operator $\bar{\Delta}$ selects $\Lambda^2 t$ terms.

But spurious terms have nonzero single cuts too!

Some general single cuts

$$I_{n,p} \equiv \frac{(2\,k \cdot A_1)\cdots(2\,k \cdot A_p)}{D_0 D_1 \cdots D_{n-1}}, \qquad f_i \equiv K_i^2 - m_i^2 + m_0^2$$

$$\begin{aligned}
\bar{\Delta}_{D_0}\,[I_{1,1}] &= 0 \\
\bar{\Delta}_{D_0}\,[I_{2,1}] &= -\frac{A_1 \cdot q}{K_1 \cdot q}\,\Delta_{D_0}\left[\frac{1}{D_0}\right] \\
\bar{\Delta}_{D_0}\,[I_{2,2}] &= -f_1 \frac{(A_1 \cdot q)(A_2 \cdot q)}{(K_1 \cdot q)^2}\Delta_{D_0}\left[\frac{1}{D_0}\right] \\
\bar{\Delta}_{D_0}\,[I_{3,1}] &= 0 \\
\bar{\Delta}_{D_0}\,[I_{3,2}] &= \frac{(A_1 \cdot q)(A_2 \cdot q)}{(K_1 \cdot q)(K_2 \cdot q)}\,\Delta_{D_0}\left[\frac{1}{D_0}\right] \\
\bar{\Delta}_{D_0}\,[I_{3,3}] &= \sum_{i=1}^{2} f_i \frac{(A_1 \cdot q)(A_2 \cdot q)(A_3 \cdot q)}{(K_1 \cdot q)(K_2 \cdot q)(K_i \cdot q)}\,\Delta_{D_0}\left[\frac{1}{D_0}\right]
\end{aligned}$$

Tadpole coefficients from single cuts

Since spurious terms contribute, need an integrand expansion instead of master integrals.

OPP expansion is a good choice.

Take single cuts of all terms.

By doing another expansion of the numerator, in a related basis, we get a system of equations involving the tadpole coefficient.

Most of the equations are unneeded for the tadpole coefficient.

Summary

- Unitarity method at one loop constructs amplitudes from branch cuts, using the fact of the expansion in known master integrals.
- Spinor integration gives analytic formulas for coefficients of master integrals.
- Massive particles bring new challenges for analytic solutions, such as tadpole coefficients.
  - Auxiliary propagators allow the usual double-cuts for tadpole coefficients; need external masses.
  - Or, we do the single cut integral directly and include spurious terms. Procedure not fully systematic.
  - Tadpole/single-cut techniques still need further study!