Resummation of large-x and small-x double logarithms in DIS and semi-inclusive e^+e^- annihilation

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- Splitting and coefficient functions and their endpoint behaviour
- Generalized threshold resummation of 1/Mellin-N contributions
- Small-x resummation of $x^{-1}\ln^{\ell}x$ (SIA) and $x^{0}\ln^{\ell}x$ (DIS) terms

Conventions and references

Double-log enhancement: two additional logs L per additional order in α_s

$$Q\big|_{\alpha_s^{n+n_a}} \sim \underset{\text{LL}}{\#L^{2n}} + \underset{\text{NLL}}{\#L^{2n-1}} + \underset{\text{NNLL}}{\#L^{2n-2}} + \ldots$$

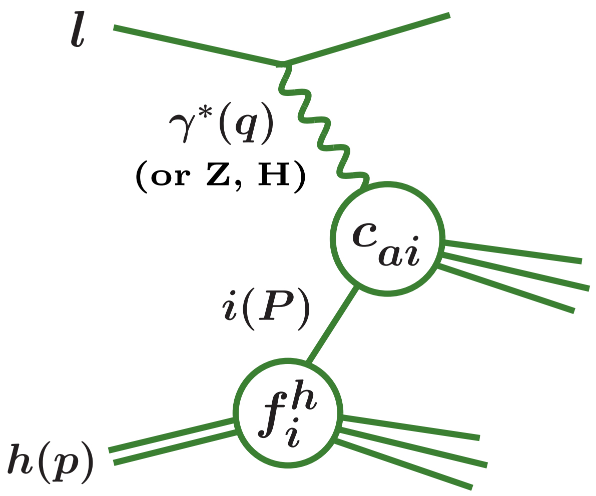
LL, NLL, . . . : leading logarithms, next-to-leading logarithms, . . .

Counting of a resummation, cf. small-x, not of a (stronger) exponentiation,
cf. soft gluons: NNLL resummation $\Leftrightarrow$ (re-expanded) NLL exponentiation

- Non-singlet NNLL (NLL for DY) resummation from physical kernels
  MV, arXiv:0902.2342 (JHEP), 0909.2124 (JHEP)
- Singlet NNLL for fourth-order splitting functions and F_L in DIS
  SMVV, 0912.0369 (NPB), 1008.0952 (Loops & Legs)
- Generalized threshold resummation in inclusive DIS and SIA
  A.V., 1005.1606 (PLB); ASV, 1012.3352 (JHEP); ALPV, 1202.5224 (Radcor), to app.
- Small-x resummation of splitting & coefficient funct's in SIA (and DIS)
  A.V., arXiv:1108.2993 (JHEP); KVY, arXiv:1207.5631 (JHEP); KV, to appear

Hard lepton-hadron processes in pQCD

Inclusive deep-inelastic scattering (DIS), semi-incl. l^+l^- annihilation (SIA)



Left $\longrightarrow$ right: DIS, q spacelike, $Q^2 = -q^2$

$P = \xi p\,,\;\; f_i^h =$ parton distributions

Top $\longrightarrow$ bottom: l^+l^-, q timelike, $Q^2 = q^2$

$p = \xi P\,,$ fragmentation distributions

Drell-Yan (DY) l^+l^- production: bottom $\longrightarrow$ top, 2$^{\text{nd}}$ hadron from right ($\{\ldots\}$)

Structure functions, fragmentation functions etc F_a: coefficient functions

$$F_a(x, Q^2) \;=\; \Big[C_{a,i\{j\}}(\alpha_s(\mu^2), \mu^2/Q^2) \otimes f_i^h(\mu^2)\{\otimes f_j^{h'}(\mu^2)\} \Big](x) + \mathcal{O}(1/Q^{(2)})$$

Scaling variables: $x = Q^2/(2p\cdot q)$ in DIS etc. μ: renorm./mass-fact. scale

Splitting and coefficient functions in pQCD

Parton/fragmentation distributions f_i : (renorm. group) evolution equations

$$\frac{d}{d\ln\mu^2} f_i(\xi,\mu^2) \;=\; \Big[P^{S/T}_{ik/ki}(\alpha_s(\mu^2)) \otimes f_k(\mu^2)\Big](\xi)\;, \qquad \otimes: \text{ Mellin convolution}$$

Initial conditions: incalculable, fit-analyses of reference processes

Expansion in α_s: splitting functions P, coefficient fct's c_a of observables

$$\begin{aligned} P &= \alpha_s P^{(0)} + \alpha_s^2 P^{(1)} + \alpha_s^3 P^{(2)} + \alpha_s^4 P^{(3)} + \dots \\ C_a &= \underbrace{\alpha_s^{n_a}\Big[c_a^{(0)} + \alpha_s c_a^{(1)}}_{} + \alpha_s^2 c_a^{(2)} + \alpha_s^3 c_a^{(3)} + \dots\Big] \end{aligned}$$

NLO: first real prediction of size of cross sections

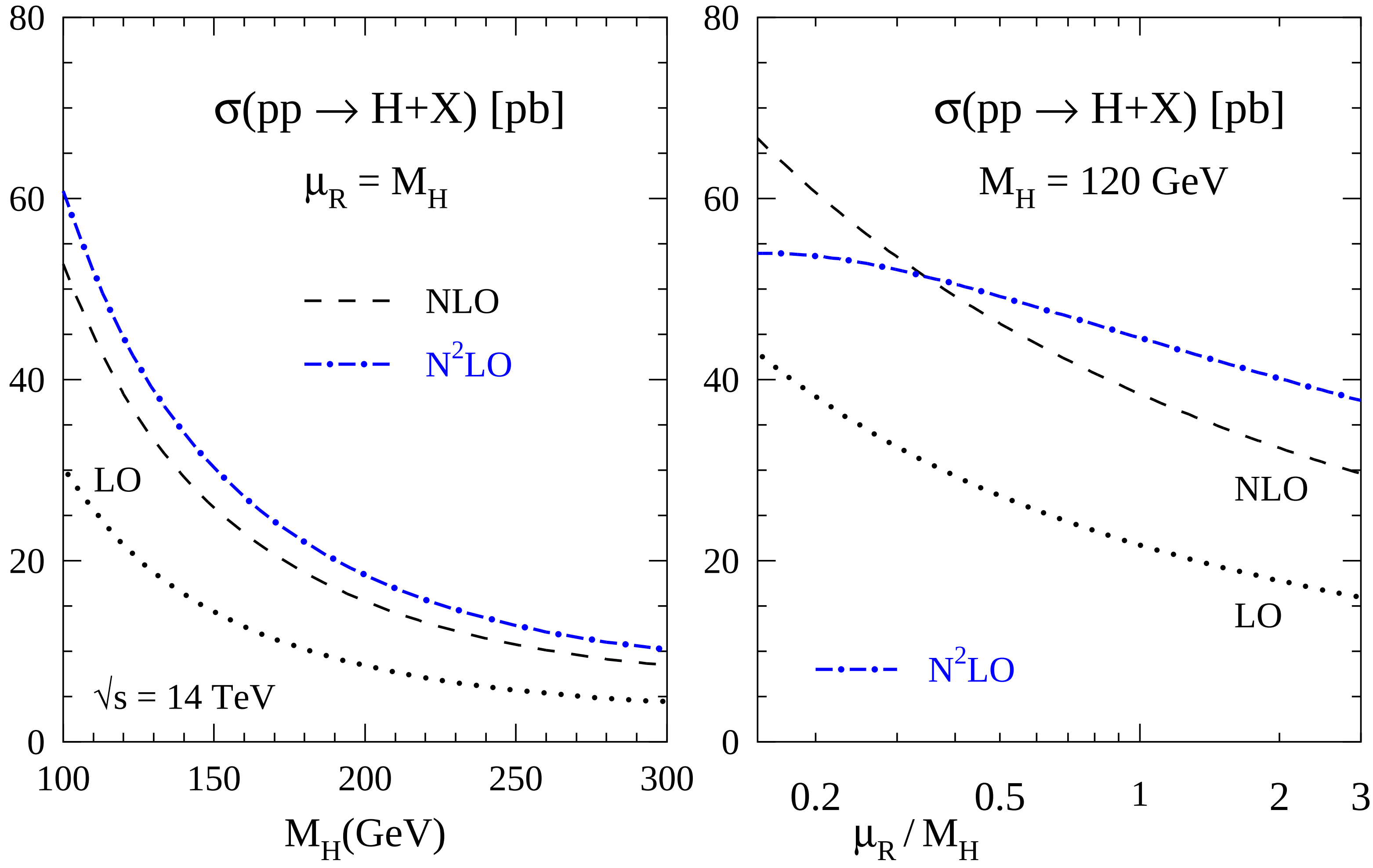
NNLO, $P^{(2)}$, $c_a^{(2)}$: first serious error estimate of pQCD predictions

N^{3}LO: for high precision (α_s from DIS), slow convergence (Higgs in $pp/p\bar{p}$)

F_2/F_3 : MVV (2005/8), $P^{(3)}_{\mathrm{ns},\,N=2}$: Baikov, Chetyrkin (06), . . . ; $\sigma_{\mathrm{H,soft}}$: MV (05), . . .

Endpoint logarithms for $x \to 0, 1$: resummation can be useful or necessary

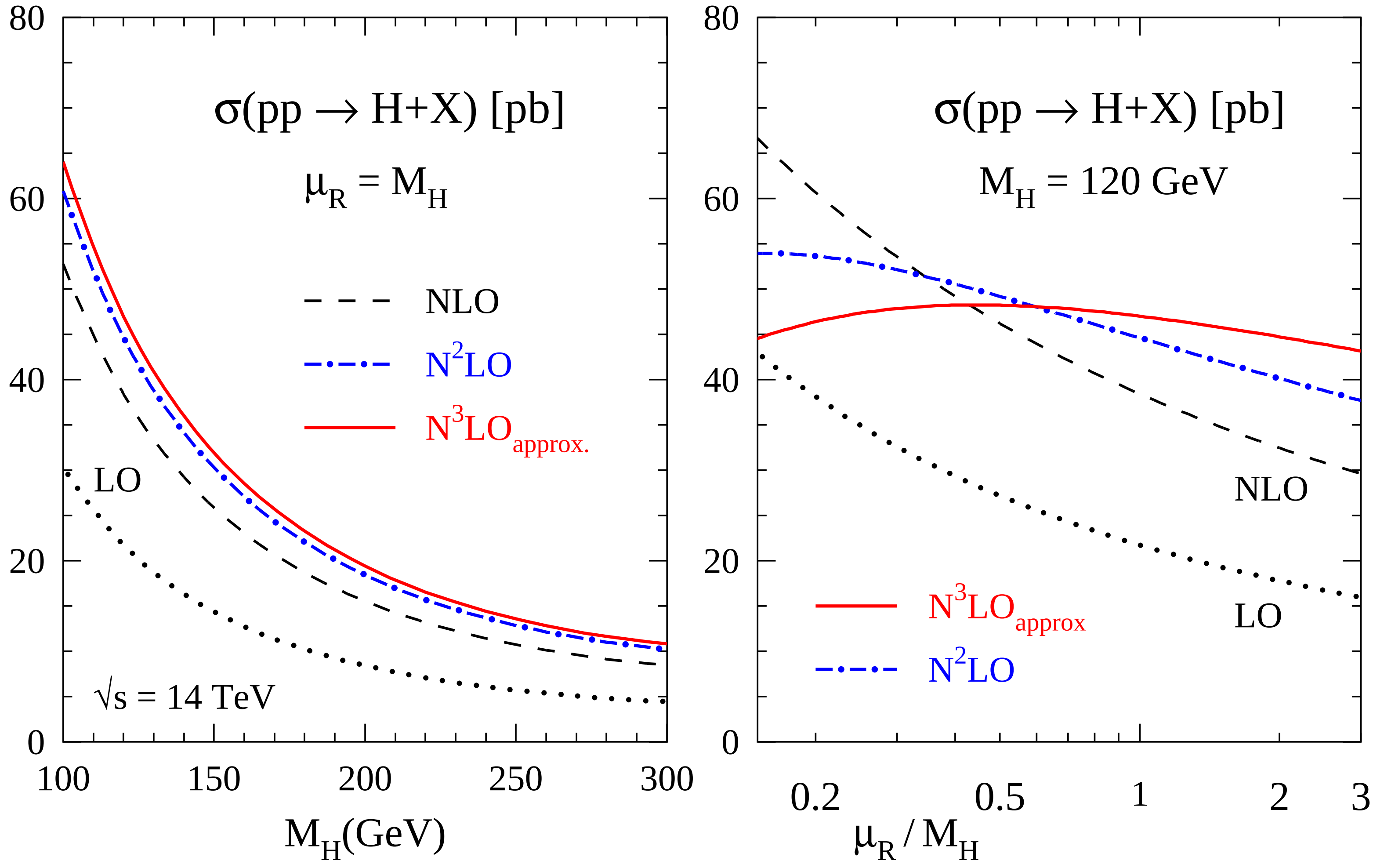
Higgs boson production at the LHC



NNLO (heavy top-quark limit):

Harlander, Kilgore; Anastasiou, Melnikov (02); Ravindran, Smith, van Neerven (03)

Higgs boson production at the LHC

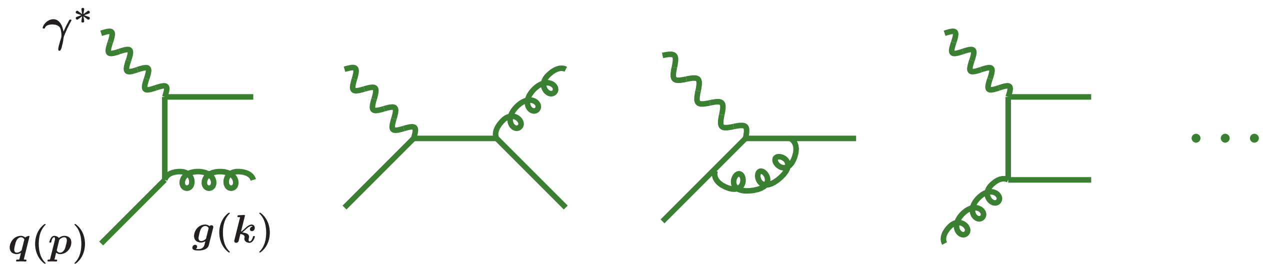


N^{3}LO increase at $\mu_r = M_H$: 5% (NNLO pdf's). μ_r variation: 4%

⇒ 5% accuracy reached by approx. N^{3}LO **Moch, A.V. (2005)**

From mass singularities to pQCD functions

First-order DIS:
(inclusive, $\int d^4k$)



Emissions collinear to the incoming partons ($m_{\mathrm{q,g}} = 0$) : denominators

$$(p-k)^2 \;=\; -2|\vec{p}\,||\vec{k}|(1-\cos\vartheta) \;\xrightarrow{\vartheta\to 0}\; -|\vec{p}\,||\vec{k}|\,\vartheta^2 \;\stackrel{\int d\vartheta}{\Rightarrow}\; \text{mass singularities}$$

Regularization (dim. $= 4-2\varepsilon$, singularities $\sim 1/\varepsilon$) and mass factorization

$$F_a(Q^2) \;=\; \hat{F}_{a,k}(\alpha_s(Q^2),\varepsilon)\otimes\hat{f}_k \;=\; C_{a,i}(\alpha_s(Q^2))\otimes\underbrace{\Gamma_{ik}(\alpha_s(Q^2),\varepsilon)\otimes\hat{f}_k}_{f_i(Q^2)}$$

$C_{a,i}$: **coefficient functions of observable a**

Γ_{ik} : **universal $1/\varepsilon$ - poles + . . . (fact. scheme). Usual: $\overline{\text{MS}}$**

Renormalized parton distributions f_i : splitting functions P_{ij}

$$\frac{\partial}{\partial\ln Q^2}f_i \;=\; \frac{\partial\,\Gamma_{ik}}{\partial\ln Q^2}\otimes\hat{f}_k \;=\; \frac{\partial\,\Gamma_{ik}}{\partial\ln Q^2}\otimes\Gamma^{-1}_{kj}\otimes f_j \;\equiv\; P_{ij}\otimes f_j$$

$\overline{\text{MS}}$ splitting functions at large x / large N

Mellin trf. $f(N) = \int_0^1 dx\,(x^{N-1}\{-1\})\,f(x)_{\{+\}}$: **M-convolutions $\rightarrow$ products**

$$\frac{\ln^n(1-x)}{(1-x)_+} \stackrel{\mathrm{M}}{=} \frac{(-1)^{n+1}}{n+1}\ln^{n+1}N + \ldots\,, \quad \ln^n(1-x) \stackrel{\mathrm{M}}{=} \frac{(-1)^n}{N}\ln^n N + \ldots$$

Diagonal splitting functions: no higher-order enhancement at N^0, N^{-1}

$$P^{(\ell-1)}_{\mathrm{qq/gg}}(N) \;=\; A^{(\ell)}_{\mathrm{q/g}}\ln N + B^{(\ell)}_{\mathrm{q/g}} + C^{(\ell)}_{\mathrm{q/g}}\,\frac{1}{N}\ln N + \ldots\,, \quad A_{\mathrm{g}} = C_A/C_F\,A_{\mathrm{q}}$$

. . . ; Korchemsky (89); MVV(04); Dokshitzer, Marchesini, Salam (05)

Off-diagonal: double-log behaviour, colour structure with $C_{AF} = C_A - C_F$

$$C_F^{-1}P^{(\ell)}_{\mathrm{gq}}\,/\,n_f^{-1}P^{(\ell)}_{\mathrm{qg}} \;=\; \frac{1}{N}\ln^{2\ell}N\;\#\,C^l_{AF}$$
$$+\,\frac{1}{N}\ln^{2\ell-1}N\,(\,\#\,C_{AF} + \#\,C_F + \#\,n_f)\,C^{l-1}_{AF} + \ldots$$

Double logs $\ln^n N$, $\ell+1 \le n \le 2\ell$ vanish for $C_F = C_A$ ($\rightarrow$ SUSY case)

Leading logarithms: maximally non-supersymmetric contributions

$\overline{\text{MS}}$ coefficient functions at large x / large N

'Diagonal' [$\mathcal{O}(1)$] coeff. fct's for $F_{2,3,\phi}$ in DIS, $F_{T,A,\phi}$ in SIA, $F_{\rm DY} = \frac{1}{\sigma_0}\frac{d\sigma_{\rm q\bar{q}}}{dQ^2}$

$$C^{(\ell)}_{2,{\rm q}/\phi,{\rm g}/\ldots} \;=\; \#\ \ln^{2\ell} N + \ldots \;+ N^{-1}(\#\ \ln^{2\ell-1} N + \ldots) + \ldots$$

N^0 parts: threshold exponentiation   Sterman (87); Catani, Trentadue (89); ...

Exponents known to next-to-next-to-next-to-leading log (N^{3}LL) accuracy - mod. $A^{(4)}$

$\Rightarrow$ highest seven (DIS, SIA), six (DY, Higgs prod.) coefficients known to all orders

DIS: MVV (05), DY/Higgs prod.: MV (05); Laenen, Magnea (05); Idilbi, Ji, Ma, Yuan (05) (+ SCET papers, from 06), SIA: Blümlein, Ravindran (06); MV, arXiv:0908.2746 (PLB)

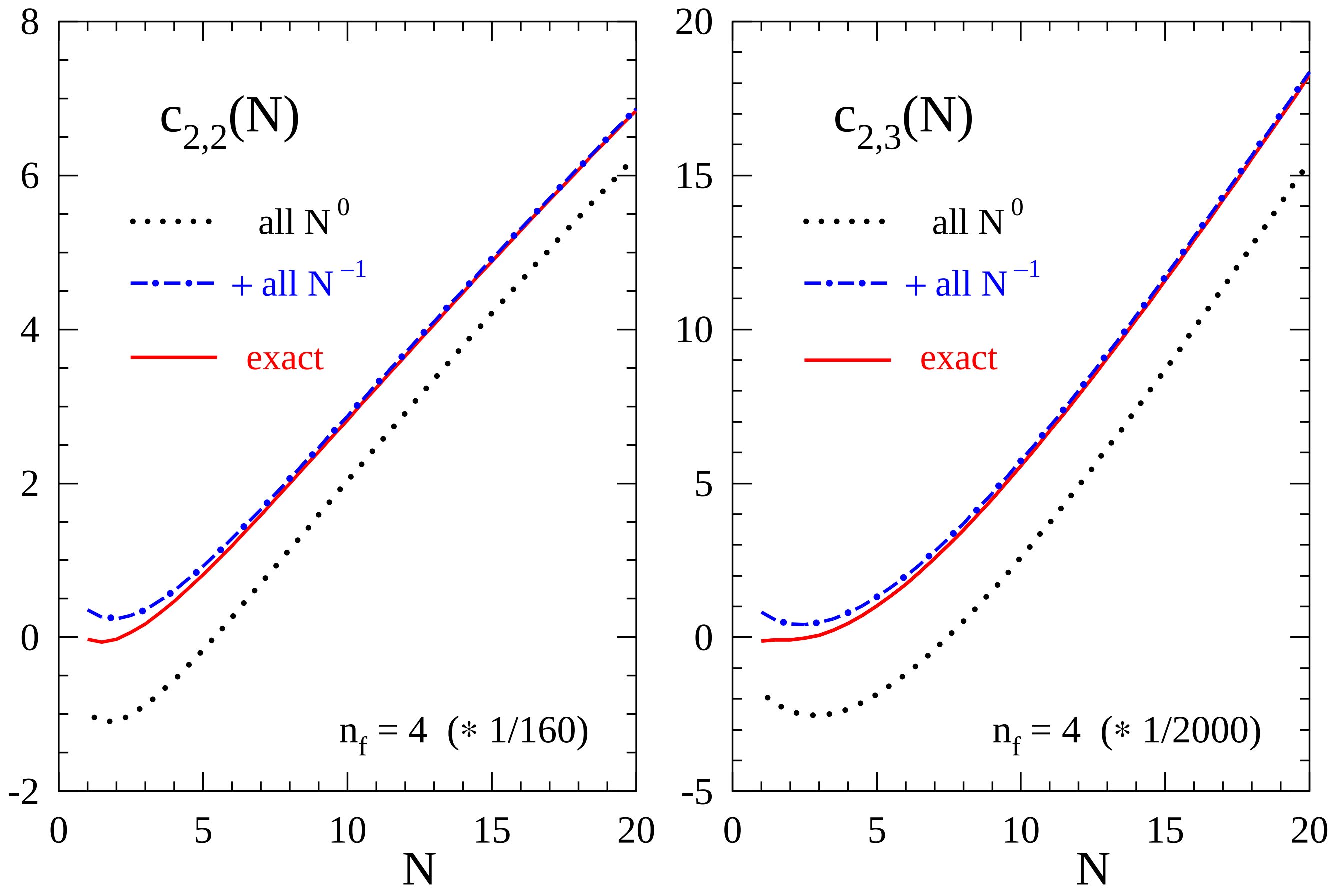
'Off-diagonal' [$\mathcal{O}(\alpha_{\rm s})$] quantities: leading N^{-1} double logarithms

$$C^{(\ell)}_{\phi,{\rm q}/2,{\rm g}/\ldots} \;=\; N^{-1}(\#\ \ln^{2\ell-1} N + \ldots) + \ldots$$

Longitudinal DIS/SIA structure functions [convention: ℓ = order in $\alpha_{\rm s}$ – 1]

$$C^{(\ell)}_{\rm L,q} \;=\; N^{-1}(\#\ \ln^{2\ell} N + \ldots) + \ldots\ ,\quad C^{(\ell)}_{\rm L,g} \;=\; N^{-2}(\#\ \ln^{2\ell} N + \ldots) + \ldots$$

Second- and third-order N-space $C_{2,\mathrm{ns}}$ in DIS



N^{-1} terms relevant over full range shown, $\mathcal{O}(N^{-2})$ sizeable only at $N < 5$

Sum of $N^{-1}\,\ln^k N$ looks almost constant: half of maximum only at $N \simeq 150$

DIS $\rightarrow$ SIA $\rightarrow$ DY : increase of the N^0 terms, N^{-1} corrections less important

$\overline{\text{MS}}$ splitting functions at small $x\,/\,N \to 1$ or 0

Logs in x-space $\Leftrightarrow$ poles in N-space, $\quad x^a \ln^n x \overset{\text{M}}{=} \dfrac{(-1)^n n!}{(N+a)^{n+1}}$

Space-like case, non-singlet: no x^{-1} terms, leading x^0 double logarithms :

LL: Kirschner, Lipatov (83); Blümlein, A.V. (95)

Singlet quantities: dominant x^{-1} terms single-log enhanced

$$P_{\rm ij}^{(\ell)S} \;=\; x^{-1}(\,\#\ln^{\ell-\delta_{\rm iq}}x + \ldots\,) + (\,\#\ln^{2\ell}x + \ldots\,) \;+\; \ldots$$

x^{-1} part: BFKL (77/78); Jaroszewicz (82); Catani, Fiorani, Marchesini (89); Catani, Hautmann (94); . . . , Fadin, Lipatov; Camici, Ciafaloni (98)

Timelike case: huge x^{-1} double logarithms

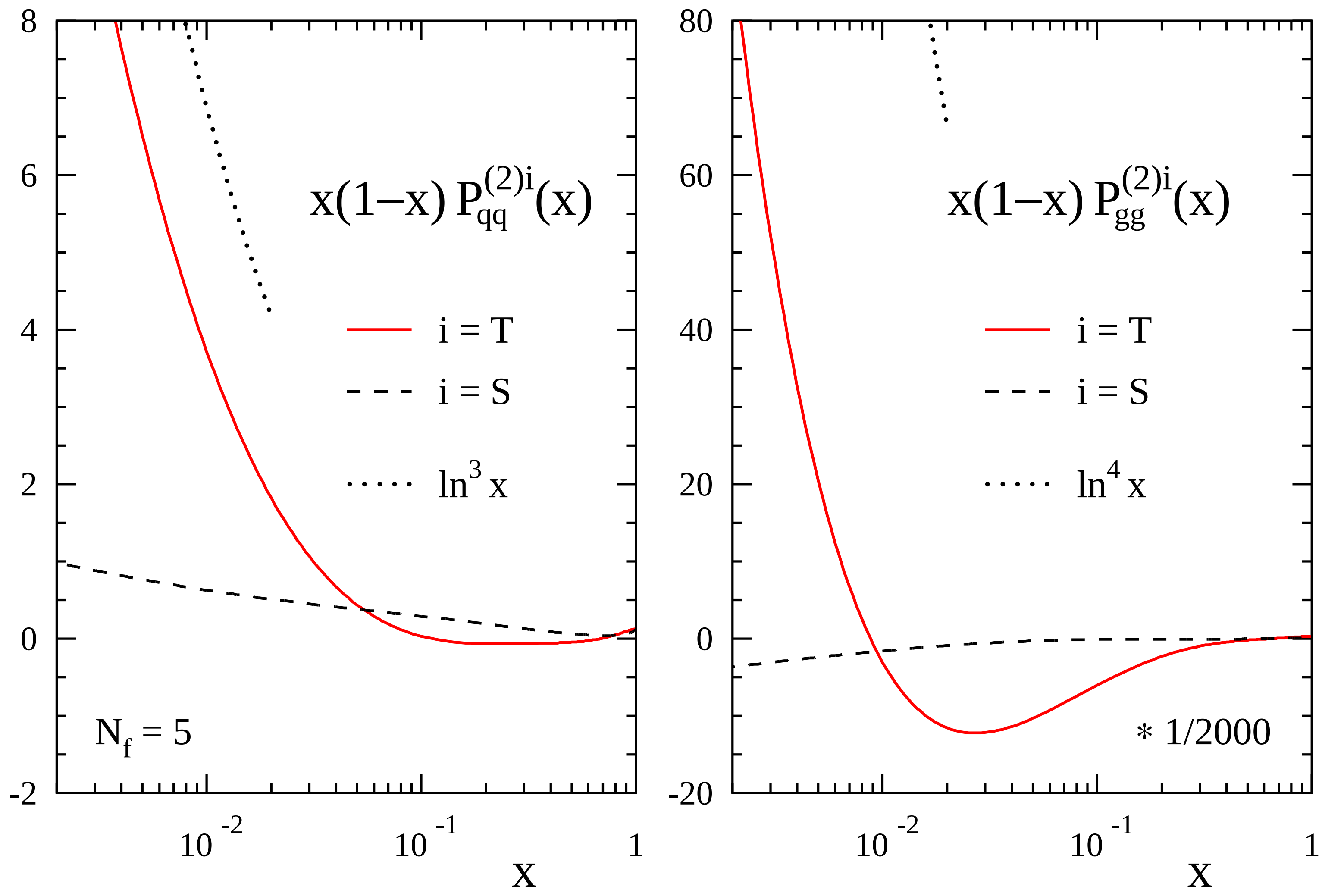
$$P_{\rm ij}^{(\ell)T} \;=\; x^{-1}(\#\;\ln^{2\ell-\delta_{\rm iq}}x + \ldots\,) + (\#\;\ln^{2\ell}x + \ldots\,) \;+\; \ldots$$

LL: Mueller (81); Bassetto, Ciafaloni, Marchesini, Mueller (82). NLL: Mueller (83)

– but latter not in $\overline{\text{MS}}$, see Albino, Bolzoni, Kniehl, Kotikov (11)

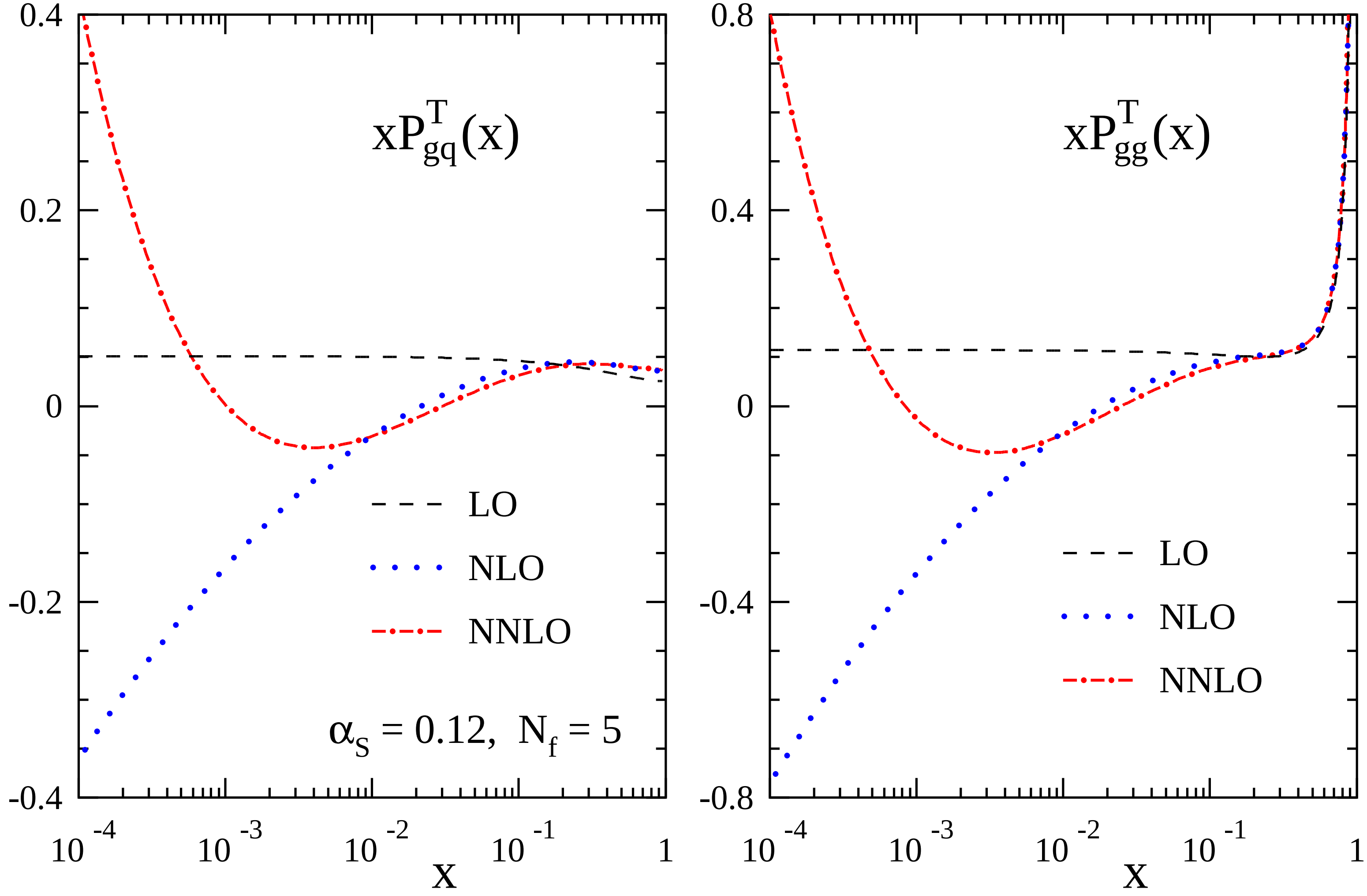
Behaviour of gauge-boson exchange coefficient functions analogous

Third-order diagonal splitting functions



T : extreme small-x rise from $x \gtrsim 10^{-2}$, in gg despite huge cancellations

NNLO approximations for $P^T_{\rm gi}(x,\alpha_s)$



NLO/NNLO: terms up to $x^{-1}\ln^2 x$ / $x^{-1}\ln^4 x$. Unstable at $x \lesssim 0.005$

Non-singlet (NS) physical evolution kernels

Eliminate quark densities from scaling violations of observables ($\mu = Q$)

$$\frac{dF_a}{d\ln Q^2} \;=\; \frac{d\,C_a}{d\ln Q^2}\,q \,+\, C_a P\,q \;=\; \Big(\beta(a_\mathrm{s})\,\frac{d\,C_a}{da_\mathrm{s}} \,+ C_a P\Big) C_a^{-1} F_a$$

$$= \;\Big(P_a \,+\, \beta(a_\mathrm{s})\,\frac{d\ln C_a}{da_\mathrm{s}}\Big) F_a \;=\; K_a F_a \;\equiv\; \sum_{\ell=0} a_\mathrm{s}^{\ell+1} K_{a,\ell}\, F_a$$

K_a: physical kernel for the NS observable F_a in N-space. For $c_{a,0} = 1$:

$$K_a \;=\; a_\mathrm{s} P_{a,0} \,+\, \sum_{l=1} a_\mathrm{s}^{\ell+1}\Big(P_{a,\ell} - \sum_{k=0}^{\ell-1} \beta_k\, \tilde{c}_{a,\ell-k}\Big), \qquad a_\mathrm{s} \equiv \alpha_\mathrm{s}/(4\pi)$$

with

$$\tilde{c}_{a,1} \;=\; c_{a,1} \;\;, \quad \tilde{c}_{a,2} = 2\,c_{a,2} - c_{a,1}^2$$

$$\tilde{c}_{a,3} \;=\; 3\,c_{a,3} - 3\,c_{a,2}\,c_{a,1} + c_{a,1}^3$$

$$\tilde{c}_{a,4} \;=\; 4\,c_{a,4} - 4\,c_{a,3}\,c_{a,1} - 2\,c_{a,2}^2 + 4\,c_{a,2}\,c_{a,1}^2 - c_{a,1}^4 \;\;, \;\; \ldots$$

Manipulations of harmonic sums/polylogarithms, (inverse) Mellin transform

FORM3 + packages: Vermaseren (00); TFORM: Tentyukov, Vermaseren (07); ...

Large-x logarithms in the physical kernels

Soft limit $1-x \ll 1 \;\; \Leftrightarrow$ large $L \equiv \ln N$: threshold exponentiation

$$C_a(N) \;=\; g_0 \exp\{L g_1(a_{\rm s}L) + g_2(a_{\rm s}L) + \ldots\} \;+\; \mathcal{O}(1/N)$$

$\Rightarrow$ single-logarithmic (SL) enhancement of physical evolution kernels K_a

$$K_a(N) \;=\; -\sum_{\ell=1} A_\ell a_{\rm s}^\ell \, L \;+\; \beta(a_{\rm s}) \, \frac{d}{da_{\rm s}} \{L g_1(a_{\rm s}L) + g_2(a_{\rm s}L) + \ldots\} \;+\; \ldots$$

Crucial observation: all K_a singly enhanced to all orders in N^{-1} or $(1-x)$

DIS/SIA $a \neq L$ leading-logarithmic kernels, with $p_{\rm qq}(x) = 2/(1-x)_+ - 1 - x$

$$\begin{aligned}
K_{a,0}(x) &= 2\, C_F \, p_{\rm qq}(x) \\
K_{a,1}(x) &= \ln(1-x)\, p_{\rm qq}(x) \; \left[-2\, C_F \beta_0 \mp 8\, C_F^2 \, \ln x\right] \\
K_{a,2}(x) &= \ln^2(1-x)\, p_{\rm qq}(x) \left[\;\, 2\, C_F \beta_0^2 \pm 12\, C_F^2 \, \beta_0 \, \ln x + \mathcal{O}(\ln^2 x)\right] \\
K_{a,3}(x) &= \ln^3(1-x)\, p_{\rm qq}(x) \left[-2\, C_F \beta_0^3 \mp 44/3 \, C_F^2 \, \beta_0^2 \, \ln x + \mathcal{O}(\ln^2 x)\right] \\
K_{a,4}(x) &= \ln^4(1-x)\, p_{\rm qq}(x) \left[\;\, 2\, C_F \beta_0^4 \pm \xi_{K_4} C_F^2 \, \beta_0^3 \, \ln x + \mathcal{O}(\ln^2 x)\right]
\end{aligned}$$

First term: leading large n_f, all orders via C_2 of Mankiewicz, Maul, Stein (97)

Higher-order non-singlet predictions

Conjecture: Single-log behaviour of K_a persists to (all) higher orders in α_s

$\Leftrightarrow$ resummation of the coefficient functions beyond soft $(1-x)^{-1}$ terms

Recall $$\underbrace{\tilde{c}_{a,4}}_{\text{SL}} = \underbrace{4\,c_{a,4}}_{\text{DL, new}} \underbrace{-\,4\,c_{a,3}\,c_{a,1} - 2\,c_{a,2}^{2} + 4\,c_{a,2}\,c_{a,1}^{2} - c_{a,1}^{4}}_{\text{DL, known for DIS/SIA}}$$

$\Rightarrow$ coefficients of highest three powers of $\ln(1-x)$ from fourth order in α_s,

i.e., $\ln^{7,6,5}(1-x)$ at order α_s^4 ,

$\ln^{9,8,7}(1-x)$ at order α_s^5 , ... for $F_{1,2,3}$ in DIS and $F_{\mathrm{T,I,A}}$ in SIA

Leading terms: $K_1 = K_2$, $K_{\mathrm{T}} = K_{\mathrm{I}}$ [total ('integrated') fragmentation fct.]

$\Rightarrow$ also three logs for space- and timelike F_L: $\ln^{6,5,4}(1-x)$ at α_s^4 etc

Alternative derivation: physical kernels for F_L, agreement non-trivial check

Drell-Yan: only NNLO known $\Rightarrow$ only two logarithms fully predicted from α_s^3

Example: α_s^4 coefficient function for F_1 in DIS

$$
\begin{aligned}
c^{(4)}_{1,\mathrm{ns}}(x) \;=\; & \Big(\ln^7(1-x)\, 8/3\, C_F^4 - \ln^6(1-x)\, 14/3\, C_F^3\, \beta_0 + \ln^5(1-x)\, 8/3\, C_F^2\, \beta_0^2\Big)\, p_{\mathrm{qq}}(x) \\
& + \ln^6(1-x)\; \Big[C_F^4\, \{p_{\mathrm{qq}}(x)\, (-14 - 68/3\, \mathrm{H}_0) + 4 + 8\, \mathrm{H}_0 - (1-x)(6 + 4\, \mathrm{H}_0)\}\Big] \\
& + \ln^5(1-x)\; \Big[C_F^4\, \Big\{p_{\mathrm{qq}}(x)\, (-9 - 8\, \widetilde{\mathrm{H}}_{1,0} + 448/3\, \mathrm{H}_{0,0} + 84\, \mathrm{H}_0 - 64\, \zeta_2) + 48\, \widetilde{\mathrm{H}}_{1,0} \\
& \qquad -22 - 96\, \mathrm{H}_{0,0} - 104\, \mathrm{H}_0 - (1-x)(13 + 24\, \widetilde{\mathrm{H}}_{1,0} - 48\, \mathrm{H}_{0,0} - 84\, \mathrm{H}_0 - 16\, \zeta_2)\Big\} \\
& \quad + C_F^3\, \beta_0\, \{p_{\mathrm{qq}}(x)\, (41 + 316/9\, \mathrm{H}_0) - 10 - 32/3\, \mathrm{H}_0 + (1-x)(41/3 + 16/3\, \mathrm{H}_0)\} \\
& \quad + C_F^3\, C_A\, \Big\{p_{\mathrm{qq}}(x)\, (16 + 8\, \widetilde{\mathrm{H}}_{1,0} + 8\, \mathrm{H}_{0,0} - 24\, \zeta_2) + 4 + (1-x)(28 - 8\, \zeta_2)\Big\} \\
& \quad + C_F^3\, (C_A - 2\, C_F)\, p_{\mathrm{qq}}(-x)\, (16\, \widetilde{\mathrm{H}}_{-1,0} - 8\, \mathrm{H}_{0,0})\Big] \;+\; \mathcal{O}\big(\ln^4(1-x)\big)
\end{aligned}
$$

First line includes identity of coefficients of leading $\ln^k(1-x)$ and $\frac{\ln^k(x-1)}{x-1}$ terms

Conjectured by Krämer, Laenen, Spira (97)

Modified basis $\widetilde{\mathrm{H}}_{m_1,m_2,\ldots} \equiv \widetilde{\mathrm{H}}_{m_1,m_2,\ldots}(x)$ of harmonic polylogarithms, e.g.,

$$
\widetilde{\mathbf{H}}_{1,0} \;=\; \mathbf{H}_{1,0} + \zeta_2 \;,\quad \widetilde{\mathbf{H}}_{1,1,0} \;=\; \mathbf{H}_{1,1,0} - \zeta_2\, \ln(1-x) - \zeta_3
$$

All $\ln(1-x)$ terms and ζ-functions taken out of expansions to all orders in $1-x$

All-order resummation of the $1/N$ terms (I)

For $F_{1,2,3}$, $F_{\mathrm{T,I,A}}$ and F_{DY}, up to terms of order $1/N^2$, with $L \equiv \ln N$

$$C_a(N) - C_a\Big|_{N^0 L^k} \;=\; \frac{1}{N}\left(\left[d_{a,1}^{(1)} L + d_{a,0}^{(1)}\right] a_s + \left[\widetilde{d}_{a,1}^{(2)} L + d_{a,0}^{(2)}\right] a_s^2 + \ldots\right)$$
$$\exp\left\{L h_1(a_s L) + h_2(a_s L) + a_s h_3(a_s L) + \ldots\right\}$$

Exponentiation functions defined by expansions $h_k(a_s L) \;\equiv\; \sum_{n=1} h_{kn}\,(a_s L)^n$

Coefficients for DIS/SIA (upper/lower sign) relative to $N^0 L^k$ resummation

$$h_{1k} \;=\; g_{1k} \qquad\qquad g_{lk} = \text{coefficients in soft-gluon exponentiation}$$
$$h_{21} \;=\; g_{21} + \frac{1}{2}\,\beta_0 \pm 6\,C_F$$
$$h_{22} \;=\; g_{22} + \frac{5}{24}\,\beta_0^2 \pm \frac{17}{9}\,\beta_0\,C_F - 18\,C_F^2$$
$$h_{23} \;=\; g_{23} + \frac{1}{8}\,\beta_0^3 \pm \left(\frac{\xi_{K_4}}{8} - \frac{53}{18}\right)\beta_0^2 C_F - \frac{34}{3}\,\beta_0\,C_F^2 \pm 72\,C_F^3$$

First term of h_3 also known, but non-universal within DIS and SIA ($\Leftrightarrow F_L$)

All-order resummation of the $1/N$ terms (II)

For space-like (-) and time-like (+) structure/fragmentation functions F_L

$$C_L^{(\pm)}(N) \;=\; N^{-1}(d_1^{(\pm)} a_s + d_2^{(\pm)} a_s^2 + \ldots)\, \exp\{L h_1(a_s L) + h_2(a_s L) + \ldots\}$$

with

$$h_{11} \;=\; 2\, C_F \;, \quad h_{12} \;=\; \frac{2}{3}\beta_0\, C_F \;, \quad h_{13} \;=\; \frac{1}{3}\beta_0^2\, C_F$$

$$h_{21} \;=\; \beta_0 \;-\; C_F \;+\; (4 - 4\,\zeta_2)(C_A - 2C_F)$$

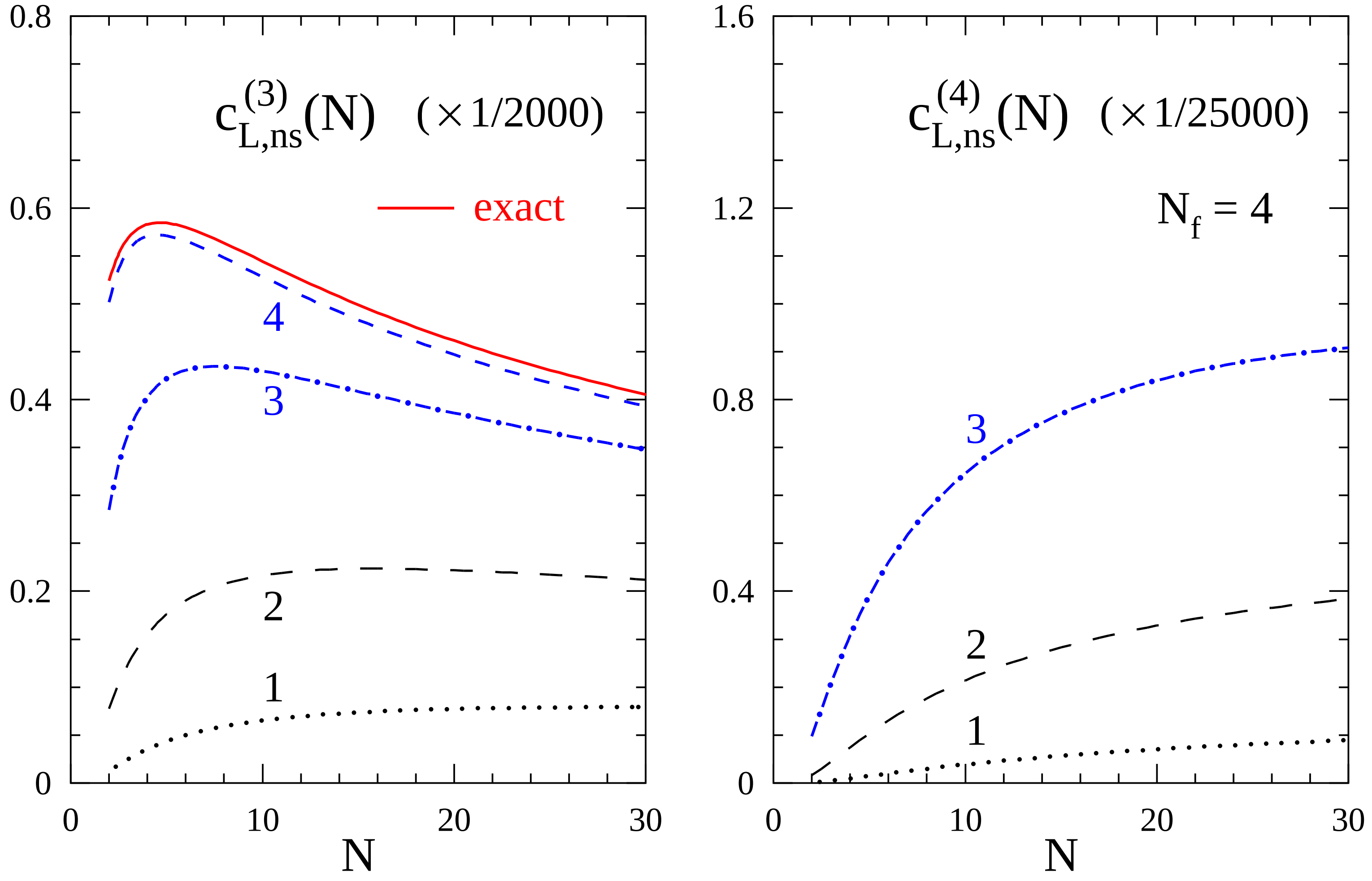
$$h_{22} \;=\; \underbrace{\frac{1}{2}(\,\beta_0\, h_{21} + A_2)}_{\text{as } g_{22} \text{ in soft-gluon exp.}} \;-\; \underbrace{8\,(C_A - 2C_F)^2(1 - 3\,\zeta_2 + \zeta_3 + \zeta_2^2\,)}_{\text{Who ordered THIS?}}$$

as g_{22} in soft-gluon exp. Who ordered THIS?

Remarks/questions

- Less predictive than $N^0 L^k$ exponentiation: nothing new, but A_2, in g_{22}
- NLL exponentiation – complete $h_2(a_s L)$ – could be feasible for $F_{a \neq L}$
- Full NNLL for $F_{1,2,3}$ etc, NLL for F_L: a log too far? h_{23} for F_L, anyone?

Third- and fourth-order C_L in DIS in N-space



1 = leading log etc. Good α_s^3 approximation by all four N^{-1} logarithms

As usual, cf. small-x: leading logs do not lead. Padé: ≈ 2.0 at $N = 20$

Large-N cross sections before factorization

Unfactorized partonic structure functions in $D = 4 - 2\varepsilon$ dimensions

$$T_{a,j} = \widetilde{C}_{a,i}\, Z_{ij}\ , \quad -\gamma \equiv P = \frac{dZ}{d\ln Q^2}\, Z^{-1}\ , \quad \frac{da_{\rm s}}{d\ln Q^2} = -\varepsilon a_{\rm s} + \beta_{D=4}$$

N^0 and N^{-1} transition functions, D-dimensional coefficient functions

$$Z\Big|_{a_{\rm s}^n} = \frac{1}{\varepsilon^n}\frac{\gamma_0^{n-1}}{n!}\Big[\gamma_0 - \frac{\beta_0}{2}n(n-1)\Big] + \sum_{\ell=1}^{n-1}\frac{1}{\varepsilon^{n-\ell}}\sum_{k=1}^{n-\ell-1}\gamma_0^{n-\ell-k-1}\gamma_\ell\gamma_0^k\,\frac{(\ell+k)!}{n!\,\ell!}$$

$$- \frac{\beta_0}{2}\sum_{\ell=1}^{n-2}\frac{1}{\varepsilon^{n-\ell}}\sum_{k=1}^{n-\ell-2}\gamma_0^{n-\ell-k-2}\gamma_\ell\gamma_0^k\,\frac{(\ell+k)!}{n!\,\ell!}(n(n-1) - \ell(\ell+k+1))$$

$+$ **NNLL contributions (explicit expressions)** $+\ \ldots$

$$\widetilde{C}_{a,i} = 1_{(\text{diagonal cases})} + \textstyle\sum_{n=1}^{\infty}\sum_{\ell=0}^{\infty} a_{\rm s}^n\,\varepsilon^\ell c_{a,i}^{(n,\ell)}\ ,$$ **ℓ additional logs at order ε^ℓ**

$\alpha_{\rm s}^n\,\varepsilon^{-n+\ell}$ off-diagonal entries: contributions up to $N^{-1}\ln^{n+\ell-1}N$

Full N$^{\rm m}$LO calc. of $T_{a,j}$: highest $m+1$ powers of ε^{-1} to all orders in $\alpha_{\rm s}$

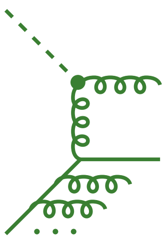
Extension to all ε for highest $n+1$ logarithms: N$^{\rm n}$LL all-order resummation

All-order off-diagonal leading-log amplitudes

Example: Leading-log (LL) $1/N$ terms of $T_{\phi,\mathrm{q}}^{(n)}$ and $T_{2,\mathrm{g}}^{(n)}$, with $L \equiv \ln N$

$$\frac{1}{C_F} T_{\phi,\mathrm{q}}^{(n)} = \frac{1}{n_f} T_{2,\mathrm{g}}^{(n)} = \frac{L^{n-1}}{N\varepsilon^n} \sum_{k=0}^{\infty} (\varepsilon L)^k \mathcal{L}_{n,k} \left(C_F^{n-1} + C_F^{n-2} C_A + \ldots + C_A^{n-1} \right)$$

$\Rightarrow$ **all-order relation for one colour structure of either amplitude sufficient**



$$T_{\phi,\mathrm{q}}^{(n)}\Big|_{C_F \text{ only}} \overset{\mathrm{LL}}{=} \frac{1}{n} T_{\phi,\mathrm{q}}^{(1)} \underbrace{T_{2,\mathrm{q}}^{(n-1)}}_{\frac{1}{(n-1)!}\,(T_{2,\mathrm{q}}^{(1)})^{n-1}} \overset{\mathrm{LL}}{=} \frac{1}{n!} T_{\phi,\mathrm{q}}^{(1)} \, (T_{2,\mathrm{q}}^{(1)})^{n-1}$$

$$\Rightarrow \quad T_{\phi,\mathrm{q}}\Big|_{C_F \text{ only}} \overset{\mathrm{LL}}{=} T_{\phi,\mathrm{q}}^{(1)} \, \frac{\exp(a_s T_{2,\mathrm{q}}^{(1)}) - 1}{T_{2,\mathrm{q}}^{(1)}}$$

Exact D-dimensional leading-log expressions for the one-loop amplitudes

$$T_{\phi,\mathrm{q}}^{(1)} \overset{\mathrm{LL}}{=} -2C_F \frac{1}{\varepsilon} (1-x)^{-\varepsilon} \overset{\mathrm{M}}{=} -\frac{2C_F}{N} \frac{1}{\varepsilon} \exp(\varepsilon \ln N)$$

$$T_{2,\mathrm{q}}^{(1)} \overset{\mathrm{LL}}{=} -4C_F \frac{1}{\varepsilon} (1-x)^{-1-\varepsilon} + \text{virtual} \overset{\mathrm{M}}{=} 4C_F \frac{1}{\varepsilon^2} \left(\exp(\varepsilon \ln N) - 1\right)$$

Leading-log splitting and coefficient functions

Expansions and iterative mass factorization to 'any' order [done in FORM]

$\Rightarrow$ All-order expressions for LL off-diagonal splitting and coefficient fct's

$$P_{\rm qg}^{\rm LL}(N,\alpha_{\rm s}) \;=\; \frac{n_f}{N}\,\frac{\alpha_{\rm s}}{2\pi}\sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}\,\tilde{a}_{\rm s}^{\,n}\;, \qquad \tilde{a}_{\rm s} \;=\; \frac{\alpha_{\rm s}}{\pi}\,(C_A - C_F)\,\ln^2 N$$

Bernoulli numbers B_n: zero for odd $n \geq 3$ $\;\Rightarrow\;$ $P_{\rm gq}^{(3)}(N) \overset{\rm LL}{=} 0$ understood

$$B_0 = 1,\; B_1 = -\tfrac{1}{2},\; B_2 = \tfrac{1}{6},\; B_4 = -\tfrac{1}{30},\; B_6 = \tfrac{1}{42},\; \ldots,\; B_{12} = -\tfrac{691}{2730},\; \ldots$$

$$C_{2,\rm g}^{\rm LL} \;=\; \frac{1}{2N\ln N}\,\frac{n_f}{C_A - C_F}\,\left\{\exp(2C_F\,a_{\rm s}\ln^2 N)\,\mathcal{B}_0(\tilde{a}_{\rm s}) - \exp(2C_A\,a_{\rm s}\ln^2 N)\right\}$$

$\exp(\ldots)$: LL soft-gluon exponentials Parisi; Curci, Greco; Amati et al. (80)

$$\mathcal{B}_0(x) \;=\; \sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}\,x^n$$

$P_{\rm gq}^{\rm LL}$, $C_{\phi,\rm q}^{\rm LL}$: same functions but with $C_F \leftrightarrow C_A$ (also in $\tilde{a}_{\rm s}$), then $n_f \rightarrow C_F$

Next-to-leading logarithmic iteration for $T_{H,\mathrm{q}}^{(n)}$

Ansatz for $T_{\phi,\mathrm{q}}^{(n)}$ in terms of first-order quantity and diagonal amplitudes

$$T_{\phi,\mathrm{q}}^{(n)} \stackrel{\mathrm{NL}}{=} \frac{1}{n}\, T_{\phi,\mathrm{q}}^{(1)} \left\{ \sum_{i=0}^{n-1} T_{\phi,\mathrm{g}}^{(i)}\, T_{2,\mathrm{q}}^{(n-i-1)} f(n,i) - \frac{\beta_0}{\varepsilon} \sum_{i=0}^{n-2} T_{\phi,\mathrm{g}}^{(i)}\, T_{2,\mathrm{q}}^{(n-i-2)} g(n,i) \right\}$$

All-order agreement with known highest four powers of ε^{-1} for

$$f(n,i) \;=\; \binom{n-1}{i}^{-1} \left[1+\varepsilon \left(\frac{\beta_0}{8C_A}\,(i+1)(n-i)\,\theta_{i1} - \frac{3}{2}\,(1-n\,\delta_{i0}) \right) \right]$$

$$g(n,i) \;=\; \binom{n}{i+1}^{-1}$$

LL: A.V. (2010)

Soft-gluon exponentiation: also $T_{\phi,\mathrm{g}}^{(n)}$ and $T_{2,\mathrm{q}}^{(n)}$ known at all powers of ε

$\Rightarrow$ next-to-leading logarithmic expression for $T_{\phi,\mathrm{q}}$ completely predicted

Mass factorization $\Rightarrow$ $P_{\mathrm{gq}}^{\mathrm{NLL}}$, $c_{\phi,\mathrm{q}}^{\mathrm{NLL}}$ to all orders. $P_{\mathrm{qg}}^{\mathrm{NLL}}$, $c_{2,\mathrm{g}}^{\mathrm{NLL}}$ analogous

Extension of this approach to higher-log accuracy (at least) cumbersome

D-dim. structure of unfactorized observables

Maximal phase space for deep-inelastic scattering/semi-incl. annihilation

$\mathrm{NLO}: \quad 2 \to 2 \ / \ 1 \to 1+2 \qquad (1-x)^{-\varepsilon} \ \ x^{\cdots} \int_0^1$ one other variable

$\mathrm{N^2LO}: \quad 2 \to 3 \ / \ 1 \to 1+3 \qquad (1-x)^{-2\varepsilon} \ x^{\cdots} \int_0^1$ four other variables

$\mathrm{N^3LO}: \quad 2 \to 4 \ / \ 1 \to 1+4 \qquad (1-x)^{-3\varepsilon} \ x^{\cdots} \int_0^1$ seven other variables

...

N^{2}LO: Matsuura, van Neerven (88), Rijken, vN (95), N$^{n\geq 3}$LO, indirectly: MV[V] (05)

Purely real contributions to unfactorized structure functions

$$T_{a,j}^{(n)\mathrm{R}} = \frac{1}{\varepsilon^{2n-1}} \sum_{\xi=0} (1-x)^{-1+\xi-n\varepsilon} \left\{ R_{a,j,\xi}^{(n)\mathrm{LL}} + \varepsilon R_{a,j,\xi}^{(n)\mathrm{NLL}} + \ldots \right\}$$

Mixed contributions ($2 \to r+1$ with $n-r$ loops in DIS)

$$T_{a,j}^{(n)\mathrm{M}} = \frac{1}{\varepsilon^{2n-1}} \sum_{\ell=r}^{n} \sum_{\xi=0} (1-x)^{-1+\xi-\ell\varepsilon} \left\{ M_{a,j,\ell,\xi}^{(n)\mathrm{LL}} + \varepsilon M_{a,j,\ell,\xi}^{(n)\mathrm{NLL}} + \ldots \right\}$$

Purely virtual part (diagonal cases, $\xi = 0$ present): $\gamma^* \mathrm{qq}$, $H\mathrm{gg}$ form factors

$$T_{a,j}^{(n)\mathrm{V}} = \delta(1-x) \frac{1}{\varepsilon^{2n}} \left\{ V_{a,j}^{(n)\mathrm{LL}} + \varepsilon V_{a,j}^{(n)\mathrm{NLL}} + \ldots \right\}$$

Resulting resummation of large-x double logs

KLN cancellation between purely real, mixed and purely virtual contributions

$$T_{a,j}^{(n)} = T_{a,j}^{(n)\mathrm{R}} + T_{a,j}^{(n)\mathrm{M}} \left(+ T_{a,j}^{(n)\mathrm{V}} \right) = \frac{1}{\varepsilon^n} \left\{ T_{a,j}^{(n)0} + \varepsilon T_{a,j}^{(n)1} + \ldots \right\}$$

$\Rightarrow$ Up to $n-1$ relations between the coeff's of $(1-x)^{-\ell\varepsilon},\ \ell = 1,\ldots,n$

Log expansion: N^{k}LL higher-order coefficients completely fixed, if first $k+1$ powers of ε known to all orders – provided by N^{k}LO calculation, see above

Present situation: (a) N^{3}LO for non-singlet $F_{a\neq L}$ in DIS – recall DMS (05)
(b) N^{2}LO for SIA, non-singlet F_L in DIS, and singlet DIS

$\Rightarrow$ resummation of the (a) four and (b) three highest $N^{-1}\ln^k N$ terms to all orders in α_s: consistent with, and extending, our previous results

Soft-gluon exponentiation of $(1-x)^{-1}/N^0$ diagonal coefficient functions:

$(1-x)^{-1-\varepsilon}, \ldots, (1-x)^{-1-(n-1)\varepsilon}$ at order n: products of lower-order quantities

$\Rightarrow$ N^{n}LO [+$A^{(n+1)}$] $\rightarrow$ N^{n}LL exponentiation; $2n[+1]$ highest logs predicted

NS results, off-diagonal splitting fct's and $C_{L,\mathrm{g}}$

NS cases: $K_{a,4}(x)$ of p. 13 confirmed with $\xi_{K_4} = \frac{100}{3}$: fourth log for $c_{a,\mathrm{ns}}^{(n\geq 4)}$

also: Grunberg (2010)

Off-diagonal splitting functions

$$\tilde{a}_{\mathrm{s}} = \frac{\alpha_{\mathrm{s}}}{\pi}(C_A - C_F)\ln^2 \tilde{N}$$

$$N P_{\mathrm{qg}}^{\mathrm{NL}}(N,\alpha_{\mathrm{s}}) = 2 a_{\mathrm{s}} n_f \mathcal{B}_0(\tilde{a}_{\mathrm{s}}) + a_{\mathrm{s}}^2 \ln\tilde{N}\, n_f \left\{ (6C_F - \beta_0)\left(\frac{2}{\tilde{a}_{\mathrm{s}}}\mathcal{B}_{-1}(\tilde{a}_{\mathrm{s}}) + \mathcal{B}_1(\tilde{a}_{\mathrm{s}})\right) + \frac{\beta_0}{\tilde{a}_{\mathrm{s}}}\mathcal{B}_{-2}(\tilde{a}_{\mathrm{s}}) \right\}$$

$$N P_{\mathrm{gq}}^{\mathrm{NL}}(N,\alpha_{\mathrm{s}}) = 2 a_{\mathrm{s}} C_F \mathcal{B}_0(-\tilde{a}_{\mathrm{s}}) + a_{\mathrm{s}}^2 \ln\tilde{N}\, C_F \left\{ (12 C_F - 6\beta_0)\frac{1}{\tilde{a}_{\mathrm{s}}}\mathcal{B}_{-1}(-\tilde{a}_{\mathrm{s}}) - \frac{\beta_0}{\tilde{a}_{\mathrm{s}}}\mathcal{B}_{-2}(-\tilde{a}_{\mathrm{s}}) + (14 C_F - 8 C_A - \beta_0)\mathcal{B}_1(-\tilde{a}_{\mathrm{s}}) \right\}$$

Gluon contribution to F_L – 'non-singlet' $C_F = 0$ part done before MV (09)

$$N^2 C_{L,\mathrm{g}}^{\mathrm{NL}}(N,\alpha_{\mathrm{s}}) = 8 a_{\mathrm{s}} n_f \exp(2 C_A a_{\mathrm{s}} \ln^2 \tilde{N}) + 4 a_{\mathrm{s}} C_F N C_{2,\mathrm{g}}^{\mathrm{LL}}(N,\alpha_{\mathrm{s}}) + 16 a_{\mathrm{s}}^2 \ln\tilde{N}\, n_f \left\{ 4 C_A - C_F + \frac{1}{3} a_{\mathrm{s}} \ln^2\tilde{N}\, C_A \beta_0 \right\} \exp(2 C_A a_{\mathrm{s}} \ln^2 \tilde{N})$$

New: also NNLL terms now in closed form ($\mathcal{B}_{-4} \ldots \mathcal{B}_2$) A. Almasy, A.V.

Resummed gluon coefficient function for F_2

$$
\begin{aligned}
N C_{2,\mathrm{g}}(N,\alpha_s) \;=\;& \\
& \frac{1}{2\ln \widetilde{N}}\;\frac{n_f}{C_A - C_F}\;\Big[\exp(2a_s C_F \ln^2 \widetilde{N})\mathcal{B}_0(a_s^3) - \exp(2a_s C_A \ln^2 \widetilde{N})\Big] \\
& - \frac{1}{8\ln^2 \widetilde{N}}\;\frac{n_f(3C_F - \beta_0)}{(C_A - C_F)^2}\;\Big[\exp(2a_s C_F \ln^2 \widetilde{N})\mathcal{B}_0(a_s^3) - \exp(2a_s C_A \ln^2 \widetilde{N})\Big] \\
& - \frac{a_s}{4}\;\frac{n_f}{C_A - C_F}\;\exp\left(2a_s C_A \ln^2 \widetilde{N}\right)\left(8C_A + 4C_F - \beta_0\right) \\
& - \frac{a_s}{4}\;\frac{n_f}{C_A - C_F}\;\exp\left(2a_s C_F \ln^2 \widetilde{N}\right)\Big[-6C_F\mathcal{B}_0(a_s^3) - (6C_F - \beta_0)\mathcal{B}_1(a_s^3) \\
& \qquad - (12C_F - 4\beta_0)\,\frac{1}{a_s^3}\,\mathcal{B}_{-1}(a_s^3) - \frac{\beta_0}{a_s^3}\,\mathcal{B}_{-2}(a_s^3)\Big] \\
& - \frac{a_s^2}{3}\,\beta_0 \ln^2 \widetilde{N}\,\frac{n_f}{C_A - C_F}\Big[C_A \exp(2a_s C_A \ln^2 \widetilde{N}) - C_F \exp(2a_s C_F \ln^2 \widetilde{N})\mathcal{B}_0(a_s^3)\Big]
\end{aligned}
$$

$+$ **known NNLL contributions (now in closed form)** $+ \ \ldots$

$C_{H,\mathrm{q}}$ analogous. Analytic form identified via the physical kernel for (F_2, F_H)

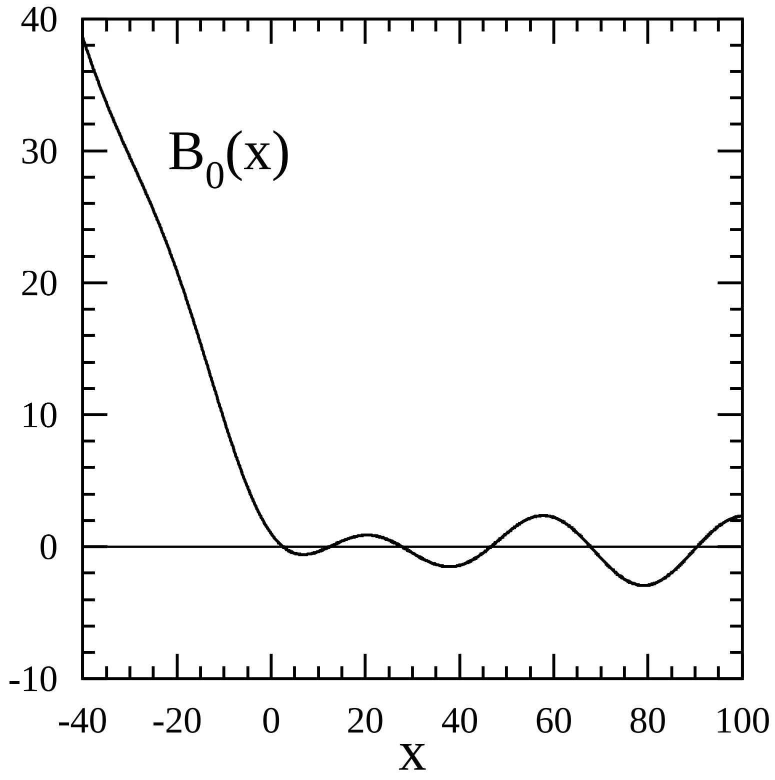
Resummed timelike splitting and coefficient functions: same structure

$\mathcal{B}$-functions: $\mathcal{B}_0$ and general definition

Relation between even-n Bernoulli numbers and the Riemann ζ-function

$$\mathcal{B}_0(x) \equiv \sum_{n=0}^{\infty} \frac{B_n}{(n!)^2} x^n = 1 - \frac{x}{2} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \zeta_{2n} \left(\frac{x}{2\pi} \right)^{2n}$$

$\mathcal{B}_0(2\pi i)$ numerically known (Wolfram MathWorld, Sloane's A093721), no closed form



Further $\mathcal{B}$-functions

$$\mathcal{B}_k(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!(n+k)!} x^n$$

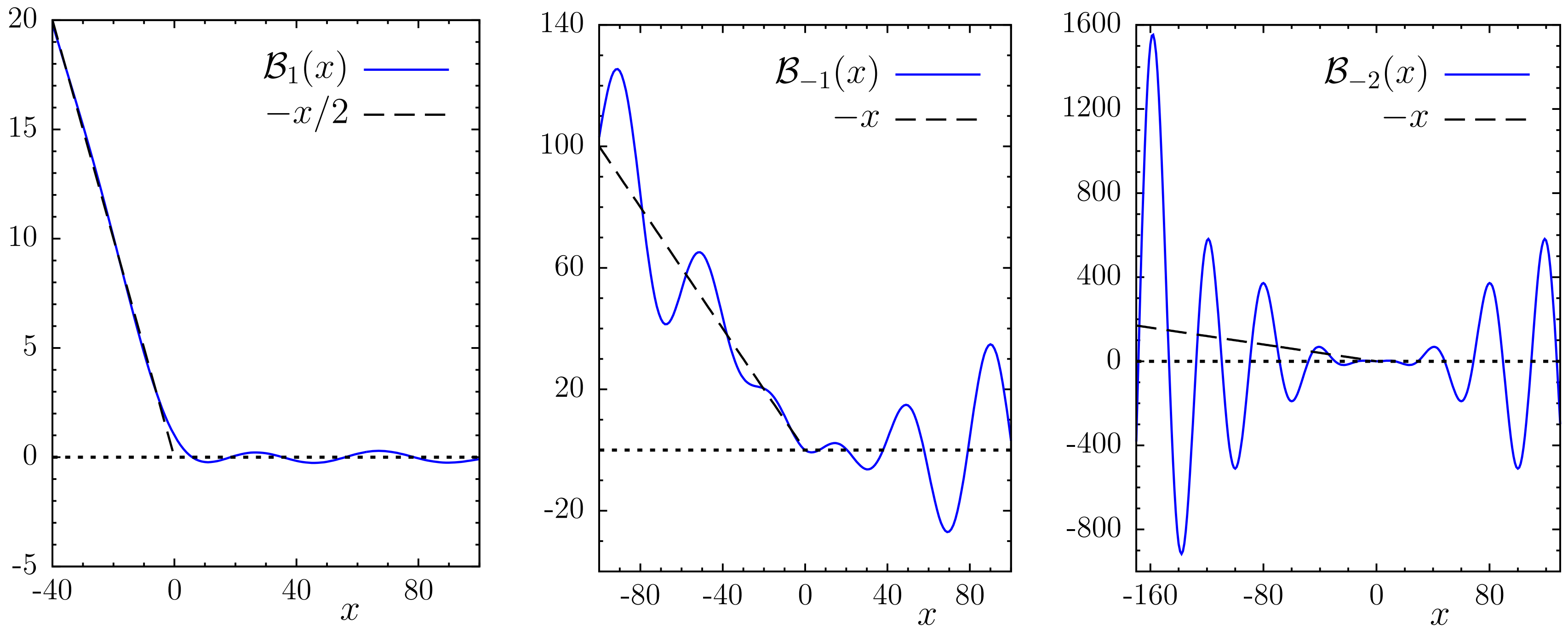
$$\mathcal{B}_{-k}(x) = \sum_{n=k}^{\infty} \frac{B_n}{n!(n-k)!} x^n$$

Relations to $\mathcal{B}_0(x)$

$$\frac{d^k}{dx^k}(x^k \mathcal{B}_k) = \mathcal{B}_0 , \quad x^k \frac{d^k}{dx^k} \mathcal{B}_0 = \mathcal{B}_{-k}$$

A.V. (2010)

$\mathcal{B}$-functions with index unequal zero



$x > 0$: all functions $\mathcal{B}_k(x)$ oscillate about $y = 0$

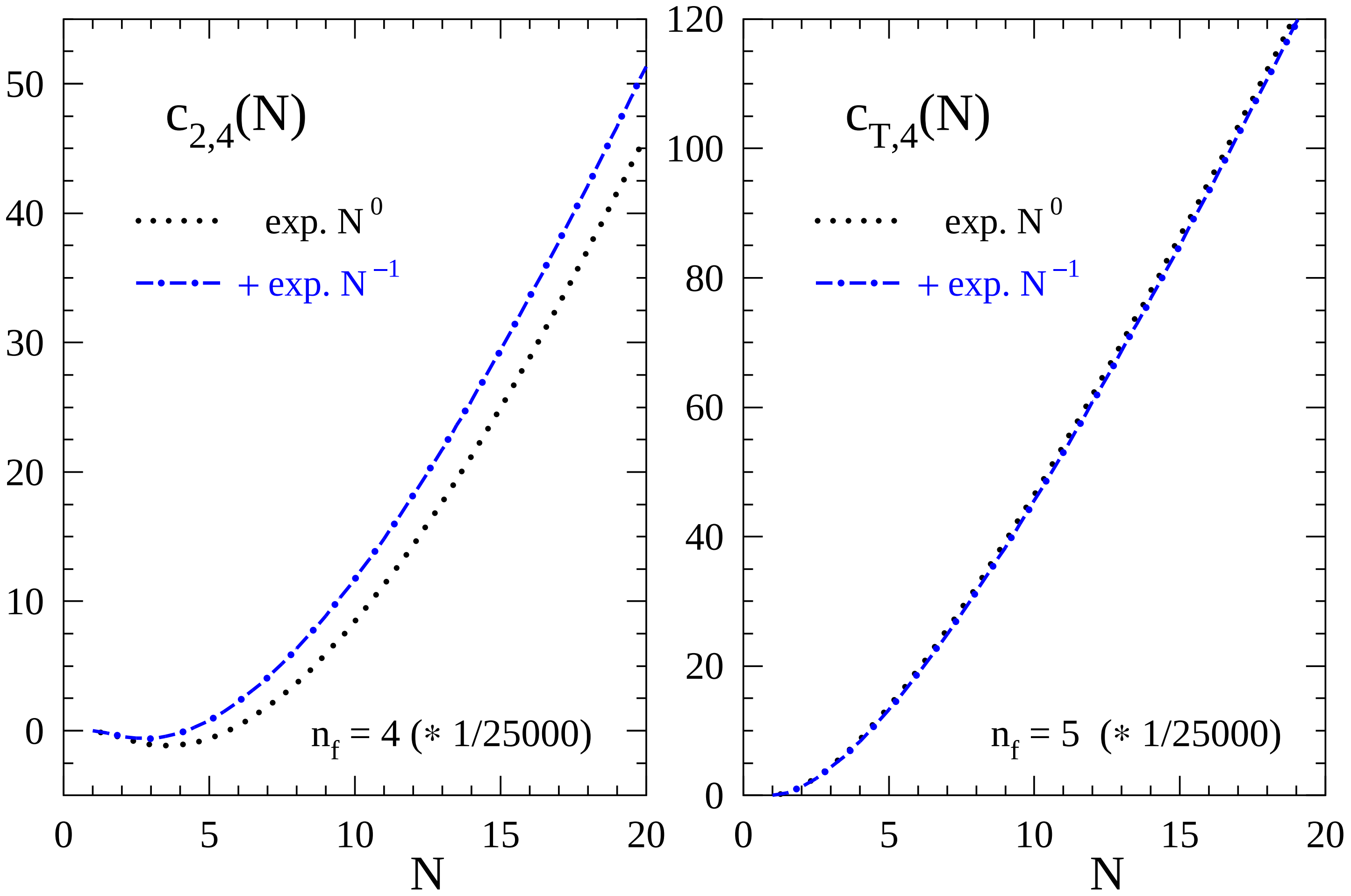
$x < 0$: oscillations about $y = -\frac{x}{(k+1)!}$ for $k \geq 0$ and $y = -x$ for $k < 0$

Amplitudes increase very rapidly with decreasing k

Oscillation of $\mathcal{B}_0$ continues (much more irregularly) to very large x

D. Broadhurst, private communication

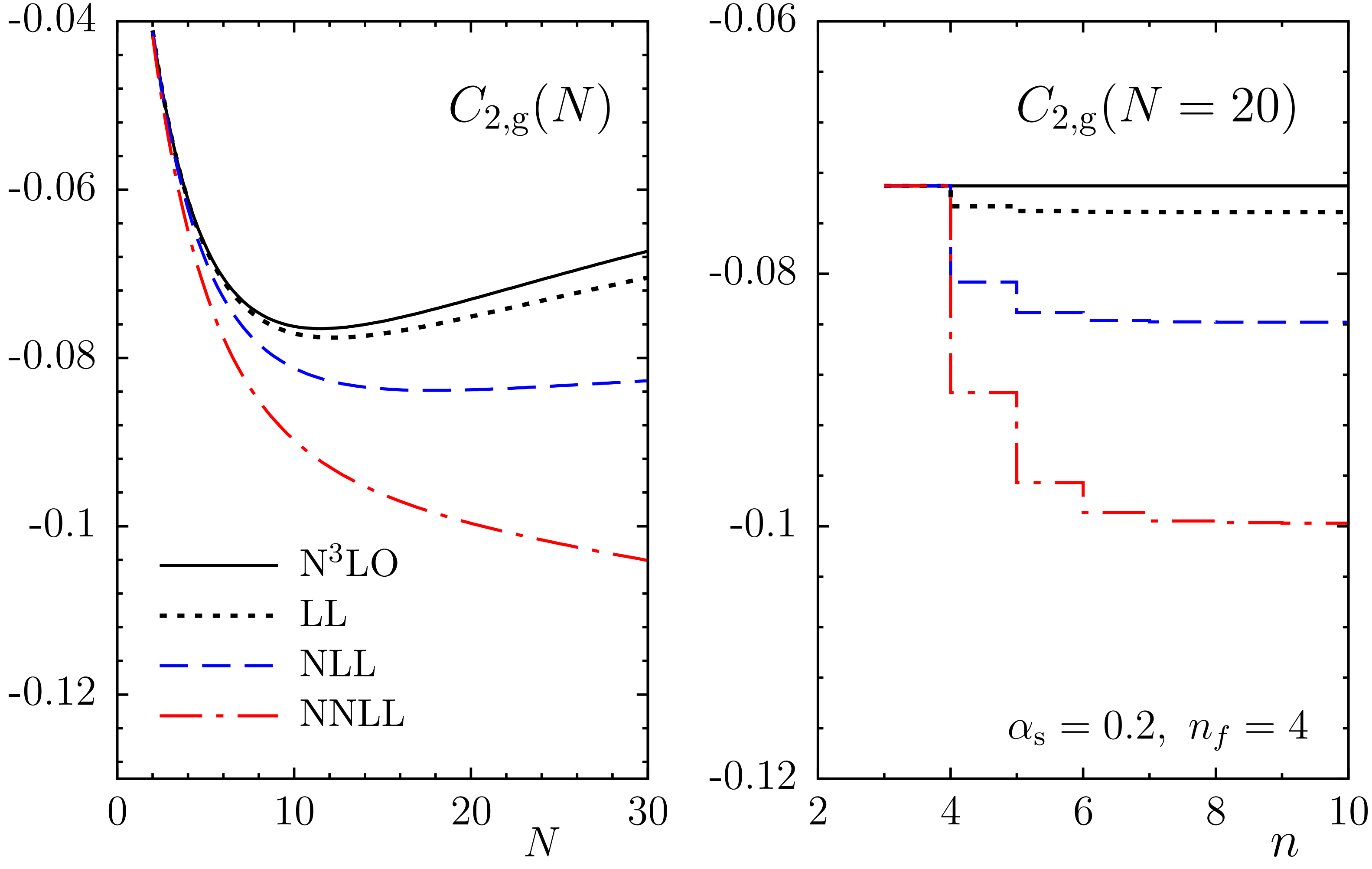
Fourth-order C_2 (DIS) and C_T (SIA) at large N



Exp. N^0: 7 of 8 logs, exp. N^{-1}: 4 of 7 logs $\Rightarrow$ large-x higher-twist analyses

N^{-1} contributions again relevant for F_2, but small for F_{T} at least at $N > 5$

Numerical illustration of $C_{2,\mathrm{g}}$



NNLL terms dominate $\Rightarrow$ impact of high orders presumably underestimated

About 35% correction at $N = 20$, 4$^{\mathrm{th}}$-order coefficient $\approx$ Padé estimate

Small-x resummation via unfactorized SIA

Phase-space integrations: $x^{a\varepsilon}$ **terms analogous to** $(1-x)^{b\varepsilon}$ **large-x factors**

2nd order: **Matsuura, van Neerven (88), Rijken, vN (95)**

Decomposition of the D-dim. partonic fragmentation functions for $a = T, \phi$

$$\widehat{F}^{(n)}_{a,\mathrm{g}} = \frac{1}{\varepsilon^{2n-1}} \sum_{\ell=0}^{n-1} x^{-1-2(n-\ell)\varepsilon} \Big\{ A^{(\ell,n)}_{a,\mathrm{g}} + \varepsilon B^{(\ell,n)}_{a,\mathrm{g}} + \varepsilon^2 C^{(\ell,n)}_{a,\mathrm{g}} + \ldots \Big\}$$

Leading log: terms of the form $x^{-1}\ln^{n+\delta-1} x$ at all orders $\varepsilon^{-n+\delta}$ with $\delta = 0,\, 1,\, 2,\, \ldots$, and $\widehat{F}^{(n)}_{a,\mathrm{g}}$ is decomposed into n contributions of the form

$$\varepsilon^{-2n+1}\, x^{-1-k\,\varepsilon} = \varepsilon^{-2n+1}\, x^{-1} \Big[1 - k\,\varepsilon \ln x + \frac{1}{2} (k\,\varepsilon)^2 \ln^2 x + \ldots \Big], \qquad k = 2,\, 4,\, \ldots,\, 2n$$

$n-1$ KLN-type cancellations – $\widehat{F}^{(n)}_{a,\mathrm{g}}$ starts at order $1/\varepsilon^n$ – plus 3 constraints from the NNLO results $\Rightarrow$ $n+2$ linear equations for n coefficients $A^{(\ell,n)}_{a,\mathrm{g}}$

Thus: N^{n}LO known $\Rightarrow$ highest $n+1$ (N^{n}LL) double logs fixed at all orders

'All-order' mass factorization: NNLL timelike splitting & coefficient functions

Splitting & coefficient functions, status 2011

$$\frac{C_A}{C_F} P^T_{\rm gq}(N,\alpha_{\rm s}) \overset{\rm LL}{=} P^T_{\rm gg}(N,\alpha_{\rm s}) \overset{\rm LL}{=} \frac{1}{4}(N-1)\left\{(1-4\xi)^{1/2}-1\right\}, \quad \xi = -\frac{8C_A a_{\rm s}}{(N-1)^2}$$

Mueller (81); Bassetto, Ciafaloni, Marchesini, Mueller (82)

NLL contributions to the $\overline{\rm MS}$ splitting functions: only partially in closed form

$$\left[P^T_{\rm gg}\right]^{\rm NLL}_{C_F=0} = \left\{(1-4\xi)^{-1/2}+1\right\} a_{\rm s}\left(\frac{11}{6}C_A + \frac{1}{3}n_f\right)$$

$$\left[\frac{C_A}{C_F} P^T_{\rm gq}\right]^{\rm NLL}_{C_F=0} = \left[P^T_{\rm gg}\right]^{\rm NLL}_{C_F=0} + \left\{(1-4\xi)^{1/2}-1\right\}\frac{1}{24}(N-1)^2(1+n_f/C_A)$$

LL coefficient functions for F_T & F_ϕ [also: Albino, Bolzoni, Kniehl, Kotikov (11)]

$$C^{\rm LL}_{T,\,\rm g} = \frac{C_F}{C_A}\left(C^{T,\,\rm LL}_{\phi,\rm g} - 1\right) = \frac{C_F}{C_A}\left\{(1-4\xi)^{-1/4}-1\right\} \quad \text{in } \overline{\rm MS}$$

'Everything else', including all of $P^T_{\rm qq}$, $P^T_{\rm qg}$, the quark coefficient fct's, $C_{L,\rm i}$:

Tables of coefficients to order $\alpha_{\rm s}^{16}$ – numerically sufficient for $x \gtrsim 10^{-4}$ – e.g.

$$P^{(n)T}_{\rm gg,\,NLL}(N) = -\frac{(-8)^n\, C_A^{n-1}}{3(N-1)^{2n}}\left[(11C_A^2 + 2\,C_A n_f)\,B^{(n)}_{\rm gg,1} - 2\,C_F n_f\,B^{(n)}_{\rm gg,2}\right]$$

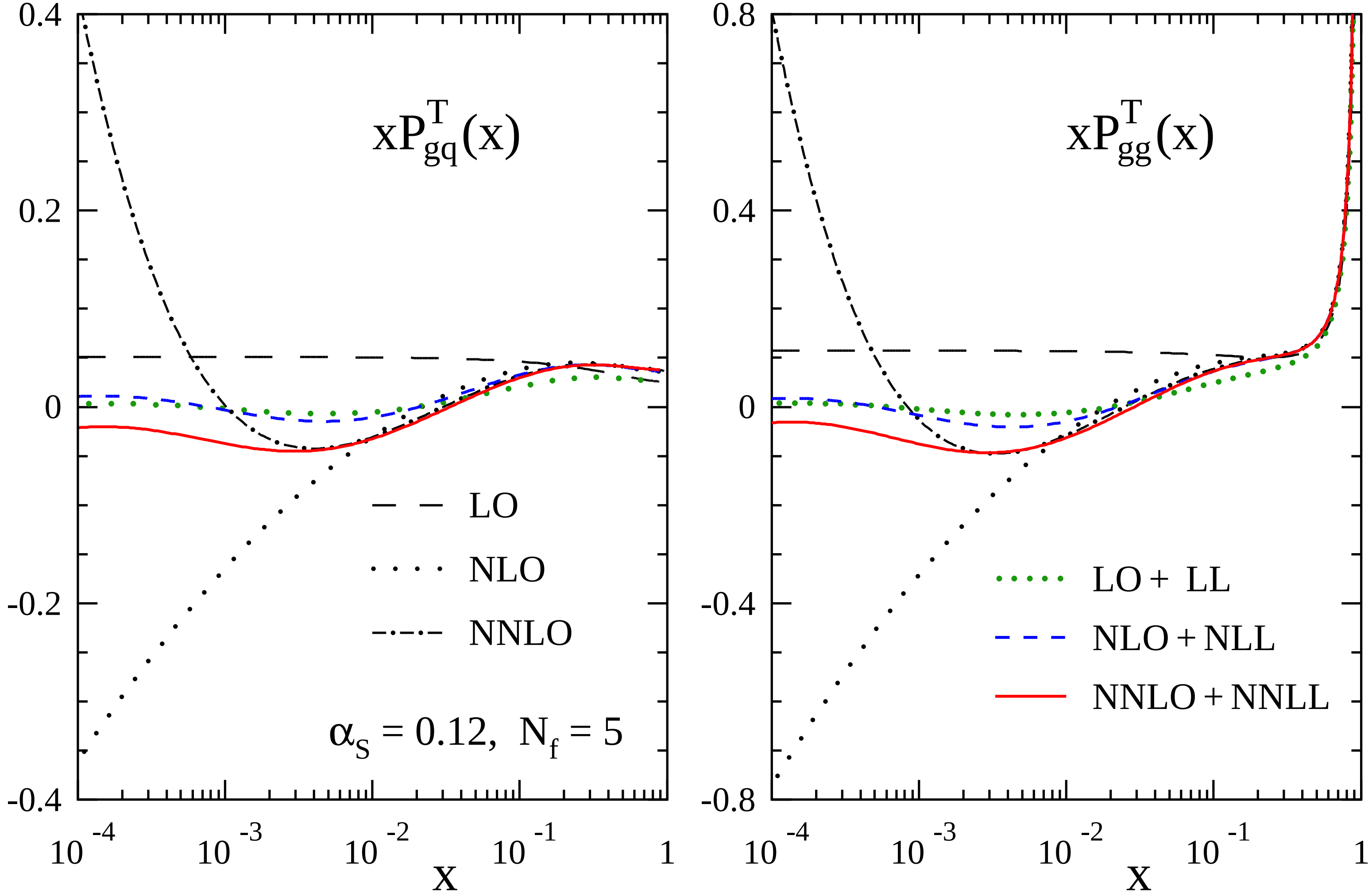
Normalized LL, NLL splitting-fct. coefficients

n	$A_{\rm gi}^{(n)}$	$B_{\rm gg,1}^{(n)}$	$B_{\rm gg,2}^{(n)}$	$B_{\rm gq,1}^{(n)}$	$B_{\rm gq,2}^{(n)}$	$B_{\rm gq,3}^{(n)}$	$A_{\rm qi}^{(n)}$
0	1	1	–	9	–	–	–
1	1	1	2	9	–	–	–
2	2	3	5	29	1	1	1
3	5	10	$\frac{49}{3}$	100	5	$\frac{19}{3}$	$\frac{11}{3}$
4	14	35	$\frac{347}{6}$	357	21	$\frac{179}{6}$	$\frac{73}{6}$
5	42	126	$\frac{6353}{30}$	1302	84	$\frac{3833}{30}$	$\frac{1207}{30}$
6	132	462	$\frac{11839}{15}$	4818	330	$\frac{7879}{15}$	$\frac{2021}{15}$
7	429	1716	$\frac{624557}{210}$	18018	1287	$\frac{444377}{210}$	$\frac{96163}{210}$
8	1430	6435	$\frac{316175}{28}$	67925	5005	$\frac{236095}{28}$	$\frac{44185}{28}$
9	4862	24310	$\frac{54324719}{1260}$	257686	19448	$\frac{42072479}{1260}$	$\frac{6936481}{1260}$

All integer series known, $B_{\rm gg,2}^{(n)} - B_{\rm gq,3}^{(n)} = 2A_{\rm gi}^{(n)},\ A_{\rm qi}^{(n)} + B_{\rm gg,2}^{(n)} = 2B_{\rm gg,1}^{(n)}$

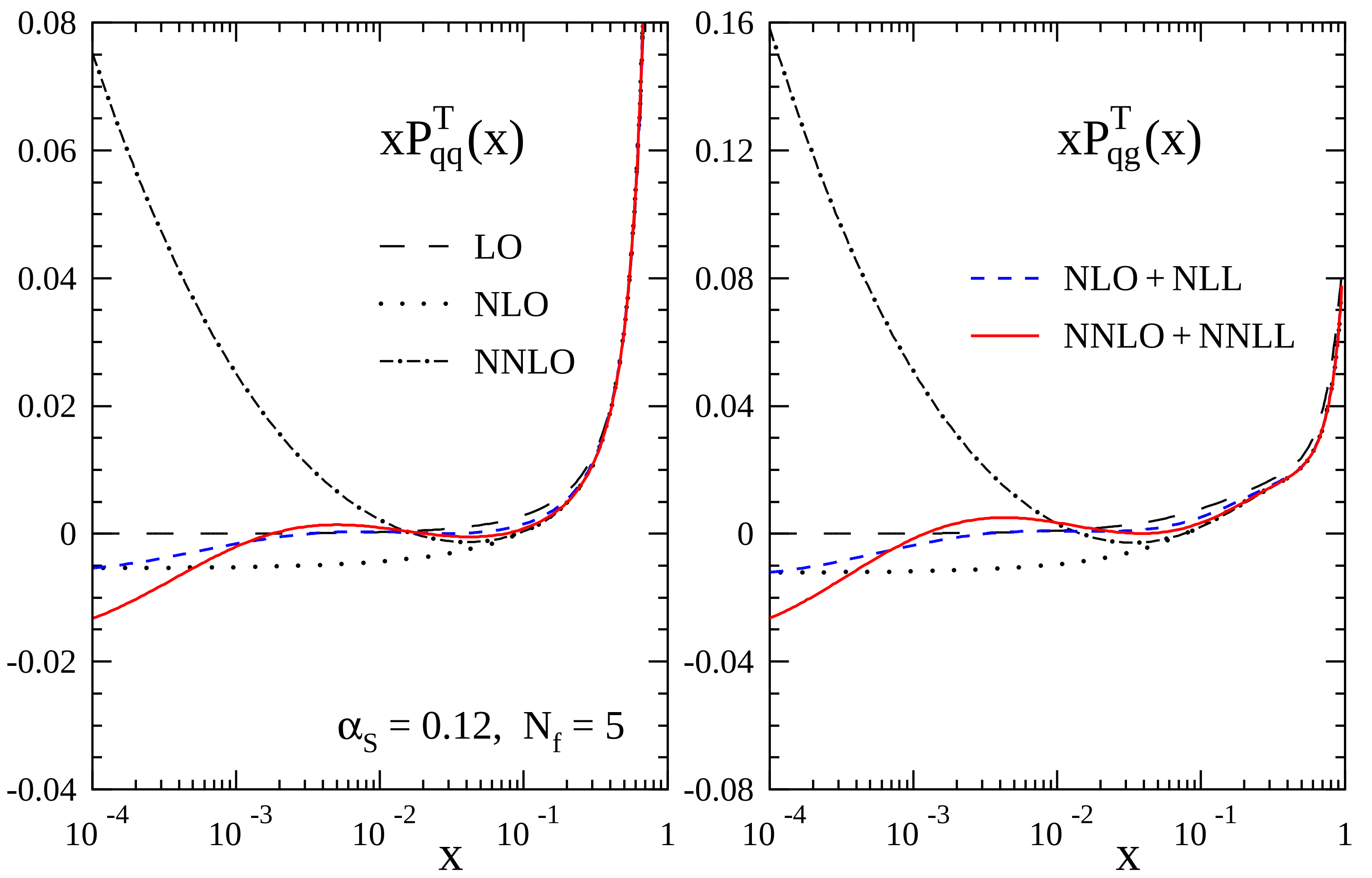
Solution of one non-integer series: analytic structure of all NLL contributions

Small-x gluon-parton splitting functions



Approximation sequence LO+LL, NLO+NLL, NNLO+NNLL rather stable to very small x

Small-x quark-parton splitting functions



Also consistent with $xP^T_{ji} \approx 0$ at $x < 10^{-2}$ (N^3LL corr's known and positive)

2012 progress: solution of the $A_{\mathrm{qi}}^{(n)}$ series

Denominators $\leftrightarrow$ triangular numbers (A025555 in OEIS) $+$ 'playing around'

$$A_{\mathrm{qi}}^{(2)} = 1 = \frac{1}{1}\,, \qquad A_{\mathrm{qi}}^{(5)} = \frac{1207}{30} = \frac{14}{10} + \frac{19}{6} + \frac{23}{3} + \frac{28}{1}\,,$$

$$A_{\mathrm{qi}}^{(3)} = \frac{11}{3} = \frac{2}{3} + \frac{3}{1}\,, \qquad A_{\mathrm{qi}}^{(6)} = \frac{2015}{15} = \frac{42}{15} = \frac{56}{10} + \frac{66}{6} + \frac{76}{3} + \frac{90}{1}\,,$$

$$A_{\mathrm{qi}}^{(4)} = \frac{73}{6} = \frac{5}{6} + \frac{7}{3} + \frac{9}{1}\,, \qquad \ldots$$

Numerators: sequence A028364, sums of products of Catalan numbers

$$\Rightarrow \quad A_{\mathrm{qi}}^{(n)} = \frac{2(2\,n-2)!}{(n-1)!(n+1)!}\left(\frac{1}{n-1} + \frac{1}{n} + \frac{6}{n+1} - 2\right) + \frac{2(2\,n)!}{n!(n+1)!}\sum_{k=n}^{2\,n-3}\frac{1}{k}$$

K. Yeats

Checked to 136 $^{\mathrm{th}}$ entries by 'brute-force' determination of $A_{\mathrm{qi}}^{(n\leq 17)}$

Generating function

$$A_{\mathrm{qi}}(\xi) = \left(\sqrt{1-4\xi} - 1\right)\left(1 + \ln\left(\frac{1}{2}\left[\sqrt{1-4\xi} + 1\right]\right)\right) + 2\xi$$

C.H. Kom

The logarithm is the key for solving all sequences in the 2011 article

Diagonal splitting functions at NNLL accuracy

Notation: $S = (1-4\xi)^{1/2}$, $\mathcal{L} = \ln\left(\frac{1}{2}(1+S)\right)$ **with** $\xi = -8C_A a_s/\bar{N}^2$, $\bar{N} \equiv N-1$

$$
\begin{aligned}
P^T_{qq}(N) \;=\;& \frac{4}{3}\frac{C_F n_f}{C_A} a_s \Big\{ \frac{1}{2\xi}(S-1)(\mathcal{L}+1)+1 \Big\} \\
&+ \frac{1}{18}\frac{C_F n_f}{C_A^3} a_s \bar{N} \Big\{ (-11\,C_A^2 + 6\,C_A n_f - 20\,C_F n_f)\,\frac{1}{2\xi}(S-1+2\,\xi) + 10\,C_A^2\,\frac{1}{\xi}(S-1)\mathcal{L} \\
&\quad - (51\,C_A^2 - 6\,C_A n_f + 12\,C_F n_f)\,\frac{1}{2}(S-1) + (11\,C_A^2 + 2\,C_A n_f - 4\,C_F n_f)\,S^{-1}\mathcal{L} \\
&\quad + (5\,C_A^2 - 2\,C_A n_f + 6\,C_F n_f)\,\frac{1}{\xi}(S-1)\mathcal{L}^2 + (51\,C_A^2 - 14\,C_A n_f + 36\,C_F n_f)\mathcal{L} \Big\}
\end{aligned}
$$

$$
\begin{aligned}
P^T_{gg}(N) \;=\;& \frac{1}{4}\bar{N}(S-1) \;-\; \frac{1}{6\,C_A} a_s\,(11\,C_A^2 + 2\,C_A n_f - 4\,C_F n_f)\,(S^{-1}-1) \;-\; \boldsymbol{P^T_{qq}(N)} \\
&+ \frac{1}{576\,C_A^3}\, a_s \bar{N} \Big\{ \Big([1193 - 576\,\zeta_2] C_A^4 - 140\,C_A^3 n_f + 4\,C_A^2 n_f^2 - 56\,C_A^2 C_F n_f - 48\,C_F^2 n_f^2 \\
&\quad + 16\,C_A C_F n_f^2 \Big)\,(S-1) \;+\; \Big([830 - 576\,\zeta_2]\,C_A^4 + 96\,C_A^3 n_f - 8\,C_A^2 n_f^2 - 208\,C_A^2 C_F n_f \\
&\quad + 64\,C_A C_F n_f^2 - 96\,C_F^2 n_f^2 \Big)\,(S^{-1}-1) \;+\; (11\,C_A^2 + 2\,C_A n_f - 4\,C_F n_f)^2 (S^{-3}-1) \Big\}
\end{aligned}
$$

First lines: LL (for P^T_{gg}) and NLL contributions. Rest: NNLL corrections

Off-diagonal splitting functions: similar. For P^T_{qi} also N^{3}LL terms known

NLO + resummed first moments

Fixed-order $N=1$ poles removed by the resummation (NLO requires NNLL)

$$
\begin{aligned}
P^T_{\mathrm{qg}}(N=1) &= \frac{8}{3}\, n_f a_{\mathsf{s}} - \frac{1}{3C_A^2}\left(17\, C_A^2\, n_f - 2\, C_A n_f^2 + 4\, C_F n_f^2\right)(2C_A a_{\mathrm{s}}^3)^{1/2} + \mathcal{O}(a_{\mathrm{s}}^2) \\
P^T_{\mathrm{qq}}(N=1) &= \frac{C_F}{C_A}\left(P^T_{\mathrm{qg}}(N=1) - \frac{4}{3}\, n_f a_{\mathsf{s}}\right) + \mathcal{O}(a_{\mathrm{s}}^2) \\
P^T_{\mathrm{gg}}(N=1) &= (2C_A a_{\mathsf{s}})^{1/2} - \frac{1}{6C_A}\left(11\, C_A^2 + 2\, C_A n_f + 12\, C_F n_f\right) a_{\mathsf{s}} \\
&\quad + \frac{1}{144\, C_A^3}\Big([1193 - 576\,\zeta_2]\, C_A^4 - 140\, C_A^3 n_f + 4\, C_A^2 n_f^2 + 760\, C_A^2 C_F n_f \\
&\quad - 80\, C_A C_F n_f^2 + 144\, C_F^2 n_f^2\Big)(2C_A a_{\mathrm{s}}^3)^{1/2} + \mathcal{O}(a_{\mathrm{s}}^2) \\
P^T_{\mathrm{gq}}(N=1) &= \frac{C_F}{C_A}\left(P^T_{\mathrm{gg}}(N=1) + \frac{4}{3}\,\frac{C_F n_f}{C_A}\, a_{\mathsf{s}}\right) + \mathcal{O}(a_{\mathrm{s}}^2)\,.
\end{aligned}
$$

Numerically for QCD with $n_f = 5$, including N^{3}LL for P^T_{qi} (α_{s}^2 contributions)

$$
\begin{aligned}
P^T_{\mathrm{qq}}(N=1) &\cong 0.2358\,\alpha_{\mathsf{s}} - 0.6773\,\alpha_{\mathsf{s}}^{3/2} + 0.5880\,\alpha_{\mathrm{s}}^2 \\
P^T_{\mathrm{qg}}(N=1) &\cong 1.0610\,\alpha_{\mathsf{s}} - 1.5240\,\alpha_{\mathsf{s}}^{3/2} + 1.8089\,\alpha_{\mathrm{s}}^2 \\
P^T_{\mathrm{gq}}(N=1) &\cong 0.3071\,\alpha_{\mathsf{s}}^{1/2} - 0.3059\,\alpha_{\mathsf{s}} + 0.2884\,\alpha_{\mathsf{s}}^{3/2} \\
P^T_{\mathrm{gg}}(N=1) &\cong 0.6910\,\alpha_{\mathsf{s}}^{1/2} - 0.9240\,\alpha_{\mathsf{s}} + 0.6490\,\alpha_{\mathsf{s}}^{3/2}
\end{aligned}
$$

First application to multiplicities: Bolzoni, Kniehl, Kotikov (Sept. 12)

Partial x-space expressions

Non-log parts (integer series): Bessel functions in $z = (32\, C_A\, a_s)^{1/2} \ln \frac{1}{x}$

$$x P_{gg}^{T} + x P_{qq}^{T}\Big|_{\rm NNLL} = \Big\{ 4\, C_A\, a_s \, + \, \frac{8}{3}\, (11 C_A^2 + 2 C_A n_f - 4 C_F n_f) a_s^2 \ln \frac{1}{x} \Big\} \, \frac{2}{z}\, J_1(z)$$

$$+ \Big\{ \frac{4}{9}\, (26\, C_F n_f - 23\, C_A n_f)\, a_s^2 + \frac{8}{9 C_A}\, (11\, C_A^2 + 2\, C_A n_f - 4\, C_F n_f)^2 a_s^3 \ln^2 \frac{1}{x} \Big\} \, \frac{2}{z}\, J_1(z)$$

$$+ \frac{32}{9 C_A} \Big([134 - 72\, \zeta_2]\, C_A^4 + 23\, C_A^3 n_f - 48\, C_A^2 C_F n_f + 4\, C_A C_F n_f^2 - 8\, C_F^2 n_f^2 \Big)\, a_s^3 \ln^2 \frac{1}{x}\, \frac{4}{z^2}\, J_2(z)$$

$$x P_{gq}^{T}(N) - \frac{C_F}{C_A}\, x P_{gg}^{T}\Big|_{\rm NLL} = \, - \, \frac{32}{3}\, \frac{C_F}{C_A}\, (C_A^2 + C_A n_f - 2\, C_F n_f)\, a_s^2 \ln \frac{1}{x} \, \frac{4}{z^2}\, J_2(z)$$

Single-logarithmic enhancement of the oscillations at extremely small x

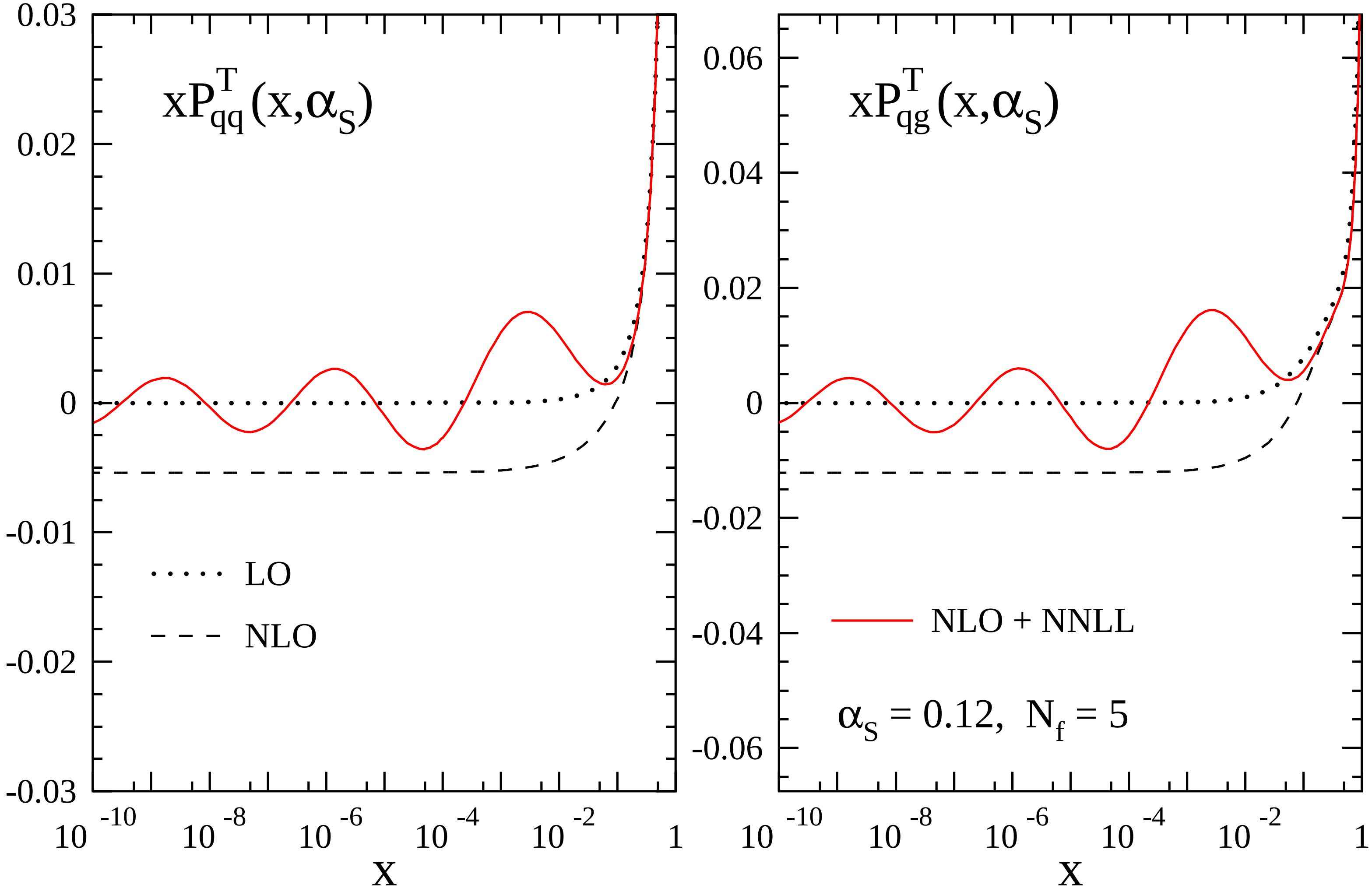
$x \rightarrow 0$ dominant $a_s (a_s \ln \frac{1}{x})^\ell \, \frac{2}{z}\, J_1(z)$ terms $\propto \, (11 C_A^2 \, - \, 2 C_A n_f \, + \, 4 C_F n_f)^\ell$

(non-C_F known to $\ell = 4$, see below). Possibility of a ‘second resummation’?

For the basic logarithm $\mathcal{L}$, unlike $A_{\rm qi}(\xi)$, we found a simple Mellin inverse

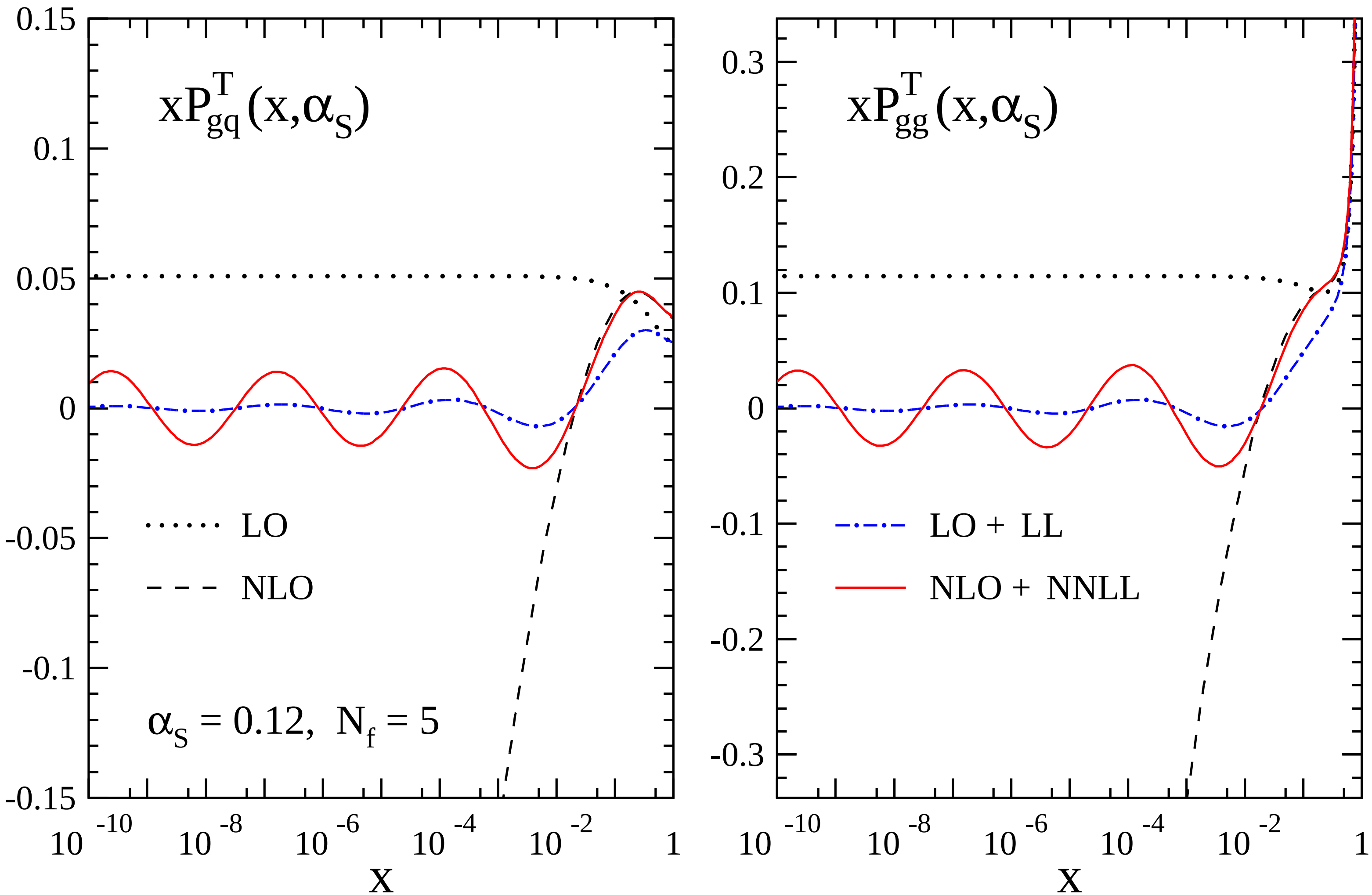
$$\int_0^1 dx\, x^{\,N-2}\, \frac{1}{\ln x}\, \left(J_0(2\sqrt{a} \ln x) - 1 \right) \; = \; \ln \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4a}{(N-1)^2}} \,\right)$$

Resummed quark-parton splitting functions



Small-x P^T_{ig} vs. P^T_{iq}: approximate 'Casimir scaling' by factor $C_A/C_F = 9/4$

Resummed gluon-parton splitting functions



LL contributions numerically small; expect large corrections beyond NNLL

Towards higher logarithmic accuracy

$N = 1$ finite NNLO $+$ small-x resummed evolution requires N^{4}LL accuracy

Relation between SIA ($\sigma = 1$, P^T) and DIS ($\sigma = -1$, P^S) parton evolution

$$\frac{\partial}{\partial \ln Q^2} f_\sigma(x, Q^2) \;=\; \left[P_u\left(\alpha_s(Q^2)\right) \otimes f_\sigma(z^\sigma Q^2)\right](x)$$

with P_u independent of σ — Dokshitzer, Marchesini, Salam (2005)

Non-singlet relation ($n_f = 0$ for P_{gg}), but found to hold for all non-C_F terms

α_s^3: Moch, A.V. (07)

In N-space :

$$\partial_{\ln Q^2} f_\sigma(N, Q^2) \;=\; P_\sigma(N)\, f_\sigma(N, Q^2) \;=\; P_u(N + \sigma\partial_{\ln Q^2})\, f_\sigma(N, Q^2)$$

$$\Rightarrow \quad P_\sigma(N) \;=\; P_u(N) + \sum_{n=1}^{\infty} \frac{\sigma^n}{n!} \frac{\partial^{n-1}}{\partial N^{n-1}} \left(\frac{\partial P_u}{\partial N} \left[P_u(N)\right]^n \right)$$

Difference $\delta P_{gg} = P_{gg}^T - P_{gg}^S$ given by lower-order quantities at any order

P_{gg}^S single-log enhanced (BFKL) $\rightarrow$ resummation of non-C_F double logs in P_{gg}^T

To NNLL as above, plus N^{3}LL and (mod. one BFKL coeff.) N^{4}LL corrections

Large-x summary and outlook

- Non-singlet physical kernels for nine observables in DIS, SIA and DY:
  Single-log behaviour $\Rightarrow$ leading three (DY: two) logs of higher-order C_a
- Singlet kernels for (F_2, F_ϕ) and (F_2, F_L) in DIS also single-logarithmic
  $\Rightarrow$ Prediction of three logs in N^{3}LO α_s^4 splitting and F_L coefficient fct's
- Iterative structure of (next-to) leading-log N^{-1} amplitudes for $C_{2,g/\phi,q}$
  $\Rightarrow$ All-order (N)LL off-diagonal splitting functions and coefficient fct's
- D-dimensional structure of unfactorized DIS$/$SIA structure functions
  Verification, extension of above results to N^{3}LL or N^{2}LL for N^{-1} terms

- Complementary: Grunberg; Laenen, Gardi, Magnea, Stavenga, White
- Applications, now: assess relevance of NS $1/N$ terms, large-x DIS fits
- Near$/$mid future: combine with other results, esp. fixed-N calculations
  (close to) feasible now: 4-loop sum rules Baikov, Chetyrkin, Kühn (10)
- Beyond-LL extension to Drell-Yan, Higgs prod'n needs more insights

Small-x summary and outlook

- D-dimensional structure of unfactorized SIA$/$DIS structure functions
  $\Rightarrow$ NNLL small-x resummation of timelike splitting & coefficient fct's
  Required for using NNLO results in SIA below $x \approx 10^{-2} \ldots 10^{-3}$

- Analogous results for (singlet case: subdominant) $x^0 \ln^\ell x$ terms in DIS
  Formally similar, numerically very different: diff. sign in roots, $(1 - \ldots)^r$

- Unlike large-x case: no direct generalization to all (higher) a in $x^a \ln^\ell x$
  But works for higher even a in SIA – DIS case not checked yet

- Does not work for the odd-N quantities F_3 and g_1 in DIS, F_A in SIA
  E.g., leading logs with group factor $d_{abc}d^{abc}$ at third order in F_3 and F_A
  cf. Dokshitzer, Marchesini (2007)

All large-x and many, but not all, small-x double logarithms in SIA and DIS appear to be 'inherited' from lower-order results.

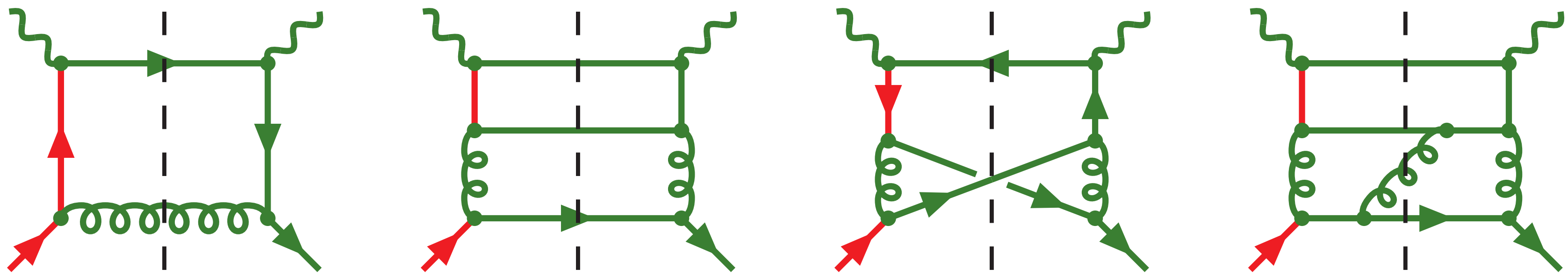
Reserve slides

Flavour singlet – non-singlet decomposition

Quark-quark splitting functions:

$$P_{\mathrm{q}_i\mathrm{q}_k} = P_{\bar{\mathrm{q}}_i\bar{\mathrm{q}}_k} = \delta_{ik} P^{\mathrm{v}}_{\mathrm{qq}} + P^{\mathrm{s}}_{\mathrm{qq}}$$

$$P_{\mathrm{q}_i\bar{\mathrm{q}}_k} = P_{\bar{\mathrm{q}}_i\mathrm{q}_k} = \delta_{ik} P^{\mathrm{v}}_{\mathrm{q}\bar{\mathrm{q}}} + P^{\mathrm{s}}_{\mathrm{q}\bar{\mathrm{q}}}$$



$P^{\mathrm{v}}_{\mathrm{qq}} = \mathcal{O}(\alpha_{\mathrm{s}})$ $\qquad$ $P^{\mathrm{s}}_{\mathrm{qq}},\, P^{\mathrm{s}}_{\mathrm{q}\bar{\mathrm{q}}} \;:\; \alpha_{\mathrm{s}}^2$ $\qquad$ $P^{\mathrm{v}}_{\mathrm{q}\bar{\mathrm{q}}} \;:\; \alpha_{\mathrm{s}}^2$ $\qquad$ $P^{\mathrm{s}}_{\mathrm{q}\bar{\mathrm{q}}} \neq P^{\mathrm{s}}_{\mathrm{qq}} \;:\; \alpha_{\mathrm{s}}^3$

Three types of difference (non-singlet) combinations: $P^{\pm}_{\mathrm{ns}} = P^{\mathrm{v}}_{\mathrm{qq}} \pm P^{\mathrm{v}}_{\mathrm{q}\bar{\mathrm{q}}}\,,\; P^{\mathrm{v}}_{\mathrm{ns}}$

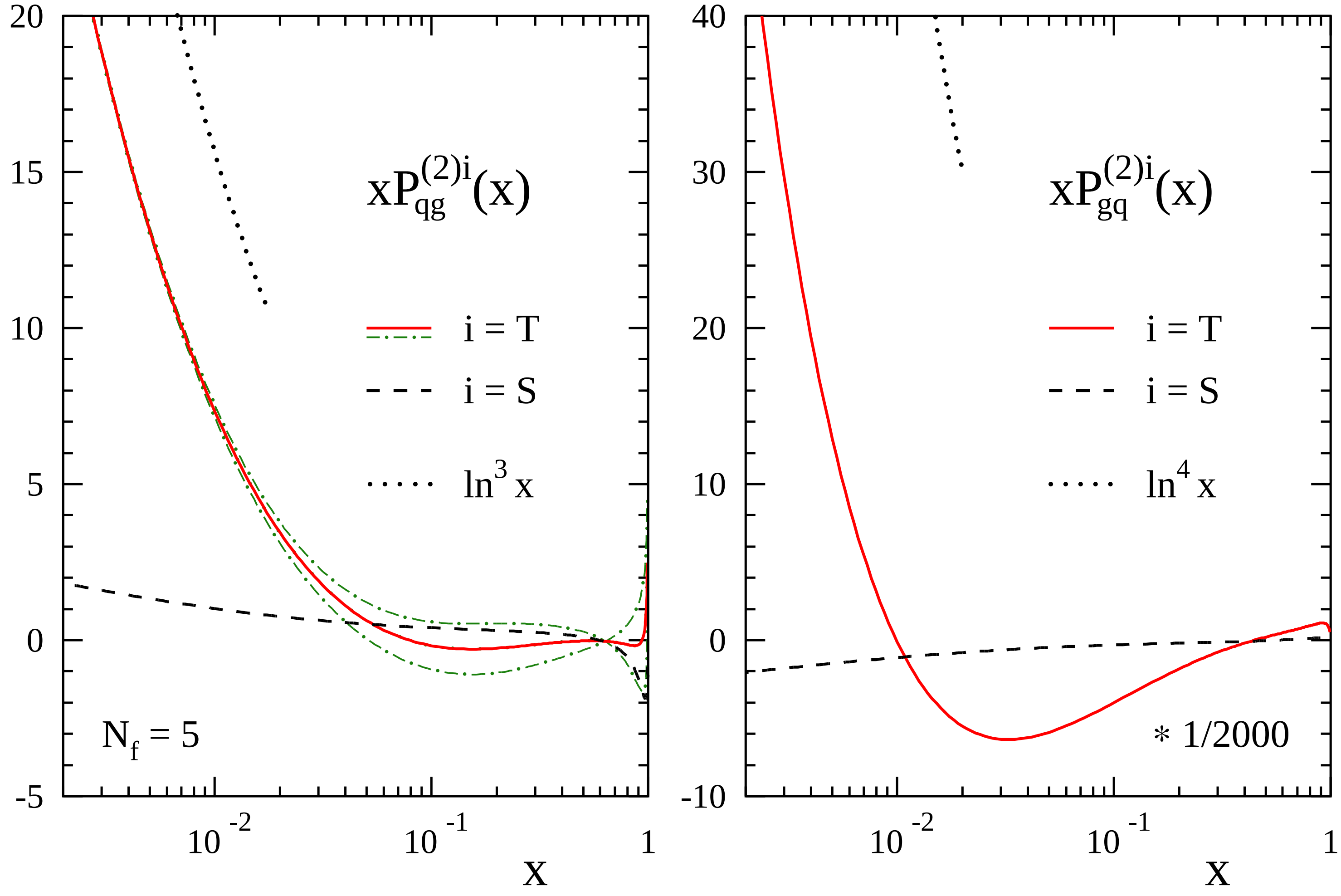
Evolution of gluon and flavour-singlet quark distributions g and q_{s}

$$q_{\mathrm{s}} = \sum_{r=1}^{n_f} (q_r + \bar{q}_r)\;, \qquad \frac{d}{d\ln\mu^2}\begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix} = \begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix} \otimes \begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix}$$

with (ps = ‘pure singlet’) $\qquad P_{\mathrm{qq}} = P^{+}_{\mathrm{ns}} + n_f (P^{\mathrm{s}}_{\mathrm{qq}} + P^{\mathrm{s}}_{\bar{\mathrm{q}}\mathrm{q}}) \equiv P^{+}_{\mathrm{ns}} + P_{\mathrm{ps}}$

Quark coefficient fct’s: analogous decomposition $C_{a,\mathrm{q}\{\bar{\mathrm{q}}\}} = C_{a,\mathrm{ns}} + C_{a,\mathrm{ps}}$

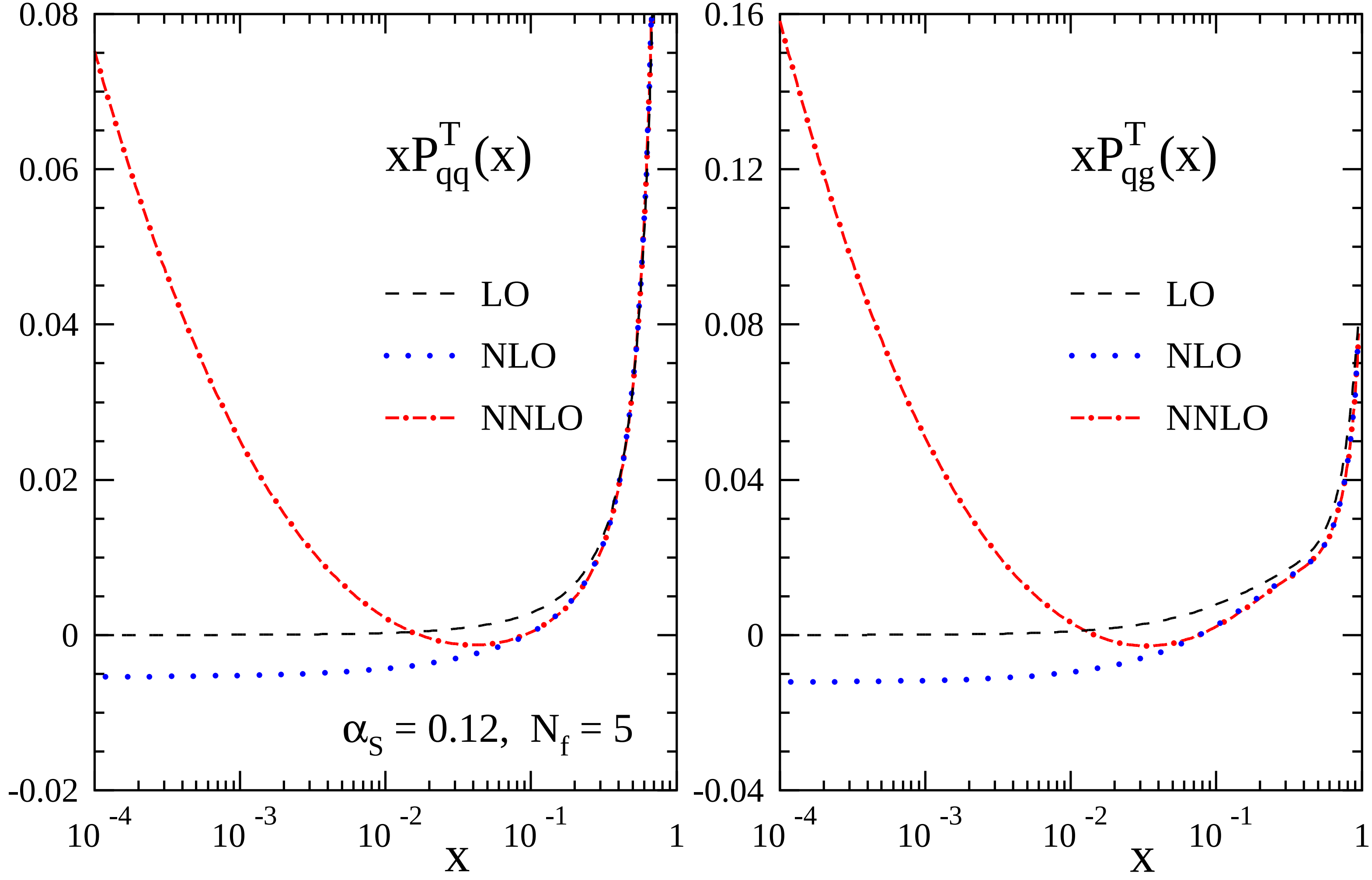
Third-order off-diagonal splitting functions



$q \to g$: not entirely fixed by Crewther-like ST-relation, $N = 2$, SUSY limit

Dash-dotted: $\delta P_{\rm qg}^{(2)T}(x) = \pm\, 2\zeta_2 \beta_0\, (C_A - C_F)\, (11 + 24\ln x)\, P_{\rm qg}^{(0)T}(x)$

NNLO approximations for $P^T_{qi}(x, \alpha_s)$



NLO: no $x^{-1} \ln x$ terms. NNLO: up to $x^{-1} \ln^3 x$. Unstable at $x \lesssim 0.02$

Singlet physical evolution kernel for (F_2, F_ϕ)

F_ϕ: Higgs-exchange DIS in heavy-top limit, to order α_s^2 also by

Daleo, Gehrmann-De Ridder, Gehrmann, Luisoni (09)

As in the non-singlet case above, but with 2-vectors$/2\times2$ matrices P_{ij} and

$$F = \begin{pmatrix} F_2 \\ F_\phi \end{pmatrix}, \quad C = \begin{pmatrix} C_{2,q} & C_{2,g} \\ C_{\phi,q} & C_{\phi,g} \end{pmatrix}, \quad K = \begin{pmatrix} K_{22} & K_{2\phi} \\ K_{\phi 2} & K_{\phi\phi} \end{pmatrix}$$

Furmanski, Petronzio (81); . . .

$$\frac{dF}{d\ln Q^2} = \frac{dC}{d\ln Q^2}\,q + CP\,q = \left(\beta(a_s)\,\frac{dC}{da_s} + CP\right)C^{-1}F$$
$$= \Big(\underbrace{\beta(a_s)\,\frac{d\ln C}{da_s}}_{\text{DL (ns + ps)}} + \underbrace{[C,P]\,C^{-1} + P}_{\text{DL (singlet only)}}\Big)F = KF$$

Observation at NLO, NNLO: single-log enhancement to all powers of $1-x$

$$K_{ab}^{(n)} \sim \ln^n(1-x) + \ \ldots\ , \quad \text{leading } K_{22/\phi\phi}^{(n)} \text{ same as NS}/C_F = 0$$

Conjecture: this behaviour persists to N^{3}LO

$\Rightarrow$ prediction of $\ln^{6,5,4}(1-x)$ of $P_{qg,gq}^{(3)}$ [and $\ln^{5,4,3}(1-x)$ of $P^{(3)}_{ps,gg|_{C_F}}$]

Example: α_s^4 splitting function $P^{(3)}_{qg}(x)$

For brevity: only $(1-x)^0$ part shown – known to all powers, $C_{AF} \equiv C_A - C_F$

$$
\begin{aligned}
P^{(3)}_{qg}(x) \quad = \quad & \ln^6(1-x) \cdot 0 \\
& + \ln^5(1-x)\left[\frac{22}{27}\,C_{AF}^3\,n_f - \frac{14}{27}\,C_{AF}^2\,C_F\,n_f + \frac{4}{27}\,C_{AF}^2\,n_f^2\right] \\
& + \ln^4(1-x)\left[\left(\frac{293}{27} - \frac{80}{9}\,\zeta_2\right)C_{AF}^3\,n_f + \left(\frac{4477}{16} - 8\zeta_2\right)C_{AF}^2\,C_F\,n_f \right. \\
& \quad \left. - \frac{13}{81}\,C_{AF}\,C_F^2\,n_f - \frac{116}{81}\,C_{AF}^2\,n_f^2 + \frac{17}{81}\,C_{AF}\,C_F\,n_f^2 - \frac{4}{81}\,C_{AF}\,n_f^3\right] \\
& + \mathcal{O}\left(\ln^3(1-x)\right)
\end{aligned}
$$

- Vanishing of the coefficient of the leading term at order α_s^4 :
  accidental (??) cancellation of contributions, for all four splitting fct's
- Remaining terms vanish in the supersymmetric case $C_A = C_F\,(= n_f)$
  Nontrivial check: same as for $P^{(2)}_{qg}$, not obvious from above construction

Singlet physical evolution kernel for (F_2, F_L)

As above, but with $F_\phi \to \widehat{F}_L = F_L / a_s c^{(0)}_{L,q}$, hence $\widehat{c}^{(n)}_{L,q/g} \sim \{1/\frac{1}{N}\} \ln^{2n} N$

$$F = \begin{pmatrix} F_2 \\ \widehat{F}_L \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 1 & \widehat{c}^{(0)}_{L,g} \end{pmatrix} + \sum_{n=1} a_s^n \begin{pmatrix} c^{(n)}_{2,q} & c^{(n)}_{2,g} \\ \widehat{c}^{(n)}_{L,q} & \widehat{c}^{(n)}_{L,g} \end{pmatrix}, \quad K = \begin{pmatrix} K_{22} & K_{2L} \\ K_{L2} & K_{LL} \end{pmatrix}$$

Catani (96), Blümlein, Ravindran, van Neerven (00) [different normalization]

Observation: single-log enhancement of N^0 part of K at NLO and NNLO

N^{3}LO conjecture + above $P^{(3)}_{qg}$: prediction of three double logs in $c^{(3)}_{L,q/g}$, e.g.

$$\begin{aligned} N^2 c^{(3)}_{L,g}(N) \quad = \quad & \ln^6 N \, \frac{32}{3} C_A^3 n_f \\ & + \ln^5 N \left[\frac{1504}{9} C_A^3 n_f - \frac{64}{9} C_A^2 n_f^2 - \frac{104}{3} C_A^2 n_f C_F - \frac{40}{3} n_f C_F^2 \right] \\ & + \ln^4 N \left[\text{known coefficients} \right] + \mathcal{O}\left(\ln^3 N\right) \end{aligned}$$

Agrees with/extends results [NS-like $C_F = 0$ part of $C_{L,g}$ only] of MV (09)

Reminder: soft limits of $q\bar{q} \to \gamma^*$, $gg \to H$

a_s^n expansion coefficients of bare partonic cross sections to $n = 3$

cf. Matsuura, van Neerven (88)

$$W_0^{\rm b} = \delta(1-x)$$

$$W_1^{\rm b} = 2\,\text{Re}\,\mathcal{F}_1\,\delta(1-x) + \mathcal{S}_1$$

$$W_2^{\rm b} = (2\,\text{Re}\,\mathcal{F}_2 + |\mathcal{F}_1|^2)\delta(1-x) + 2\,\text{Re}\,\mathcal{F}_1\mathcal{S}_1 + \mathcal{S}_2$$

$$W_3^{\rm b} = (2\,\text{Re}\,\mathcal{F}_3 + 2\,|\mathcal{F}_1\mathcal{F}_2|)\delta(1-x) + (2\,\text{Re}\,\mathcal{F}_2 + |\mathcal{F}_1|^2)\mathcal{S}_1 + 2\,\text{Re}\,\mathcal{F}_1\mathcal{S}_2 + \mathcal{S}_3$$

$\mathcal{F}_\ell$: bare ℓ-loop time-like q or g form factor, $\mathcal{S}_\ell$ includes soft real emissions

$$\mathcal{S}_k = \mathsf{S}_k(\varepsilon)\cdot\varepsilon[\,(1-x)^{-1-2k\varepsilon}\,]_+ = \mathsf{S}_k(\varepsilon)\Big[-\frac{1}{2k}\,\delta(1-x) + \sum_{i=0}\frac{(-2k\varepsilon)^i}{i\,!}\,\varepsilon\,\mathcal{D}_i\Big]$$

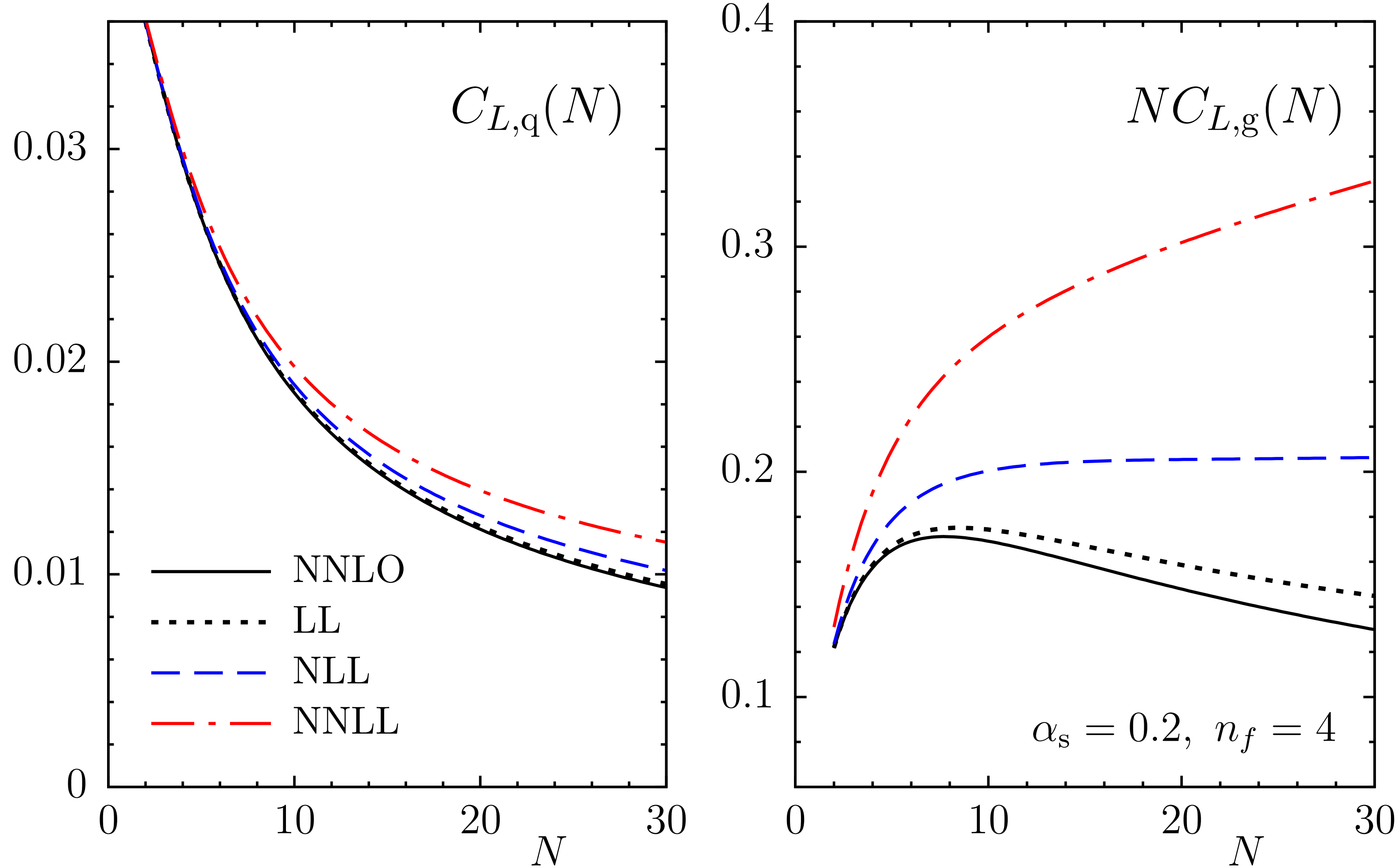
Poles in $\varepsilon = 2 - D/2$: KLN, renormalization, mass factorization

$1/\varepsilon$ pieces of $\mathcal{F}_n$ + n-loop splitting functions $\longrightarrow$ $1/\varepsilon$ coefficients of S_n

$\longrightarrow$ $\mathcal{D}_{2n,\ldots,0}$ terms of coefficient fct's c_n $\longrightarrow$ N^{n}LL resummation coeff's D_n

$n = 3$: Moch, A.V. (2005)

Numerical illustration of $C_{L,\mathrm{q}}$ and $C_{L,\mathrm{g}}$



Corrections smaller and convergence with order n faster in quark case(s)

$\simeq$ 15% NNLL correction at $N = 20$ for $C_{L,\mathrm{q}}$ vs. 100% for $C_{L,\mathrm{g}}$ ($\approx$ Padé)

Results for $\ln^\ell x$ contributions in DIS

Analogous to SIA: highest three $x^0 \ln^\ell x$ double logarithms (to order α_s^{16}) derived for non-singlet$^+$ and flavour-singlet splitting & coefficient functions

Splitting functions $P_{\mathrm{NS}+}^{(n\geq 1)}$: all-order expressions for coefficients to NNLL

LL: $\quad 2^{n+1} C(n)\, C_F^{n+1}$ — as in Blümlein, A.V. (95)

NLL: $\quad 2^{n} C(n)\, (n+1)\, C_F^{n} (C_F - \frac{1}{2}\,\beta_0)$

NNLL:
$$2^{n} C(n-1) \Big\{ [n(n+1) - 4 - \zeta_2(48n - 44)]\, C_F^{n+1} + \Big[\frac{10}{3} n + 48\zeta_2 (n-1)\Big]\, C_F^{n} C_A + n \Big(n - \frac{14}{3}\Big)\, C_F^{n} \beta_0 - 15\zeta_2\, (n-1)\, C_F^{n-1} C_A^2 + \frac{1}{4}\, n(n-1)\, C_F^{n-1} \beta_0^2 \Big\}$$

in terms of the Catalan numbers $C(n) = (2n)!\, [\, n!\, (n+1)!\,]^{-1}$

Also other quantities now in closed all-n form beyond leading logarithms

NNLL expressions alone insufficient for stable results – details another time

Combine with future fixed Mellin-N fourth-order calculations, . . .