The three-loop four-point correlator and single-valued polylogarithms

Claude Duhr
University of Edinburgh, 06/02/2013
based on work in collaboration with James Drummond, Burkhard Eden, Paul Heslop, Jeffrey Pennington and Volodya Smirnov

Motivation

- Question: Why a talk about the three-loop four-point correlator in planar N=4 Super Yang-Mills in a phenomenology group..?

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- Answer: Because we have to start somewhere...
- Planar N=4 SYM is a very 'clean' environment to study scattering amplitudes and Feynman integrals.
  - ➡ Ideal playground to investigate new ideas.
- In the last couple of years, it has become more and more clear that there is a deep connection between scattering amplitudes and modern pure mathematics.
- 'Holy grail': Get results for Feynman integrals without having to go through the pain of computing complicated integrals!

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- Multi-loop computations are generically considered to be extremely complicated.
  - ➡ Integrals are divergent (UV and IR).
  - ➡ Complicated analytical structures:

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- The 4-point correlator in N=4 SYM is an ideal test ground for new ideas.
  - ➡ N=4 SYM is UV finite.
  - ➡ Correlator is IR finite (4-point off-shell function).
  - ➡ Functions are highly constraint.

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Highly constraint!
(e.g., max. transcendentality)

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- Aim of this talk: A first example where we succeeded in computing complicated 3 and 4 loop integrals solely based on

  ➡ Symmetries of the integral.

  ➡ Unitarity (Cutkosky rules) & algebraic geometry.

  ➡ Number theory and modern algebra.

  ➡ Asymptotic expansions (~ boundary condition).

- While the example we discuss is a correlator in N=4 SYM, the mathematics is generic!

Outline

- The four-point correlator in planar N=4 SYM.
- Input from algebraic geometry:
  - ➡ Leading singularities and residues.
- Input from number theory:
  - ➡ Single-valued polylogarithms.
- The three-loop correlator.
- Going beyond three loops.

The four-point correlator in N=4 Super Yang-Mills

The correlator

- The N=4 on-shell supermultiplet:
  - ➡ the gluon (2 helicities).
  - ➡ four gluinos (2 helicities each).
  - ➡ 6 real scalars.

$$\Phi(p,\eta) = G^+(p) + \eta_I\, \tilde{g}_I^+(p) + \frac{1}{2!}\epsilon^{IJKL}\,\eta_I\,\eta_J\,\phi_{KL}(p)$$
$$+ \frac{1}{3!}\epsilon^{IJKL}\,\eta_I\,\eta_J\,\eta_K\,\tilde{g}_K^-(p) + \frac{1}{4!}\epsilon^{IJKL}\,\eta_I\,\eta_J\,\eta_K\,\eta_L\,G^-(p)$$

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- In the following we consider the correlator

$$\langle \mathcal{O}(x_1)\tilde{\mathcal{O}}(x_2)\mathcal{O}(x_3)\tilde{\mathcal{O}}(x_4)\rangle$$

$$\mathcal{O} = \mathrm{Tr}(\phi_{12}\phi_{12}) \qquad\qquad \tilde{\mathcal{O}} = \mathrm{Tr}(\bar{\phi}^{12}\bar{\phi}^{12})$$

The correlator

- This correlator is finite, as long as $x_{ij}^2 \equiv (x_i - x_j)^2 \neq 0$.
- N=4 SYM is conformal at the quantum level, and so the correlator can only depend on conformal cross ratios:

$$\frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

- There are 6 conformal cross ratios one can form out of 4 points, but only two are independent:

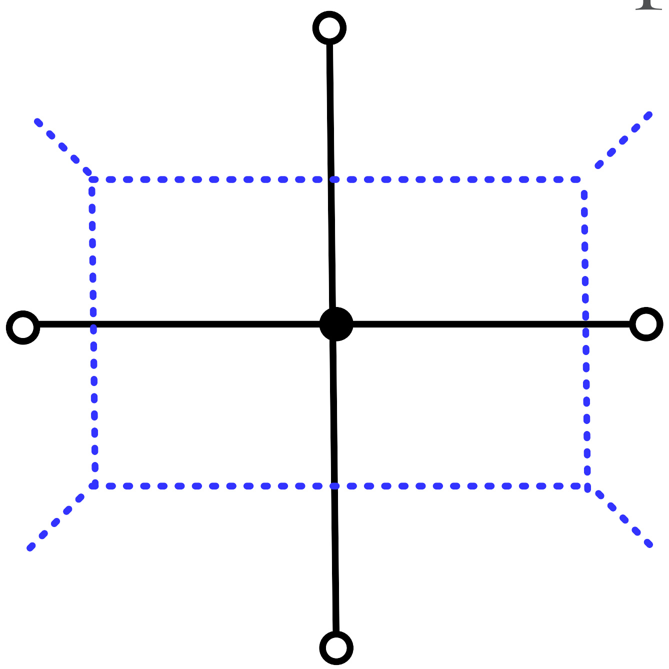
$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad\qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

- The correlator admits the perturbative expansion (normalized to tree-level)

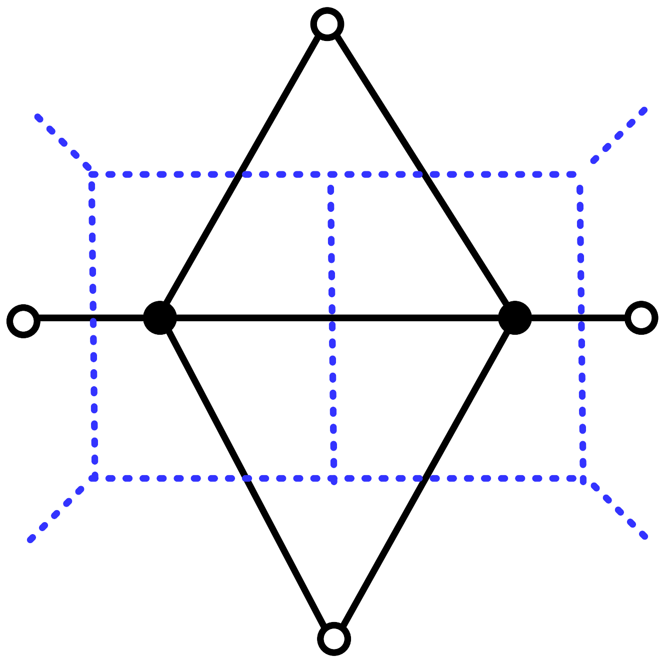
$$1 + a\, x_{13}^2 x_{24}^2\, g_1 + a^2\, x_{13}^2 x_{24}^2\, g_2 + \ldots$$

The correlator

- At one and two loop order:



$$\frac{1}{4\pi^2}\int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$



$$\frac{x_{34}^2}{(4\pi^2)^2}\int \frac{d^4x_5\, d^4x_6}{(x_{15}^2 x_{35}^2 x_{45}^2) x_{56}^2 (x_{26}^2 x_{36}^2 x_{46}^2)}$$

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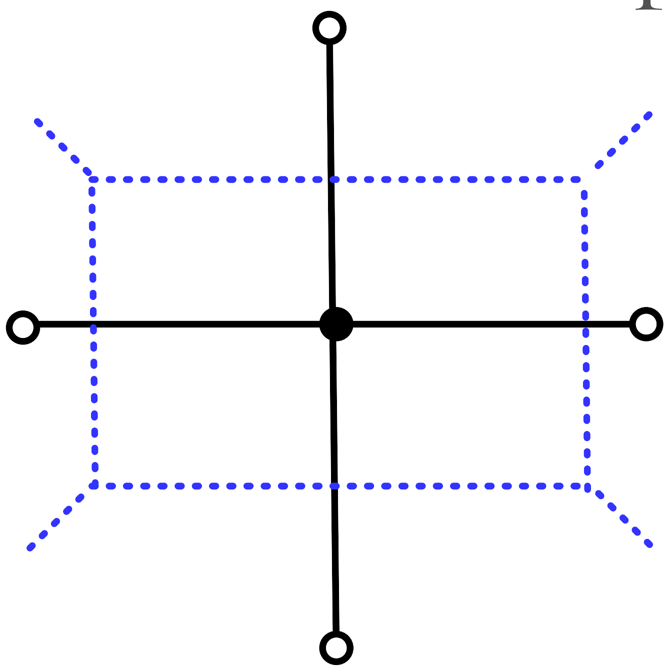
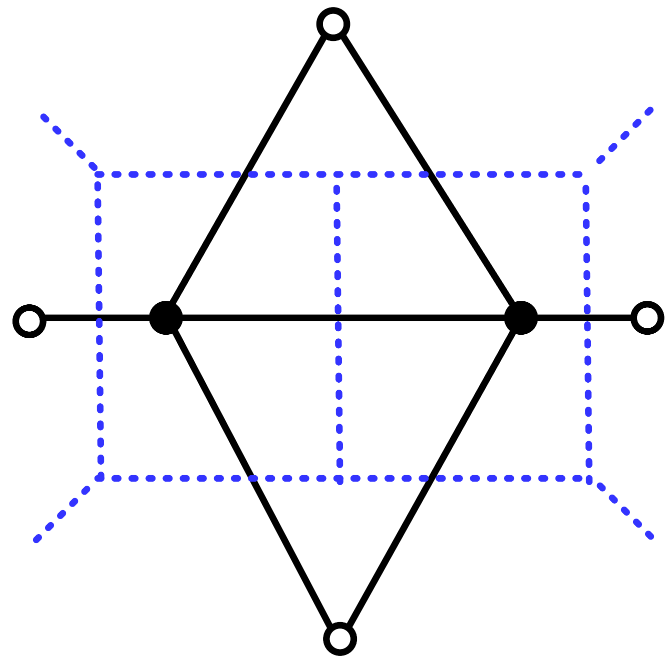




$$\frac{1}{4\pi^2}\int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \qquad \frac{x_{34}^2}{(4\pi^2)^2}\int \frac{d^4x_5\, d^4x_6}{(x_{15}^2 x_{35}^2 x_{45}^2)x_{56}^2(x_{26}^2 x_{36}^2 x_{46}^2)}$$

- These are special cases of ladder integrals:

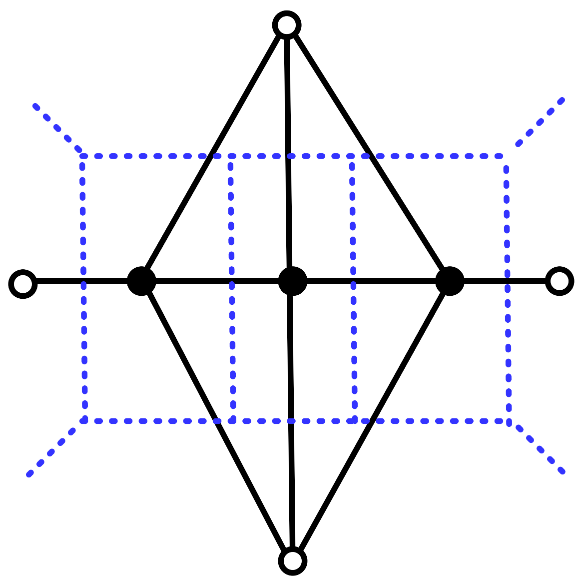
$$\Phi^{(L)}(u,v) = \frac{(-1)^{L+1}}{x-\bar{x}}\sum_{k=0}^{L}\frac{(-1)^r(2L-r)!}{r!(L-r)!L!}\log^r(x\bar{x})\,\left(\mathrm{Li}_{2L-r}(x)-\mathrm{Li}_{2L-r}(\bar{x})\right)$$

$$u = x\,\bar{x} \qquad v = (1-x)(1-\bar{x})$$

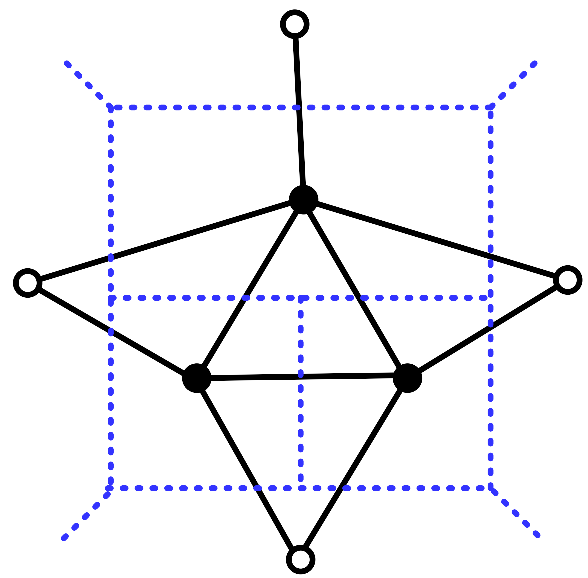
[Davydychev, Usyukina]

The correlator

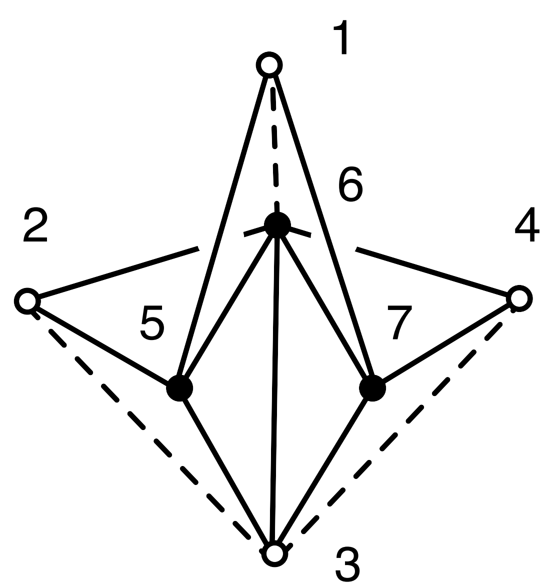
- At three loop order: [Eden, Heslop, Korchemsky, Sokatchev]



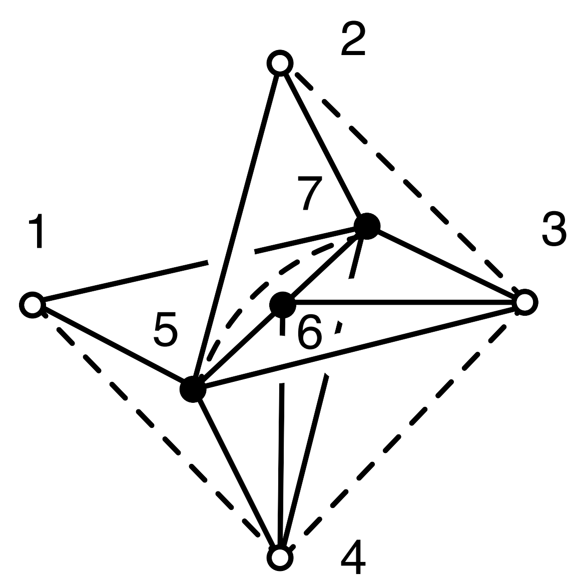
3-loop ladder



Tennis court



'Easy' integral



'Hard' integral

The correlator

- The 3-loop ladder integral is known. [Davydychev, Usyukina]
- The tennis court can be reduced to the 3-loop ladder. [Drummond, Henn, Smirnov, Sokatchev]
- The ‘easy’ and ‘hard’ integrals are unknown.
  - ➡ Integrals are too complicated...

The correlator

- The 3-loop ladder integral is known. [Davydychev, Usyukina]
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- The 'easy' and 'hard' integrals are unknown.
  - ➡ Integrals are too complicated...
  - ➡ Solution: Simply don't compute them!

$$I = \sum_i R_i(x,\bar{x})\, P_i(x,\bar{x})$$

- If we have an ansatz for the rational functions R and the polylogarithms P, then we might guess what the function is.

$$\Phi^{(L)}(u,v) = \frac{(-1)^{L+1}}{x-\bar{x}} \sum_{k=0}^{L} \frac{(-1)^r (2L-r)!}{r!(L-r)!L!} \log^r(x\bar{x}) \left(\mathrm{Li}_{2L-r}(x) - \mathrm{Li}_{2L-r}(\bar{x})\right)$$

Input from algebraic geometry

Leading singularities and residues

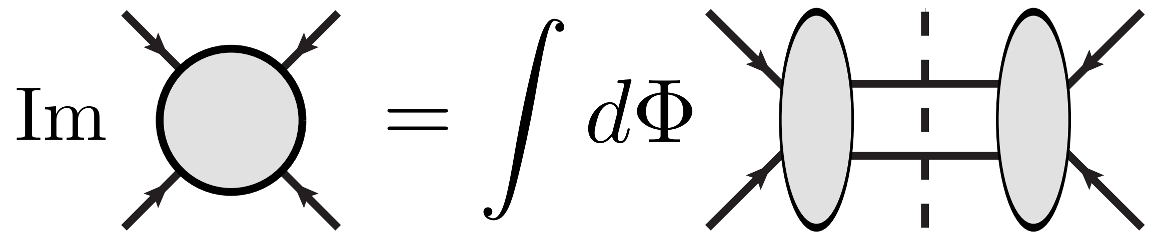
Leading singularities

$$I = \sum_i R_i(x,\bar{x})\, P_i(x,\bar{x})$$

- One of the main differences between the rational coefficients R and the polylogarithmic terms P:
  - ➡ R is meromorphic.
  - ➡ P has discontinuities.
- In other words: if we take 'enough' discontinuities, there is nothing left of the polylogarithmic part P!
  - ➡ Project out the rational coefficients R.
- In terms of Feynman diagrams: the rational coefficients are the leading singularities of the integral!

Unitarity and discontinuities

- Discontinuities of Feynman integrals are given by unitarity:



- Cutkosky rules: discontinuities arise from propagators going on shell:

$$\mathrm{Disc}\,\frac{1}{q^2} = 2\pi i\,\delta_+(q^2)$$

- Leading singularities (LS): all propagators are on shell.
    - ➡ This must be an algebraic function, because there is no discontinuity left!

- More correct way of thinking about it: LS are residues of Feynman integrals.

[Cachazo; Skinner; Spradlin, Volovich]

Multi-dimensional residues

- Consider the integral
$$\int \frac{d^n x}{P_1(x)\dots P_n(x)}$$
where the $P_i(x)$ are polynomials.
- Let x_0 be the simultaneous zero of all the polynomials.
- The residue at x_0 can be computed by changing variables to
$$p_i = P_i(x)$$
and the residue is defined by
$$\mathrm{Res}_{x_0} \int \frac{d^n p}{p_1 \dots p_n\, J} = \frac{1}{J}_{|p=0}$$
where J is the jacobian of the change of variables,
$$J = \det \frac{\partial P_i}{\partial x_j}$$

Example: 4-mass box

- Consider the integral

$$\int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

- We change variables to $p_i = x_{i5}^2$. The jacobian is

$$J = \det\left(\frac{\partial p_i}{\partial x_5^\mu}\right) = \det\left(-2x_{i5}^\mu\right)$$

$$J^2 = \det\left(4x_{i5} \cdot x_{j5}\right) = 16 \det\left(x_{ij}^2 - x_{i5}^2 - x_{j5}^2\right)$$

- After some algebra, we find that the leading singularity is

$$\frac{1}{x_{13}^2 x_{24}^2 \sqrt{\lambda(1,u,v)}} = \frac{1}{x_{13}^2 x_{24}^2 (x-\bar{x})} \qquad \begin{array}{l} u = x\,\bar{x} \\ v = (1-x)(1-\bar{x}) \end{array}$$

$$\Phi^{(L)}(u,v) = \frac{(-1)^{L+1}}{x-\bar{x}} \sum_{k=0}^{L} \frac{(-1)^r (2L-r)!}{r!(L-r)!L!} \log^r(x\bar{x})\, \left(\mathrm{Li}_{2L-r}(x) - \mathrm{Li}_{2L-r}(\bar{x})\right)$$

Residues of the correlator integrals

- Leading singularities of the ladder integrals:

$$\frac{1}{x-\bar{x}}$$

- Leading singularities of the 'easy' integral:

$$\frac{1}{x-\bar{x}} \qquad \frac{1}{(1-u)(x-\bar{x})} \qquad \frac{u}{(1-u)(x-\bar{x})}$$

- Leading singularities of the 'hard' integral:

$$\frac{1}{(x-\bar{x})^2} \qquad \frac{1}{(u-v)(x-\bar{x})}$$

- Conjecture: These are the only rational prefactors of these integrals!

Input from number theory

Single-valued polylogarithms

Multiple polylogarithms

- Feynman integrals can often be expressed in terms of polylogarithms:

$$\log z = \int_1^z \frac{dt}{t} \qquad \mathrm{Li}_n(z) = \int_0^z \frac{dt}{t}\mathrm{Li}_{n-1}(t)$$

- For multi-scale integrals also multiple polylogarithms appear:

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{\mathrm{d}t}{t-a_1}\,G(a_2,\ldots,a_n;t)$$

- Complication: polylogarithms satisfy complicated relations among themselves.

$$-\mathrm{Li}_2(z) - \ln z \ln(1-z) = \mathrm{Li}_2(1-z) - \frac{\pi^2}{6}$$

Multiple polylogarithms

$$I = \sum_i R_i(x,\bar{x})\, P_i(x,\bar{x})$$

- We know all the rational coefficients R.
- Ideally: Write down an ansatz of independent polylogarithms (with rational numbers as coefficients), and determine their coefficients by matching to some asymptotic expansion.

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- Ideally: Write down an ansatz of independent polylogarithms (with rational numbers as coefficients), and determine their coefficients by matching to some asymptotic expansion.
- However:
  - ➡ Which polylogarithms? with which arguments?
  - ➡ What is a 'basis' for polylogarithms?

$$-\mathrm{Li}_2(z) - \ln z \ln(1-z) = \mathrm{Li}_2(1-z) - \frac{\pi^2}{6}$$

Number theory meets QFT

- Polylogarithms and their generalizations have been studied by Euler, Nielsen, Poincaré,...
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Number theory meets QFT

- Polylogarithms and their generalizations have been studied by Euler, Nielsen, Poincaré,...
  - ➡ 'Mathematics of the 19th century'.
- No! Very active field of research in pure mathematics in the last 20 years.
- Mathematics of polylogarithms governed by powerful algebraic structures (Hopf algebra).
- Hopf algebra controls, conjecturally, all the relations among polylogarithms.

Symbols

[Goncharov, Spradlin, Vergu, Volovich]

- Consider an iterated integral

$$F_k = \int F_{k-1}\, d\log R$$

- If its total differential satisfies

$$dF_k = \sum_i F_{k-1,i}\, d\log R_i$$

then we define the symbol of F by

$$\mathcal{S}(F_k) = \sum_i \mathcal{S}(F_{k-1,i}) \otimes d\log R_i$$

- Example: $d\mathrm{Li}_n(z) = \mathrm{Li}_{n-1}(z)\, d\log z$

$$\mathcal{S}(\mathrm{Li}_n(z)) = \mathcal{S}(\mathrm{Li}_{n-1}(z)) \otimes z = -(1-z) \otimes \underbrace{z \otimes \ldots \otimes z}_{n-1}$$

Symbols

- In general:

$$dG(a_{n-1},\ldots,a_1;a_n)=\sum_{i=1}^{n-1}G(a_{n-1},\ldots,a_{i-1},a_{i+1},\ldots,a_1;a_n)\,d\ln\left(\frac{a_i-a_{i+1}}{a_i-a_{i-1}}\right)$$

$$\mathcal{S}(G(a_{n-1},\ldots,a_1;a_n))=\sum_{i=1}^{n-1}\mathcal{S}(G(a_{n-1},\ldots,a_{i-1},a_{i+1},\ldots,a_1;a_n))\otimes\left(\frac{a_i-a_{i+1}}{a_i-a_{i-1}}\right)$$

- Properties:

$$\ldots\otimes(a\cdot b)\otimes\ldots=\ldots\otimes a\otimes\ldots+\ldots\otimes b\otimes\ldots$$

$$\ldots\otimes(\pm 1)\otimes\ldots=0$$

$$\mathcal{S}(\zeta_n)=0$$

- Concequence: Complicated identities among polylogarithms become symbol algebraic identities among symbols.

Symbols of ladder integrals

$$\Phi^{(L)}(u,v) = \frac{(-1)^{L+1}}{x-\bar{x}} \sum_{k=0}^{L} \frac{(-1)^r (2L-r)!}{r!(L-r)!L!} \log^r(x\bar{x}) \left(\mathrm{Li}_{2L-r}(x) - \mathrm{Li}_{2L-r}(\bar{x})\right)$$

$$\mathcal{S}(\mathrm{Li}_n(z)) = -(1-z) \otimes \underbrace{z \otimes \ldots \otimes z}_{n-1} \qquad \mathcal{S}(\log z) = z$$

- The symbols of ladder integrals have all their entries drawn from $\{x, 1-x, \bar{x}, 1-\bar{x}\}$.

- Idea: To find a basis, work with the tensors!
  - ➡ Pure linear algebra.
  - ➡ All identities are resolved.

Integrability condition

- Is every tensor the symbol of a function?

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- No! The tensor

$$\sum_{i_1,\dots,i_n} c_{i_1\dots i_n} \omega_{i_1} \otimes \dots \otimes \omega_{i_n}$$

is the symbol of a function if and only if the following integrability condition is fulfilled

$$\sum_{i_1,\dots,i_n} c_{i_1\dots i_n}\, d\log\omega_{i_k} \wedge d\log\omega_{i_{k+1}}\, \omega_{i_1} \otimes \dots \otimes \omega_{i_{k-1}} \otimes \omega_{i_{k+2}} \otimes \dots \otimes \omega_{i_n} = 0$$

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- This is a very strong constraint!
- In other words, we only need to work with a subspace of all tensor.
- But this space is still too large for Feynman integrals...

Discontinuities

- The symbol encodes the discontinuities of a function in its first entry.
- Example: If the symbol of a function F is

$$\mathcal{S}(F) = (a_1 - x) \otimes \ldots \otimes (a_n - x)$$

then F has a branch cut starting at $x = a_1$, and the discontinuity across the cut is

$$\mathcal{S}(\mathrm{Disc}_{x=a_1} F) = 2\pi i (a_2 - x) \otimes \ldots \otimes (a_n - x)$$

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- If we take a random combination of (integrable) tensors, then we get a random collection of cuts.
- But the cuts of Feynman integrals are all but random...

First entry condition

- The branch cuts of a massless Feynman integral start at points where one of the Mandelstam invariants is zero.
- As a consequence, the first entry of the symbol of a massless Feynman integral must be a Mandelstam invariant!

[Gaiotto, Maldacena, Sever, Vieira]

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  [Gaiotto, Maldacena, Sever, Vieira]
- In our case, all terms in the symbol must be of the form

$$x_{ij}^2 \otimes \ldots$$

  and conformal invariance implies that the symbols have the form

$$u \otimes S_u + v \otimes S_v$$

- But in our case we can still do better!

Single-valuedness

- We introduce the parametrization

$$u = x\,\bar{x} \qquad\qquad v = (1-x)(1-\bar{x})$$

$$u \otimes S_u + v \otimes S_v = (x\bar{x}) \otimes S_u + [(1-x)(1-\bar{x})] \otimes S_v$$

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- Let us now compute the discontinuity around x=0:

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- Conclusion: off-shell conformal four-point functions are single-valued in the complex x plane!

- Polylogarithms are highly constraint!

  ➡ only combinations where all branch cuts cancel are allowed.

Single-valuedness

$$\Phi^{(L)}(u,v) = \frac{(-1)^{L+1}}{x-\bar{x}} \sum_{k=0}^{L} \frac{(-1)^r (2L-r)!}{r!(L-r)!L!} \log^r(x\bar{x}) \left(\mathrm{Li}_{2L-r}(x) - \mathrm{Li}_{2L-r}(\bar{x})\right)$$

- These functions are indeed single-valued in the complex x plane!
- More generally, all single-valued polylogarithms whose symbols have their entries drawn from $\{x, 1-x, \bar{x}, 1-\bar{x}\}$ have been completely classified. [Brown]

  ➡ Single-valued harmonic polylogarithms.

- Infinite classes of generalized ladder integrals are known to evaluate to single-valued harmonic polylogarithms. [Drummond]

The three-loop correlator

The ‘easy’ and ‘hard’ integrals

General strategy

$$I = \sum_i R_i(x,\bar{x})\, P_i(x,\bar{x})$$

- The rational coefficients R correspond to leading singularities.
- The polylogarithms are constraint to be single-valued in the complex x plane.
  ➡ Minimal ansatz: Single-valued harmonic polylogarithms.
- Asymptotic expansions for these integrals are known in the limit where u is small. [Eden]

General strategy

$$
\begin{aligned}
E_{14;23} \;=\;& \frac{\log u}{x}\Big[-6\zeta_3 H(0,1;x) - 6\zeta_3 H(1,1;x) + H(0,1,0,1,1;x) \\
&\qquad - H(0,1,1,0,1;x) + H(1,0,0,1,1;x) + 2H(1,0,1,1,1;x) \\
&\qquad - H(1,1,0,0,1;x) - 2H(1,1,1,0,1;x)\Big] \\
-\;& \frac{2}{x}\Big[-6\zeta_3 H(0,0,1;x) + 2\zeta_3 H(0,1,1;x) - 4\zeta_3 H(1,0,1;x) \\
+\;& 4\zeta_3 H(1,1,1;x) + H(0,0,1,0,1,1;x) - H(0,0,1,1,0,1;x) \\
+\;& H(0,1,0,0,1,1;x) - H(0,1,1,0,0,1;x) + 2H(1,0,0,0,1,1;x) \\
+\;& 2H(1,0,0,1,1,1;x) + 2H(1,0,1,0,1,1;x) - 2H(1,1,0,0,0,1;x) \\
-\;& 2H(1,1,0,1,0,1;x) - 2H(1,1,1,0,0,1;x)\Big] + \mathcal{O}(u)\,,
\end{aligned}
$$

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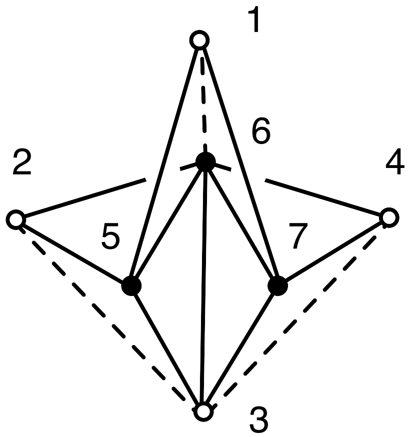
- The polylogarithms are constraint to be single-valued in the complex x plane.
  - ➡ Minimal ansatz: Single-valued harmonic polylogarithms.
- Asymptotic expansions for these integrals are known in the limit where u is small. [Eden]
- Strategy: Write an ansatz using the leading singularities and single-valued polylogarithms and match the coefficients.

‘Easy’ integral

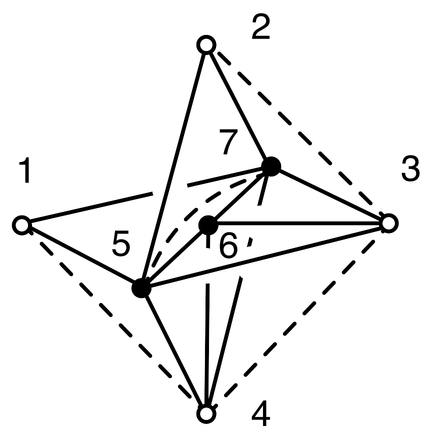


$$= \frac{1}{(1-u)(x-\bar{x})} E(x,\bar{x}) + \frac{u}{(1-u)(x-\bar{x})} E(1/x,1/\bar{x})$$

- Making an ansatz for E in terms of single valued harmonic polylogarithms we find

$$\begin{aligned} E(x,\bar{x}) &= 4\,L_{3,1,2} - 64\,L_{5,1} - 16\,L_{4,2} - 4\,L_{3,2,1} + 4\,L_0^2\,L_{3,1} - 4\,L_0^2\,L_{2,1,1} - 3\,L_1\,L_0\,L_{3,1} \\ &- 3\,L_1^2\,L_4 + 4\,L_1\,L_4\,L_0 + 2\,L_0\,L_2\,L_{2,1} + 4\,L_0\,L_2\,L_3 + 2\,L_1\,L_2\,L_3 \\ &- \frac{4}{3}\,L_1\,L_0^3\,L_2 + L_1^2\,L_0^2\,L_2 - \frac{1}{3}\,L_2^3 - 8\,\zeta_3\,L_0\,L_2 + 2\,\zeta_3\,L_1\,L_2\,. \end{aligned}$$

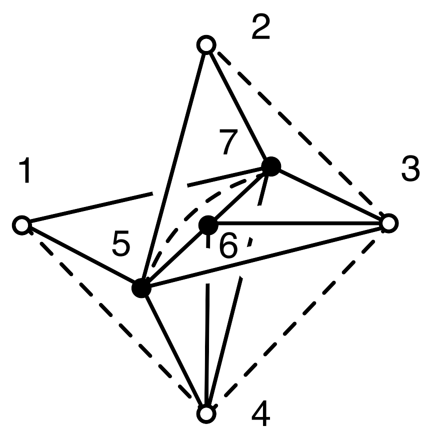
‘Hard’ integral



$$= \frac{H^{(a)}(x,\bar{x})}{(x-\bar{x})^2} + \frac{H^{(b)}(x,\bar{x})}{(v-1)(x-\bar{x})}$$

- Making an ansatz for H in terms of single valued harmonic polylogarithms we find

‘Hard’ integral



$$= \frac{H^{(a)}(x,\bar{x})}{(x-\bar{x})^2} + \frac{H^{(b)}(x,\bar{x})}{(v-1)(x-\bar{x})}$$

- Making an ansatz for H in terms of single valued harmonic polylogarithms we find

...nothing...

- The space of functions is not big enough!

- For two-loop three-point functions, it is known that other single-valued function appears, whose symbols have entries drawn from $\{x, 1-x, \bar{x}, 1-\bar{x}, x-\bar{x}\}$.

[Chavez, CD]

‘Hard’ integral

- Example:

$$
\begin{aligned}
\mathcal{Q}_3(z) =& \frac{1}{2}\left[G\left(0,\frac{1}{\bar z},\frac{1}{z},1\right)-G\left(0,\frac{1}{z},\frac{1}{\bar z},1\right)\right]+\frac{1}{4}\ln|z|^2\left[G\left(\frac{1}{z},\frac{1}{\bar z},1\right)-G\left(\frac{1}{\bar z},\frac{1}{z},1\right)\right]\\
&+\frac{1}{2}\Big[\mathrm{Li}_3(1-z)-\mathrm{Li}_3(1-\bar z)\Big]+\mathrm{Li}_3(z)-\mathrm{Li}_3(\bar z)+\frac{1}{4}\Big[\mathrm{Li}_2(z)+\mathrm{Li}_2(\bar z)\Big]\ln\frac{1-z}{1-\bar z}\\
&+\frac{1}{4}\Big[\mathrm{Li}_2(z)-\mathrm{Li}_2(\bar z)\Big]\ln|1-z|^2+\frac{1}{16}\ln\frac{z}{\bar z}\ln^2\frac{1-z}{1-\bar z}+\frac{1}{8}\ln^2|z|^2\ln\frac{1-z}{1-\bar z}\\
&+\frac{1}{4}\ln|z|^2\ln|1-z|^2\ln\frac{1-z}{1-\bar z}+\frac{1}{16}\ln^2|1-z|^2\ln\frac{z}{\bar z}-\frac{\pi^2}{12}\ln\frac{1-z}{1-\bar z}\,.
\end{aligned}
$$

- If we enlarge the space of functions to include these functions as well, we can find a solution for the hard integral!
  - ➡ Result rather long, so will not be shown here.
- But extension of the space of functions seems rather ad hoc...
  - ➡ More on this shortly!

The correlator

- Conclusion: We have now the full analytic result for the three-loop four-point correlator.
- The remaining integrals were obtained without computing any actual integral!
  - ➡ Residues of loop integrals.
  - ➡ Basis for the space of polylogarithms.
  - ➡ Asymptotic expansions.
- Were we just lucky..?
- What about the rather ad hoc extension of the space of functions..?

Going beyond three loops

A specific four-loop integral

A 4-loop integral

- To see how robust our method is, we went to the simplest non-trivial 4-loop integral

$$\int \frac{d^4x_5 d^4x_6 d^4x_7 d^4x_8 x_{14}^2 x_{24}^2 x_{34}^2}{x_{15}^2 x_{18}^2 x_{25}^2 x_{26}^2 x_{37}^2 x_{38}^2 x_{45}^2 x_{46}^2 x_{47}^2 x_{48}^2 x_{56}^2 x_{67}^2 x_{78}^2}$$

- How far do we get..?

A 4-loop integral

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- How far do we get..?
- There is only one residue (up to conjugation):

$$\frac{1}{x - \bar{x}}$$

➡ We are looking for a function of the form:

$$\frac{\phi(x, \bar{x})}{x - \bar{x}}$$

- What about the space of polylogarithms?

A 4-loop integral

- The asymptotic expansions for this integral can be computed.
- Using the same ansatz for the space of polylogarithms as for the 'easy' and 'hard' integrals we find

A 4-loop integral

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...nothing...

- So it seems hopeless...

A 4-loop integral

- The asymptotic expansions for this integral can be computed.
- Using the same ansatz for the space of polylogarithms as for the 'easy' and 'hard' integrals we find

...nothing...

- So it seems hopeless...
- But there is a differential equation for the 4-loop integral:

$$\partial_x \partial_{\bar{x}} \phi = -\frac{x - \bar{x}}{x\bar{x}} E(x, \bar{x}) = -\frac{1}{x\bar{x}(1 - x\bar{x})} \mathcal{E}(x, \bar{x})$$

➡ The leading singularity of the 'easy' integral enter as the kernel of the differential equation

A 4-loop integral

# Loops	LS	Polylogarithms
1 & 2	$\dfrac{1}{x-\bar{x}}$	$\{x, 1-x, \bar{x}, 1-\bar{x}\}$ $G\left(0,\frac{1}{x},\frac{1}{x};1\right) \quad G\left(0,\frac{1}{\bar{x}},\frac{1}{\bar{x}};1\right)$

A 4-loop integral

# Loops	LS	Polylogarithms
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3	$\frac{\frac{1}{x-\bar{x}}}{\frac{1}{(x-\bar{x})(1-x\bar{x})}} + \dots$	$\{x, 1-x, \bar{x}, 1-\bar{x}, x-\bar{x}\}$ $G\left(0, \frac{1}{x}, \frac{1}{\bar{x}}; 1\right)$

A 4-loop integral

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3	$\frac{\frac{1}{x-\bar{x}}}{\frac{1}{(x-\bar{x})(1-x\bar{x})}} + \dots$	$\{x, 1-x, \bar{x}, 1-\bar{x}, x-\bar{x}\}$ $G\left(0, \frac{1}{x}, \frac{1}{\bar{x}}; 1\right)$
4	$\frac{1}{x-\bar{x}} + \dots$	$\{x, \bar{x}, 1-x, 1-\bar{x}, 1-x\bar{x}\}$ $G\left(0, \frac{1}{x}, \frac{1}{x\bar{x}}; 1\right) \quad G\left(0, \frac{1}{\bar{x}}, \frac{1}{x\bar{x}}; 1\right)$

A 4-loop integral

# Loops	LS	Polylogarithms
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Conclusion

- We have computed the fully analytic result for the three-loop four-point correlator in planar N=4 SYM, solely by using
  - ➡ Symmetries.
  - ➡ Leading singularities - algebraic geometry.
  - ➡ Symbol - number theory - modern algebra.
  - ➡ Asymptotic expansions.

- Four-loop analysis seems to suggest that space of function is related to the leading singularities at lower loop orders.

- While the computation was performed for N=4 SYM, the technique might also apply outside this theory.
  - ➡ New way to compute Feynman integrals.

Hopf algebras

- Algebras

➡ 'Two become one'

$$\mu : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$$

$$\mu(a \otimes b) = a \cdot b$$

- Coalgebras

➡ 'One becomes two'

$$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$$

$$\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$$

- In a Hopf algebra these two operations are compatible.

$$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$$

- Idea: if combinatorics of some object is too complicated, 'break' it into smaller pieces and work with these.