New ideas in lattice supersymmetry

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Why study supersymmetry on lattice ?

- Rigorous definition of theory - like lattice QCD ($a \to 0$, $V \to \infty$) for QCD
- Fascinating theories; exhibit confinement, chiral symmetry breaking, dualities
  BUT key issue is SUSY breaking...
- $\mathcal{N} = 4$ super Yang-Mills. Finite non-trivial QFT. AdS/CFT - provides description of black holes, quark-gluon plasma ...
- Lattice allows:
  - Strong coupling calculations
  - Monte Carlo simulations.
  - New ideas/approaches - eg. (quantum) geometry from gauge theory

Lattice SUSY - problems

- SUSY extends Poincaré – broken by discretization. $\{Q, \overline{Q}\} = \gamma.p$. No p.
- Leads to (very) difficult fine tuning – lots of relevant SUSY breaking counterterms in effective action.
- Folklore: Impossible to put SUSY on lattice exactly.
- $\mathcal{N} = 4$ particularly difficult – contains scalar fields

Way out?

Options

Let SUSY emerge as accidental symmetry in continuum limit.
Limited possibilities:

- Chiral symmetry protect against dangerous SUSY violating operators eg gaugino mass in $\mathcal{N} = 1$ SYM (Veneziano)
  - Wilson fermions - tune to chiral limit
  - Or better (but maybe harder) Overlap/DW fermions
- With GW fermions then super QCD may be possible – but great numerical challenge .. fine tune scalar sector still ... future ?

Or preserve some SUSY exactly in lattice theory

Exact lattice SUSY - new ideas

- Discretize a topologically twisted form of SYM theory
- Build lattice theory by orbifolding supersymmetric matrix model

Two approaches produce identical lattice theories!

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- Preserve subset of continuum SUSY exactly on lattice. Boson/fermion spectrum degenerate, vacuum energy zero, ...
- Warning: Approaches work only if $\mathcal{Q}$ multiple of 2^D

In $D = 4$ unique theory: $\mathcal{N} = 4$ SYM

Twisting - basic idea

- Consider (extended) SUSY theories possessing additional flavor (R) symmetries.
- If flavor contains an appropriate subgroup can Twist: decompose fields under $G = \mathrm{Diag}(SO_{\mathrm{Lorentz}}(D) \times SO_{\mathrm{R}}(D))$
- Fermions: spinors under both factors – become integer spin after twisting.
- Scalars transform as vectors under R-symmetry – vectors after twisting.
- Gauge fields remain vectors – combine with scalars to make complex gauge fields. Still just $U(N)$ gauge symmetry.

Important: flat space: just a change of variable

Example – 2D $\mathcal{N}=2$ YM

- Fields: gauge field, 2 scalars, 2 Majorana fermions
- Twist: consider 2 fermions as matrix

$$\lambda^i_\alpha \rightarrow \Psi_{\alpha\beta}$$

Expand:

$$\Psi = \frac{\eta}{2} I + \psi_\mu \gamma_\mu + \chi_{12}\gamma_1\gamma_2$$

- $\eta, \psi_\mu, \chi_{\mu\nu}$ twisted fermions
- Scalar fermion - scalar supersymmetry $\mathcal{Q}$ with $\mathcal{Q}^2 = 0$

Twisted actions

- Original SUSY algebra implies

$$\{Q, Q_\mu\} = p_\mu$$

- Momentum Q-exact. Plausible that energy-momentum tensor also Q-exact.
- General conclusion: Twisted theories typically have Q-exact actions. Very important for discretization
- Aside: Actually $\mathcal{Q}$-exact structure also implies $\mathcal{Q}$ invariant sector is topological

Example - twisted form of 2D $\mathcal{N}=2$ action

Twisted form of action (adjoint fields AH generators)

$$S = \frac{1}{g^2}\mathcal{Q}\int \mathrm{Tr}\left(\chi_{\mu\nu}\mathcal{F}_{\mu\nu} + \eta[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu] - \frac{1}{2}\eta d\right)$$

$$\begin{aligned}
\mathcal{Q}\,\mathcal{A}_\mu &= \psi_\mu \\
\mathcal{Q}\,\psi_\mu &= 0 \\
\mathcal{Q}\,\overline{\mathcal{A}}_\mu &= 0 \\
\mathcal{Q}\,\chi_{\mu\nu} &= -\overline{\mathcal{F}}_{\mu\nu} \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned}$$

Note: complexified gauge field $\mathcal{A}_\mu = A_\mu + iB_\mu$, $\mathcal{F}_{\mu\nu}(\mathcal{A})$

Untwisting

$\mathcal{Q}$-variation, integrate d:

$$S = \frac{1}{g^2}\int \mathrm{Tr}\,\left(-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 - \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} - \eta\overline{\mathcal{D}}_\mu\psi_\mu\right)$$

Rewrite as

$$S = \frac{1}{g^2}\int \mathrm{Tr}\,\left(-F_{\mu\nu}^2 + 2B_\mu D_\nu D_\nu B_\mu - [B_\mu, B_\nu]^2 + L_F\right)$$

where

$$L_F = \begin{pmatrix} \chi_{12} & \frac{\eta}{2} \end{pmatrix} \begin{pmatrix} -D_2 - iB_2 & D_1 + iB_1 \\ D_1 - iB_1 & D_2 - iB_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Lattice theory

How to discretize ?

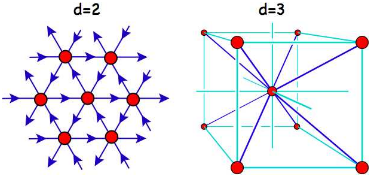
- Bosons: $\mathcal{A}_\mu(x) \to \mathcal{U}_\mu(n)$. Complexified Wilson link variables.
- Fermions: η on sites, ψ_μ same links as $\mathcal{U}_\mu$, $\chi_{\mu\nu}$ diagonal links $x+\mu+\nu \to x$.

Gauge transformation (adjoint rep $f(x) = \sum_a T^a f^a(x)$):

$$\begin{aligned}
\mathcal{U}_\mu(\mathbf{x}) &\to G(\mathbf{x})\mathcal{U}_\mu(\mathbf{x})G^\dagger(\mathbf{x}+\mu)\\
\eta(\mathbf{x}) &\to G(\mathbf{x})\eta(\mathbf{x})G^\dagger(\mathbf{x})\\
\psi_\mu(\mathbf{x}) &\to G(\mathbf{x})\psi_\mu(\mathbf{x})G^\dagger(\mathbf{x}+\mu)\\
\chi_{\mu\nu}(\mathbf{x}) &\to G(\mathbf{x}+\mu+\nu)\chi_{\mu\nu}(\mathbf{x})G^\dagger(\mathbf{x})
\end{aligned}$$

Lattice structure

- Lattice determined by required link fields. eg in 2D ex need square lattice with diagonal links
- Other more exotic possibilities eg triangular in 2D, bcc in 3D and A_4^* in 4D. Lattice structure dictated by exact lattice supersymmetry...



Lattice actions, SUSY, ...

- Nilpotent SUSY as in continuum eg. $\mathcal{Q}\,\mathcal{U}_\mu = \psi_\mu$
- Derivatives replaced with covariant differences compatible with lattice G.I

Eg.

$$\mathcal{F}_{\mu\nu} = \mathcal{D}^{(+)}_\mu \mathcal{U}_\nu = \mathcal{U}_\mu(x)\mathcal{U}_\nu(x+\mu) - \mathcal{U}_\nu(x)\mathcal{U}_\mu(x+\nu)$$

G.T:

$$\mathcal{U}_\mu(x)\mathcal{U}_\nu(x+\mu) \to G(x)\mathcal{U}_\mu(x)G(x+\mu)^\dagger G(x+\mu)\mathcal{U}_\nu(x+\mu)G(x+\mu+\nu)^\dagger$$

transforms as lattice 2-form

Contract with $\chi_{\mu\nu}$ – gauge invariant loop.

Twisted lattice fermions

- Twisted fermions governed by Kähler-Dirac action

$$S = \sum \eta \mathcal{D}_\mu \psi_\mu + \chi_{\mu\nu} \mathcal{D}_{[\mu} \psi_{\nu]}$$

- Free theory can be mapped into action of reduced staggered fermions - thus one Dirac field in 2D. No doubling.
- Twisting is nothing but the usual flavor-spin mixing of staggered quarks..

Twisted lattice bosons

- Twisted boson action $\sum F^\dagger F + \frac{1}{2}\left(\overline{\mathcal{D}}_\mu \mathcal{U}_\mu\right)^2$ is simply

$$S_B = \sum_x \sum_{\mu<\nu} \mathcal{U}_\mu(x)\mathcal{U}_\nu(x+\mu)\mathcal{U}^\dagger_\mu(x+\nu)\mathcal{U}^\dagger_\nu(x) + \sum_\mu \mathcal{U}_\mu \mathcal{U}^\dagger_\mu - I$$

- In unitary limit (set scalars to zero) just Wilson plaquette!

Twisted continuum $\mathcal{N} = 4$ SYM

- Lattice theory again arises from discretization of twisted theory $SO(4)' = SO_R(4) \times SO_{rot}(4)$
- Twisted fields:
  - 16 fermions: $\Psi = (\eta, \psi_\mu, \chi_{\mu\nu}, \theta_{5\mu}, \kappa_5)$
  - 10 bosons: $\mathcal{A}_\mu = A_\mu + iB_\mu, \oplus (\phi, \overline{\phi})$
- Compactly expressed as:
  - $\Psi = (\eta, \psi_a, \chi_{ab}), a, b = 1 \ldots 5$
  - $\mathcal{A}_a, a = 1 \ldots 5$
  - Action $S = \mathcal{Q} \int \left(\chi_{ab} F_{ab} + \eta[\overline{\mathcal{D}}_a, \mathcal{D}_a] - 1/2\eta d\right) + S_{\rm closed}$

(Almost) same as 2D example !

Lattice $\mathcal{N} = 4$ theory

- Place $\mathcal{U}_a$ on links of $\boxed{A_4^*}$ lattice - 5 basis vectors correspond to vectors from center of hypertetrahedron to vertices (Weyl group of $SU(5)$)
- Fermions:
  - $\eta \quad x \rightarrow x$
  - $\psi_a \quad x \rightarrow x + \mu_a$
  - $\chi_{ab} \quad x + \mu_a + \mu_b \rightarrow x$
- All fields transform like links: $X_p \rightarrow G(x)\, X_p\, G^\dagger(x + p)$
- Single exact lattice SUSY $\mathcal{Q}^2 = 0$

Lattice action

$$S = \beta(S_{\rm exact} + S_{\rm closed})$$

$$\begin{aligned} S_{\rm exact} &= \sum_{\mathbf{x}} {\rm Tr}\left(\mathcal{F}^{\dagger}_{ab}\mathcal{F}_{ab} + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_{a}\mathcal{U}_{a}\right)^{2}\right. \\ &- \left.\chi_{ab}\mathcal{D}^{(+)}_{[a}\psi_{\,b]} - \eta\overline{\mathcal{D}}^{(-)}_{a}\psi_{a}\right) \\ S_{\rm closed} &= -\frac{1}{2}\sum_{\mathbf{x}} {\rm Tr}\ \epsilon_{abcde}\chi_{de}(\mathbf{x}+\mu_a+\mu_b+\mu_c)\overline{\mathcal{D}}^{(-)}_{c}\chi(\mathbf{x}+\mu_c) \end{aligned}$$

Outstanding questions

Lattice theories are:

local, gauge invariant, doubler free and invariant under one SUSY

Two questions:

- Is rotational symmetry restored as $a \rightarrow 0$?
- What about restoration of full SUSY ?

Must understand how lattice theory renormalizes ...

Two approaches

- Examine using p. theory
- Attempt a non-perturbative tuning by measuring broken SUSY Ward identities

Renormalization

Lattice symmetries:

- Gauge invariance
- $\mathcal{Q}$-symmetry.
- Point group symmetry - eg. S^5 for A_4^* - subgroup of $SO(4)'$
- Exact fermionic shift symmetry $\eta \rightarrow \eta + \epsilon I$

Conclusion:

- S^5 PGS guarantees twisted $SO(4)'$ restored as $a \rightarrow 0$
- No mass terms can appear to all orders in perturbation theory! $\Gamma_{\rm eff}(\mathcal{A}^{\rm classical}) = 0$
- Power counting: only relevant ops correspond to log renormalizations of 4 coefficients α_i in bare lattice action

Why so few counterterms ?

$$S = \sum \mathcal{Q}\left(\alpha_1 \chi F + \alpha_2 \eta \overline{\mathcal{D}} \psi + \frac{1}{2}\alpha_3 \eta d\right) + \alpha_4 S_{\text{closed}}$$

- Lattice gauge invariance very restrictive. Most fermion bilinears not gauge invariant eg $\psi_a \eta$.
- Also, $\mathcal{Q}$-symmetry requires all counterterms to be $\mathcal{Q}$-exact.
- General structure: all counterterms must correspond to short (relevant) closed loops on lattice countaining a single fermion link field.
- The bare action essentially employs all these ...

Sketch of $\Gamma_{\rm eff}=0$

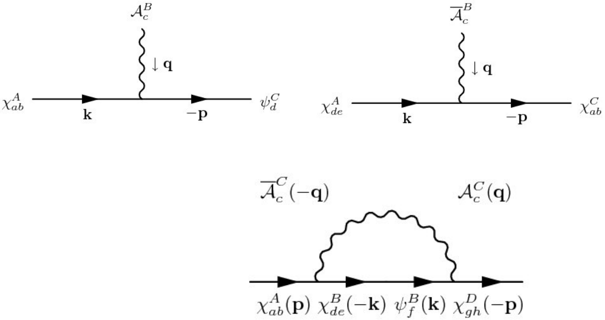
- Classical vacua constant commuting complex matrices $\mathcal{U}_\mu$
- Expand to quadratic order about generic vacuum $\mathcal{U}_b(x)=I+\mathcal{A}^c_b+a_b(x)$. Integrate
- Bosons $\det^{-5}\left(\overline{\mathcal{D}}^{(-)}_a\mathcal{D}^{(+)}_a\right)$
- Ghosts+Fermions: $\det\left(\overline{\mathcal{D}}^{(-)}_a\mathcal{D}^{(+)}_a\right)+\left(Pf(M_F)\overset{Maple}{=}\det^4\left(\overline{\mathcal{D}}^{(-)}_a\mathcal{D}^{(+)}_a\right)\right)$
- Thus $Z_{\rm pbc}=1$ at 1-loop. $\mathcal{Q}$-exact structure – result good to all orders! Exact quantum moduli space
- Witten index: all states cancel except vacua. Counting indep of g.

Ingredients for perturbation theory

Lattice rules for A_4^* lattice (Feynman gauge):

- Boson propagator $< \overline{\mathcal{A}}_a^C(k)\mathcal{A}_b^D(-k) >= \frac{1}{\hat{k}^2}\delta_{ab}\delta^{CD}$ with $\hat{k}^2 = 4\sum_a \sin^2(k_a/2)$
- Fermion propagator $M^{-1}_{\mathrm{KD}}(k) = \frac{1}{\hat{k}^2}M_{\mathrm{KD}}(k)$ with $M(k)$ a 16×16 block matrix acting on $(\eta, \psi_a, \chi_{ab})$
- Vertices: $\psi\eta$, $\psi\chi$ and $\chi\chi$.
- Four one loop Feymann graphs needed to renormalize three fermion propagators. Yields 3 α's.
- One additional bosonic propagator for remaining α.

Example: chi-chi propagator



- $\Sigma_i(0) = 0; \quad \frac{\partial \Sigma_i}{\partial p} = A g^2 \ln \mu a + \text{finite} + \mathcal{O}(a)$
- Implies $\sqrt{\alpha_i} = Z_i = 1 + \frac{A}{2} g^2 \ln \mu a + \ldots$

Why so simple ?

- One loop lattice diagrams in 1-1 correspondence with continuum diagrams and have only log divergences.
- This comes from region near $pa \sim 0$ where lattice propagators and vertices approach continuum expressions
- Thus (divergent part of) 1-loop renormalization of lattice - same as continuum.
- In continuum twisted theory equivalent to usual - has full supersymmetry. Requires common wavefunction renormalization all fermions/bosons ...

Lattice ...?

- Lattice divergence structure same - hence
  $\boxed{\text{no fine tuning to target } \mathcal{N} = 4}$ at weak coupling.
- Furthermore expect $\boxed{\beta_{\text{lattice}}(g) = 0}$
- Line of fixed pts at weak coupling

Strong coupling ?

- Unfortunately this correspondance to continuum does not hold at higher loops.
  - Lattice couplings α_i may flow differently at strong coupling/large lattice spacing.
  - May need to (log) fine tune for coarse lattices. beta function non-zero.
- To fully understand the question of fine tuning requires non-perturbative methods.
- OK - can simulate using usual methods of lattice QCD

Simulations

- Integrate out (twisted) fermions - $\mathrm{Pf}(\mathrm{M})$ where

$$M^{AB;cd}_{x,x'}, \quad A,B=(\eta,\ldots,\chi_{ab})\,;c,d=1\ldots N^2$$

- Represent using *pseudofermions* $F,\overline{F}$

$$\mathrm{Pf}(M)=\int \mathcal{D}F\mathcal{D}\overline{F}e^{-\overline{F}\left(M^\dagger M\right)^{-\frac{1}{4}}F}$$

- Rational hybrid Monte Carlo algorithm (RHMC) to sample:

$$x^{-\frac{1}{4}}\sim\alpha_0+\sum_{i=1}^{P}\frac{\alpha_i}{x+\beta_i}$$

coefficients (α_i,β_i) determined by Remez alg. to minimise error in some interval $\epsilon<x<1$.

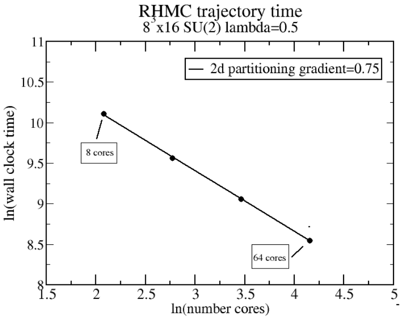
Continuing

RHMC alg. proceeds by:

- Promote $S \rightarrow H$
- Evolve classical EOM using eg leapfrog
- Metropolis test to accept trajectories
- Yields unbiased sampling of distribution e^{-S}.
- Pseudofermion forces: solve $(M^{\dagger}M + \beta_i)x = F_i$ using multimass CG.

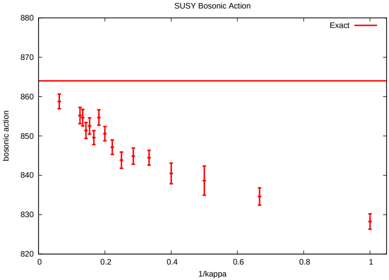
Code

- Parallized, multiple time step RHMC code developed.
- Uses MDP libraries within FermiQCD for communication.



Tests of exact supersymmetry

SUSY predicts: $\kappa < S_B >= \frac{9}{2}(N^2 - 1)V$



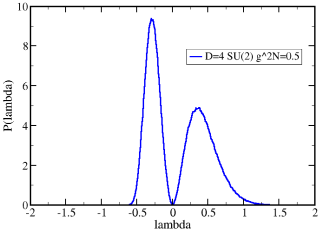
$4^3 \times 8$ lattice (apbc)

κ	κS_B	exact
1.0	13.67(4)	13.5
10.0	13.52(2)	13.5
100.0	13.48(2)	13.5

$D = 0$ $SU(2)$

Divergent path integral ?

Infinite number classical vacua corresponding to constant diagonal matrices



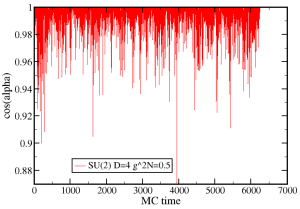
$SU(2)$ $P(\lambda)$ vs λ

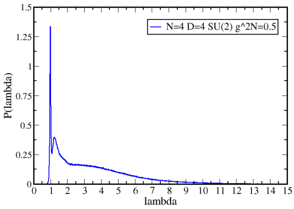
no divergence scalars localized close to origin with power law tails

caveat : still need to introduce mass for scalar trace mode

Sign problem ?

- Observed no significant sign problem with small volume simulations over most parameter range.
- In the case of periodic bcs we understand this: $< e^{i\alpha(\mathrm{Pf}(M))} >_{\text{phase quenched}} = Z_{\text{unquenched}}$ But Z is a topological invariant - can be computed exactly at 1-loop where one finds $< e^{i\alpha} >= 1$.





Summary

- Exciting time for lattice SUSY - much activity, many developments.
- Lattice actions retaining some exact SUSY possible.
- Much progress in understanding renormalization continuum theory; counterterm structure simple - at most 3 ops need log tuning to see full SUSY as $a \to 0$.
- One loop structure of $\mathcal{N} = 4$ even simpler; no fine tuning at weak coupling - lattice has vanishing beta function!
- Possibility for nonperturbative exploration $\mathcal{N} = 4$ YM. Make contact with RHIC physics, black holes and quantum gravity (via AdS/CFT)

Cautionary note - vacuum stability

- To stabilize vacuum $\mathcal{U}_a = I + \mathcal{A}_a + \ldots$ need introduce mass for U(1) trace mode using eg mass term $m = O(1/a)$.
- Breaks exact $\mathcal{Q}$-symmetry. May have consequences for
  - Moduli space.
  - Sign problem - Pf complex - phase ignored in simulations ...

Issue under investigation.

Neuberger problem

Two parametrizations of gauge links have been used

- $\mathcal{U}_a = e^{\mathcal{A}_a}$. Exponential form
- $\mathcal{U}_a = I + \mathcal{A}_a$. Linear (orbifold) form. Gauge invariance only kept if realized dynamically - by using mass terms to fix vev of trace mode.

Imply different treatments of measure (Haar or flat).
Haar measure *may* be problematic
Neuberger showed that $\mathcal{Q}$-exact actions based on compact groups have $Z_{\text{gauge fixing}} = 0$.
May be restated as $Z_{\text{gauge fixing}} = \chi$ Euler number of group manifold.