Muon $g-2$ on the lattice: The Challenge

(A friendly incursion into the lattice by a phenomenologist)

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Based on a collaboration with C. Aubin, T. Blum, M. Golterman and K. Maltman

(but all the misconceptions are mine).

Generalities

In a world with $\mathcal{P}$ symmetry, a fermion, mass m_f, $q = p' - p$

$$\langle f, p' | J^\mu(0) | f, p \rangle = \overline{u}(p') \left[F_1(q^2) \gamma^\mu + \frac{i}{2\, m_f} \, F_2(q^2) \, \sigma^{\mu\nu} q_\nu \right] u(p)$$

Charge: $F_1(0) = 1$

Anomalous magnetic moment: $F_2(0) = \frac{(g-2)_f}{2} = a_f$

Dimensional analysis:

- Only f in loops $\Rightarrow a_f$ indep. of mass and universal $f = e, \mu, \tau$.
- Mass $m << m_f \Rightarrow a_f \sim \log \frac{m_f}{m}$
- Mass $M >> m_f \Rightarrow a_f \sim \frac{m_f^2}{M^2} \log \frac{M}{m_f}$

$g-2$ *for electron, muon and tau*

- The measurement of a_τ :

$$a_\tau^{EXP} = -0.018(17) \qquad \text{(DELPHI '04)}$$

is not very constraining. Compare with

$$a_\tau^{TH} = 117721(5) \times 10^{-8} \qquad \text{(Eidelman et al.'07)}$$

- There is a very good measurement of a_e :

$$a_e^{EXP} = 1159652180.73(28) \times 10^{-12} \quad [0.24\, ppb] \qquad \text{(Hanneke et al. '11)}$$

but we need it to <u>define</u> α , (Aoyama et al. '12):

$$\alpha^{-1}(a_e) = 137.0359991727 \underbrace{(68)}_{\alpha^4}\underbrace{(46)}_{\alpha^5}\underbrace{(26)}_{QCD+EW}\underbrace{(331)}_{exp} \quad [0.25\, ppb]$$

to be able to make predictions.

(Notice that the error due to QCD begins to show up).

$g-2$ *for electron, muon and tau*

- Only then does a_μ become calculable in the SM, :

$$a_\mu^{\rm SM} = 11\,659\,182.8\,(4.9) \times 10^{-10} \quad (\text{Hagiwara et al.}'11)$$

and with

$$a_\mu^{EXP} = 11\,659\,208.9\,\underline{(6.3) \times 10^{-10}}\,[0.54\,ppm] \quad (\text{Bennett et al.}'06)$$

one gets

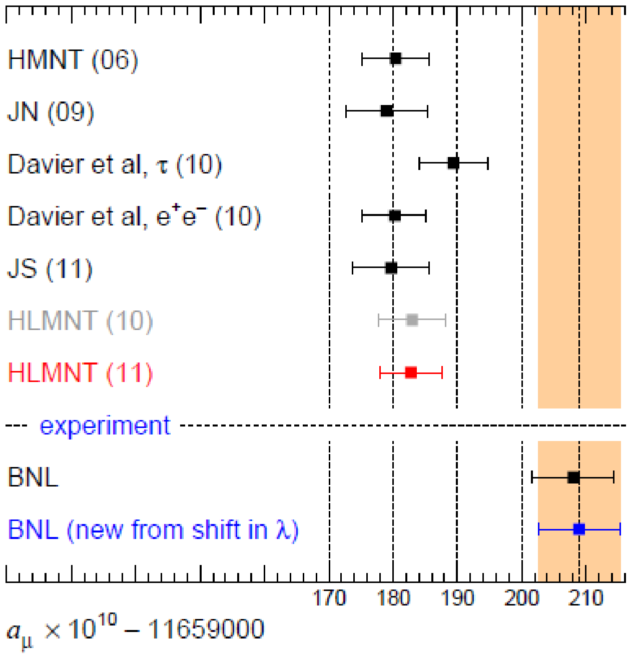
$$a_\mu^{EXP} - a_\mu^{SM} = 26.1\,(8.0) \times 10^{-10} \quad \underline{[3.3\,\sigma]}$$

with the prospects of reducing the exp. error by a factor of ~ 4 to 0.14 ppm.

(FERMILAB E989, circa ’17 ??). See also (J-PARC E34).

a_μ Current Status

(K. Hagiwara et al. ’11)



All SM determinations consistently below the exp. result.

(Precise discrepancy depends on details, though).

QED & EW & QCD

a_μ: QED contributions

(Aoyama et al. '12 and many refs therein)

E.g. at order α^5:

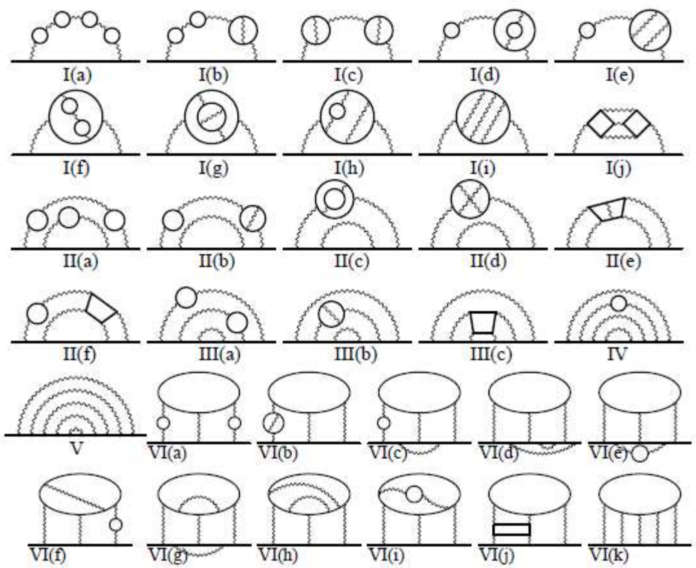


FIG. 2. Typical self-energy-like diagrams representing 32 gauge-invariant subsets contributing to the tenth-order lepton $g-2$. Solid lines represent lepton lines propagating in a weak magnetic field.

$A^{(2)}$	0.5
$A^{(4)}$	0.765857425(17)
$A^{(6)}$	24.05050996(32)
$A^{(8)}$	130.8796(63)
$A^{(10)}$	753.29(1.04)

- $A^{(2n)}, n \leq 3$ are known analytically (errors are due to e, μ, τ masses).
- $A^{(8)}, A^{(10)}$ known only numerically.

12672 diagrams later:

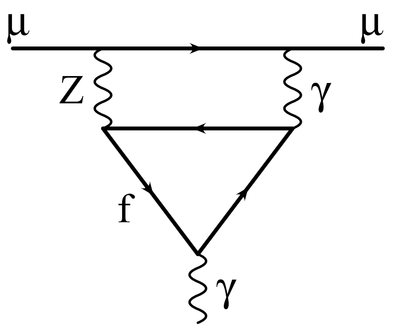
$$a_\mu^{QED} = A^{(2)}\left(\frac{\alpha}{\pi}\right) + A^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + A^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + A^{(8)}\left(\frac{\alpha}{\pi}\right)^4 + A^{(10)}\left(\frac{\alpha}{\pi}\right)^5$$

$$= 116584718845(9)_{\ell\ \mathrm{mass}}(19)_{\alpha^4}(7)_{\alpha^5}(30)_{\alpha(a_e)} \times 10^{-14}$$

a_μ: *EW contributions*

(Czarnecki, Krause and Marciano ’96)
(Knecht, SP, Perrottet and de Rafael ’02)
(Czarnecki, Marciano and Vainshtein ’03)
(Gnendiger, Stöckinger, Stöckinger-Kim. ’13)

E.g. at two loops:



$$a_\mu^{EW} \sim G_F\, m_\mu^2 \Big[1 + (\tfrac{\alpha}{\pi})\, \Phi(M_t, M_H, \text{chiral anomaly}, ...)\Big]$$

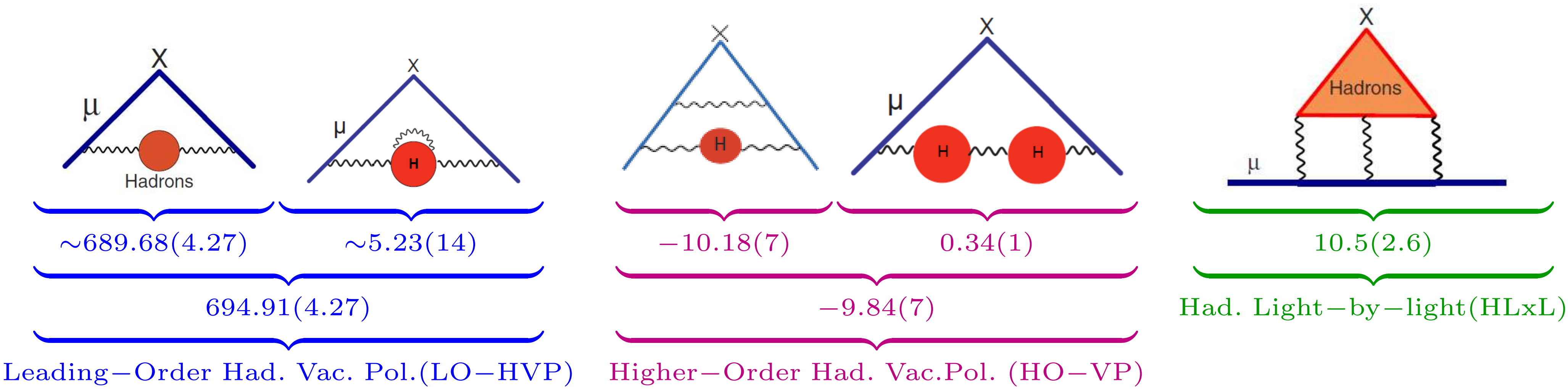
$$a_\mu^{EW} = 15.36(10) \times 10^{-10}$$

a_μ: *Anatomy of QCD contributions*

- Classification by Calmet et al. '77; Krause '97; see also Greynat and de Rafael '12.
- Numbers from, e.g., Hagiwara et al. '11.

(Recall $\Delta a_\mu^{EXP} = 6.3 \times 10^{-10}$.)

$a_\mu(\times 10^{-10})$:

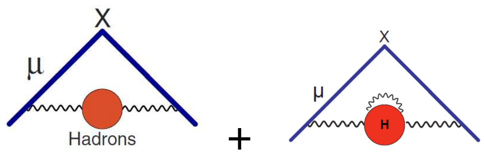


$$a_\mu^{\rm SM} = 11\ 659\ 182.8\ (4.9) \times 10^{-10}$$

- LO-HVP and HO-HVP are data based: $\sigma(e^+e^- \to had)$.
- HLxL is model based. Systematic error?
- HLxL is larger than Δa_μ^{EXP}. Without the lattice, there's little hope to get a model-independent calculation.

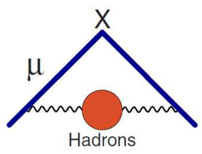
LO-HVP

- Standard Method (Gourdin and de Rafael '69):



$$\sim \int_{m_\pi^2}^{\infty} \frac{dt}{t} K(t) \overbrace{\mathrm{Im}\Pi(t)}^{e^+e^-\to\mathrm{had}(\gamma)} , \quad K(t) \sim m_\mu^2/t, \quad t \to \infty$$

- Lattice method (Blum '03; Lautrup, Peterman, de Rafael '72):



$$\sim \int_0^{\infty} dQ^2 \underbrace{f(Q^2)}_{\text{known}} \left[\Pi(0) - \Pi(Q^2)\right]$$

- Current goal: to compute this with less than 1% error.

- Warning: This is not the same as what is obtained with the Standard Method (the "handbag" diagram is missing).

- Lots of work devoted to this calculation...

LO-HVP: The works

- T. Blum, Phys. Rev. Lett. **91**, 052001 (2003)
- C. Aubin and T. Blum, Phys. Rev. D **75**, 114502 (2007)
- X. Feng, K. Jansen, M. Petschlies and D. B. Renner, Phys. Rev. Lett. **107**, 081802 (2011)
- P. Boyle, L. Del Debbio, E. Kerrane and J. Zanotti, Phys. Rev. D **85**, 074504 (2012)
- M. Della Morte, B. Jager, A. Juttner and H. Wittig, JHEP **1203**, 055 (2012)
- G. M. de Divitiis, R. Petronzio and N. Tantalo, Phys. Lett. B **718**, 589 (2012)
- X. Feng, S. Hashimoto, G. Hotzel, K. Jansen, M. Petschlies and D. B. Renner, arXiv:1305.5878 [hep-lat].
- A. Francis, B. Jaeger, H. B. Meyer and H. Wittig, arXiv:1306.2532 [hep-lat].
- C. Aubin, T. Blum, M. Golterman and S.P., Phys. Rev. D **86**, 054509 (2012).
- C. Aubin, T. Blum, M. Golterman and S.P., Phys. Rev. D **88**, 074505 (2013).
- M. Golterman, K. Maltman and S.P., arXiv:1309.2153 [hep-lat].
- etc, etc... (Apologies to all those I missed)

LO-HVP: The problem

LO-HVP: The problem

- Integrand strongly peaked at $Q^2 \sim m_\mu^2/4 \sim 0.003\mathrm{GeV}^2$.
- Current typical lattice data reaches down to $Q^2 \sim (2\pi/aT)^2 \sim 0.021$ (e.g. for $1/a = 3.3\mathrm{GeV}$ and $T = 144$).
- Integral as a Riemann sum is not an option. One must fit lattice data for $\Pi(Q^2)$ to a function and then integrate this function.
- Integrand is largest where there is no data.

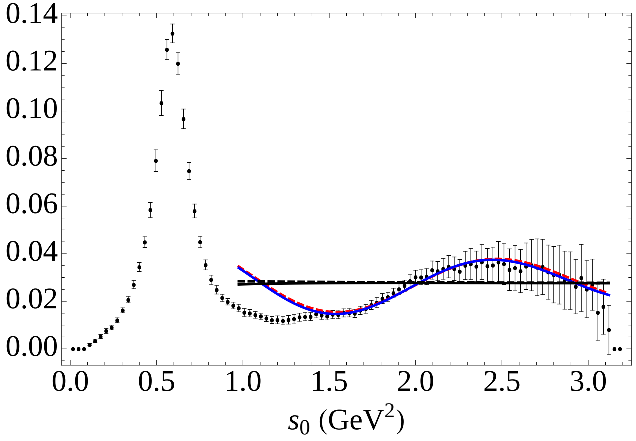
How to assess the systematic error ?

“IN GOD WE TRUST,

ALL THE OTHERS MUST BRING DATA.”

(W. Edwards Deming, American Statistician, 1900-1993.)

LO-HVP: A τ-based model for $I = 1$ part



Boito, Cata, Golterman, Jamin, Mahdavi, Maltman, Osborne, SP '11 + '12

$t \leq 1.5\ \mathrm{GeV}^2 \rightarrow$ OPAL data.

$t \geq 1.5\ \mathrm{GeV}^2$:

$$\mathrm{Im}\Pi(t) = \rho_{\ \mathrm{Pert.Th.}}(t) + e^{-\delta-\gamma\ t}\sin(\alpha+\beta t)$$

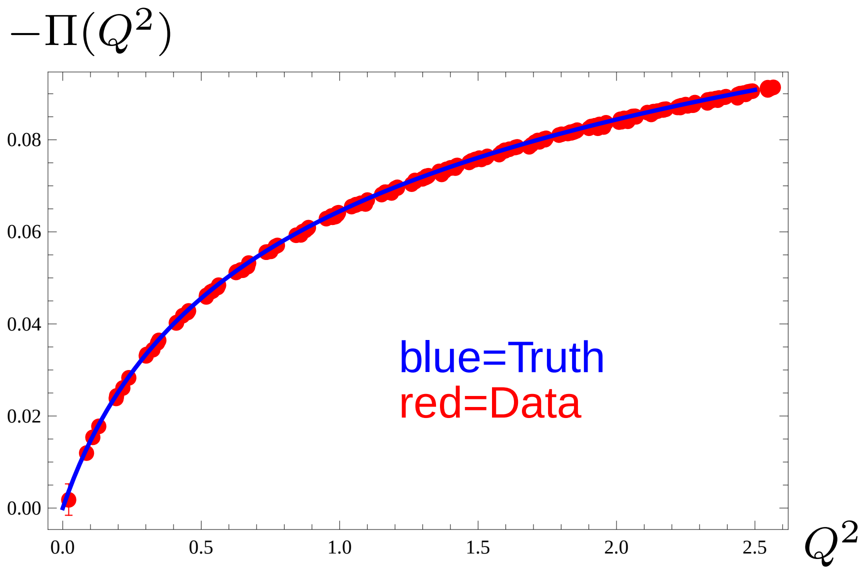
$$\Pi(Q^2) = -Q^2 \int_{4m_\pi^2}^{\infty} \frac{dt}{\pi}\ \frac{\mathrm{Im}\Pi(t)}{t+Q^2}$$

Goal: test fitting ansätze accuracy in lattice determinations

- Take typical lattice Q^2 values + lattice Covariance matrix

(e.g. $64^3 \times 144$, a = 0.06 fm, periodic BCs, MILC Asqtad ensemble; Aubin et al. '12).

$\Longrightarrow$ generate <u>fake</u> lattice data for $\Pi(Q^2)$ and compare with true answer from model



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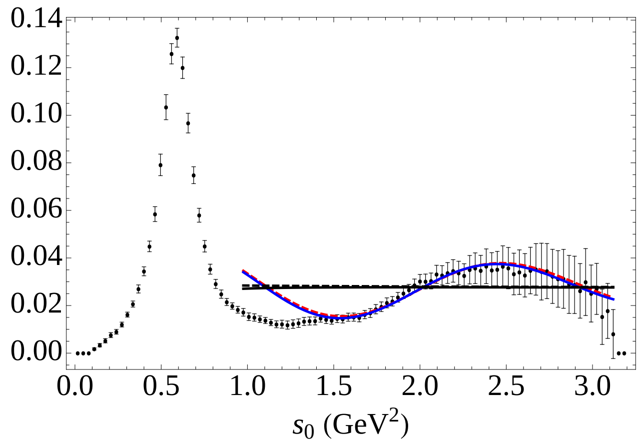

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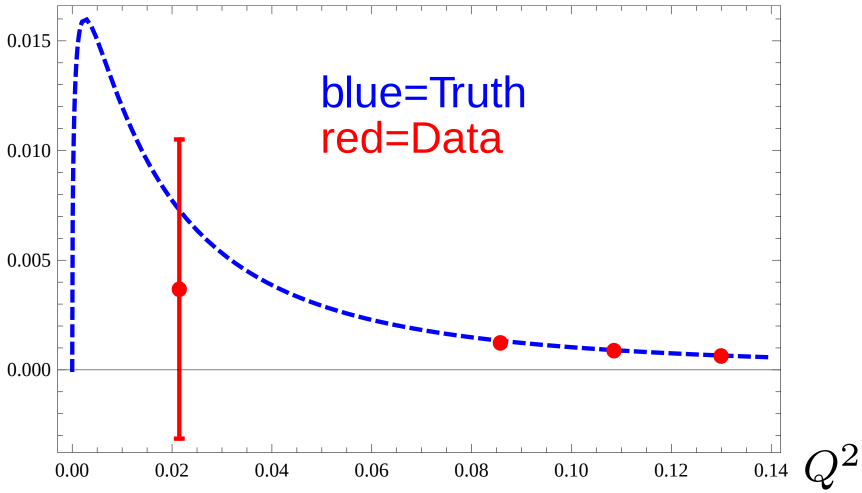
$(g-2)$ integrand



LO-HVP: Fitting functions

Aubin, Blum, Golterman, SP '12

★ Padés, model independent, they enjoy a convergence theorem for $N \to \infty$:

$$\Pi(Q^2) = \Pi(0) + Q^2 \underbrace{\left(a_0 + \sum_{r=1}^{N} \frac{a_r}{Q^2 + b_r} \right)}_{\text{Pade}}$$

$\Pi(0)$, $a's$ and $b's$ are fitting parameters.

★ VMD is not a Pade, since you fix $b_1 = M_\rho^2$. (true $\Pi(Q^2)$ has cut starting at $4m_\pi^2$...)

We have: $a_0 \neq 0 \Longrightarrow [N, N]$ Pade; $a_0 = 0 \Longrightarrow [N-1, N]$ Pade.

For instance:

- $\frac{a_1}{Q^2+b_1}$ is a [0,1] Pade $\Longrightarrow \Pi(Q^2) = \Pi(0) + Q^2 \left(\frac{a_1}{Q^2+b_1}\right)$
- $a_0 + \frac{a_1}{Q^2+b_1}$ is a [1,1] Pade $\Longrightarrow \Pi(Q^2) = \Pi(0) + Q^2 \left(a_0 + \frac{a_1}{Q^2+b_1}\right)$

etc...

LO-HVP: Model Fits

Golterman, Maltman, SP '13

"Exact result": $a_\mu^{HVP}|_{Q^2 \leq 1\ \mathrm{GeV}^2} = 1.204 \times 10^{-7}$.

Fit interval $0 < Q^2 \leq 1$ GeV^2 , (49 points).

Pull = (exact - fit) / error

	$a_\mu \times 10^7$	Error$\times 10^7$	χ^2/dof	Pull
VMD			2189/47 ×	
VMD+			67.4/46	
$[0,1]$			285/46	
$[1,1]$			61.4/45	
$[1,2]$			55.0/44	
$[2,2]$			54.6/43	

VMD "flavors" :

- VMD: [0,1] Pade with $b_1 = M_\rho^2$.
- VMD+: [1,1] Pade with $b_1 = M_\rho^2$ (i.e. VMD + linear polynomial)

LO-HVP: Model Fits

Golterman, Maltman, SP '13

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	$a_\mu \times 10^7$	Error$\times 10^7$	χ^2/dof	Pull
VMD	1.3201	0.0052	2189/47	-
VMD+			67.4/46	
$[0, 1]$			285/46	
$[1, 1]$			61.4/45	
$[1, 2]$			55.0/44	
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VMD "flavors" :

- VMD has a bad χ^2 and $(g-2)$.

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VMD	1.3201	0.0052	2189/47	-
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$[0,1]$			285/46	
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$[1,2]$			55.0/44	
$[2,2]$			54.6/43	

- VMD has a bad χ^2 and $(g-2)$.
- VMD+ also gets it wrong although the χ^2 is good $\Longrightarrow$ DANGER !

LO-HVP: Model Fits

Golterman, Maltman, SP '13

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	$a_\mu \times 10^7$	Error$\times 10^7$	χ^2/dof	Pull
VMD	1.3201	0.0052	2189/47	-
VMD+	1.0658	0.0076	67.4/46	18
$[0,1]$	0.8703	0.0095	285/46	-
$[1,1]$	1.116	0.022	61.4/45	4
$[1,2]$	1.182	0.043	55.0/44	0.5
$[2,2]$	1.177	0.058	54.6/43	0.5

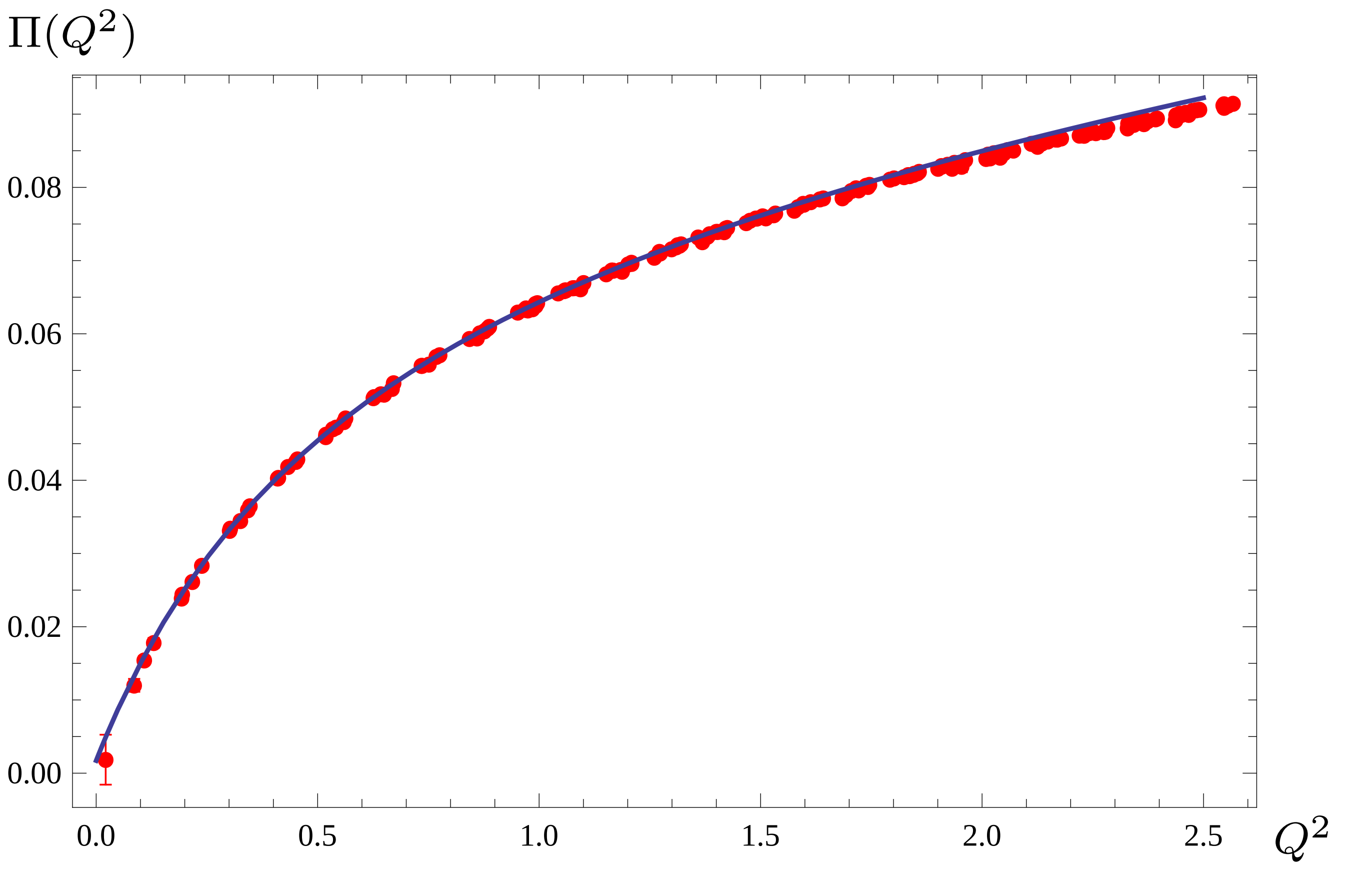
- VMD has a bad χ^2 and $(g-2)$.
- VMD+ also gets it wrong although the χ^2 is good $\Longrightarrow$ DANGER !
- Pades [1,2] and [2,2] get it right, but the error is $\sim 4\%$.

LO-HVP: Moral

Take, e.g., the VMD+ case

You may think this is a good fit for an accurate $(g-2)_\mu$:

solid line=VMD+ Fit

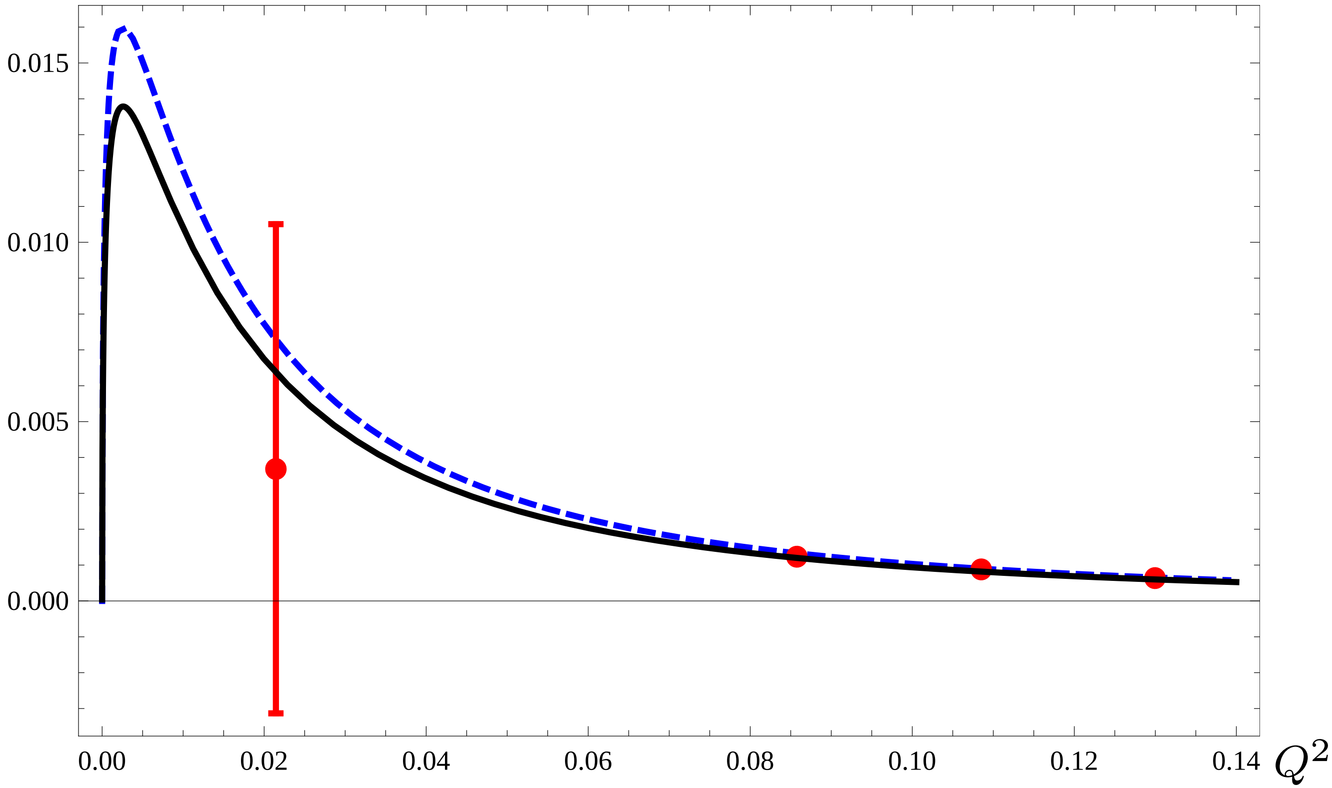


LO-HVP: Moral

while, in fact, this is what you should be looking at:

dashed blue=Truth
solid line=VMD+ Fit

(g-2) Integrand



LO-HVP: Science Fiction

(Recall exact value: $a_\mu^{HVP}|_{Q^2 \leq 1 \text{ GeV}^2} = 1.204 \times 10^{-7}$.)

Reduce the previous covariance matrix by 10^4:

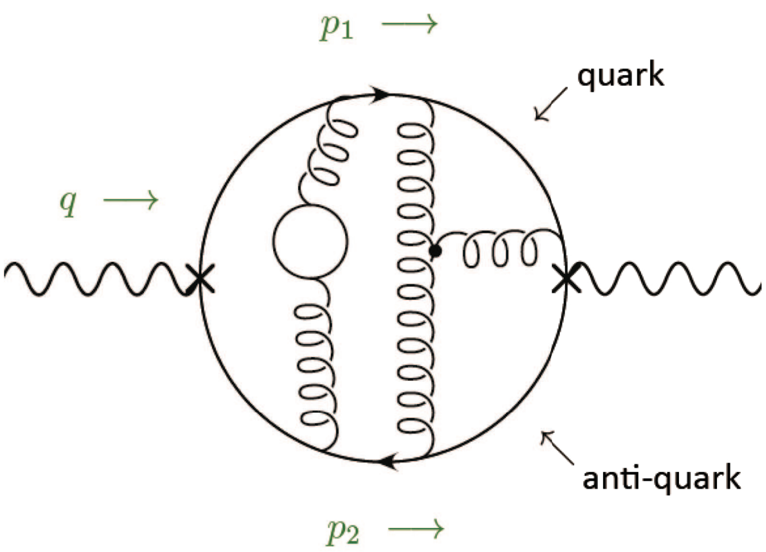
	$a_\mu \times 10^7$	Error$\times 10^7$	χ^2/dof	Pull
VMD	1.31861	0.00005	$2 \times 10^7/47$	-
VMD+	1.07117	0.00008	$7 \times 10^4/46$	-
$[0,1]$	0.87782	0.00009	$2 \times 10^7/46$	-
$[1,1]$	1.0991	0.0002	$5 \times 10^4/45$	-
$[1,2]$	1.1623	0.0004	$1340/44$	-
$[2,2]$	1.1862	0.0015	$76.4/43$ (?)	12
$[2,3]$	1.1965	0.0028	$42.0/42$	2

- VMD-type fits are very bad.
- Pade fits are eventually better, but need good data around the peak for χ^2 errors to be reliable.
- $[2,3]$ reaches error comparable to present e^+e^- and τ-data based determination.
- To reduce “Pull”, need twisting (see later) or larger volumes to have very good data in the region of curvature *of the integrand.* See also the strategies in de Divitiis et al. ’13 and Feng et al. ’13.

LO-HVP: Twisted bc's

LOHVP: Twisted bc's

(Bedaque '04; de Divitiis et al. '04; Sachrajda et al. '05)
(Della Morte, Jager, Juttner and Wittig '12)



$$
\begin{aligned}
q_t(x) &= e^{-i\theta_\mu}\, q_t(x+\widehat{\mu}\, L_\mu) \\
q(x) &= q(x+\widehat{\mu}\, L_\mu) \\
q &= p_1 + p_2 = \frac{2\pi n_1 + \theta}{L} + \frac{2\pi n_2}{L} \\
&= \;= \frac{2\pi(n_1+n_2)+\theta}{L}
\end{aligned}
$$

$\Rightarrow$ q varies continuously

One can think of this as an explicit breaking of an isospin-like symmetry for

$$\Psi = \begin{pmatrix} q \\ e^{-i\theta x/L}\; q_t \end{pmatrix}$$

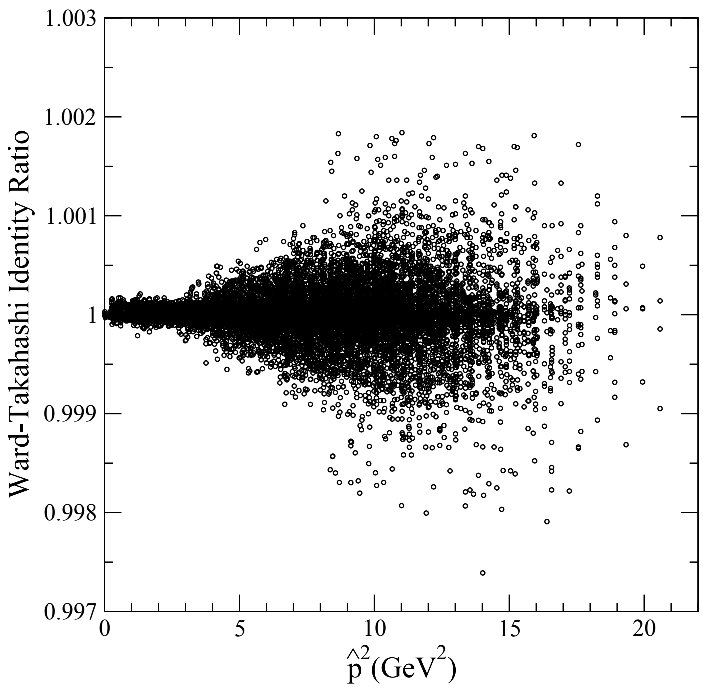
$$\delta q(x) = i\alpha^+(x)\; e^{-i\theta x/L} q_t(x) \quad , \quad \delta q_t(x) = i\alpha^-(x)\; e^{i\theta x/L} q(x)$$

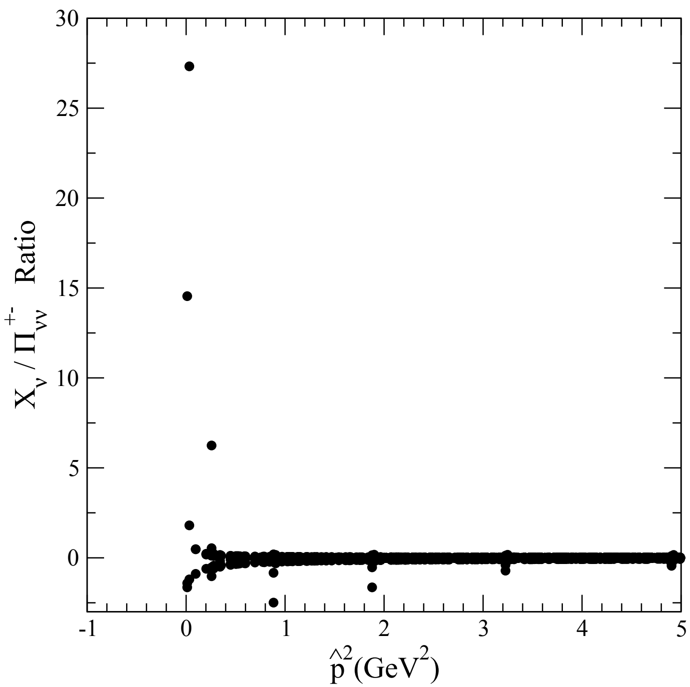
that would be exact if $\theta = 0$.

LO-HVP: Twisted bc's

(Aubin, Blum, Golterman, SP '13)

$$\Pi_{\mu\nu}(\widehat{p}) = (\widehat{p}^2\delta_{\mu\nu} - \widehat{p}_\mu\widehat{p}_\nu)\,\Pi(\widehat{p}^2) \,+\, \frac{\delta_{\mu\nu}}{a^2}\,X_\nu(\widehat{p}) \quad \Rightarrow \quad \sum_\mu \widehat{p}_\mu \Pi_{\mu\nu}(\widehat{p}) = \frac{\widehat{p}_\nu}{a^2} X_\nu(\widehat{p})$$





1 config. Asqtad MILC

$L^3 \times T = 48^3 \times 144$

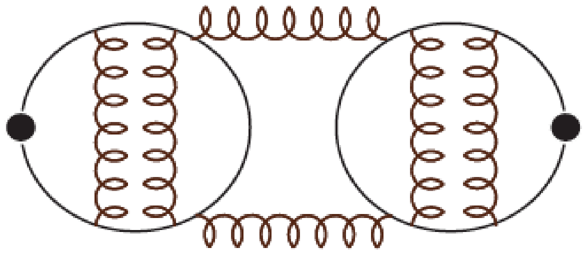
$1/a = 3.35$ GeV

$am = 0.0036$

$\theta_x = \theta_y = \theta_z = 0.28\pi$

$\theta_t = 0$

LO-HVP: Remaining issues

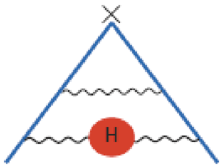


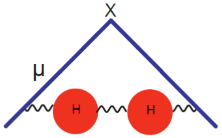
- Disconnected diagrams? Vanish in the $SU(3) \times SU(3)$ limit. Expected to be small ($\sim 10\%$ of the connected diagram).
  (Della Morte, Juttner '10; Francis, Jager, Meyer and Wittig '13).

- $m_u \neq m_d \neq m_s$, important: recall goal is less-than-1% precision for LO-HVP.

HO-HVP

- Standard Method (Hagiwara et al. '11):



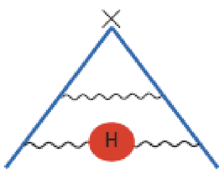


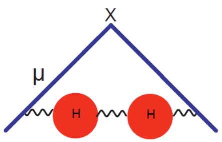
$$\sim \int_{4m_\pi^2}^{\infty} \frac{dt}{t}\, \widetilde{K}(t) \overbrace{\mathrm{Im}\Pi(t)}^{e^+e^-\to \mathrm{had}(\gamma)} \quad , \quad \widetilde{K}(t) \sim m_\mu^2/t, \quad t\to\infty$$

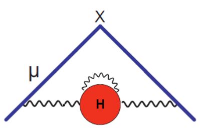
$$\sim \int_{4m_\pi^2}^{\infty} \frac{dt}{t}\, K(t) \left[\mathrm{Re}\Pi(t)\, \mathrm{Im}\Pi(t)\right] \quad , \quad K(t) \sim m_\mu^2/t, \quad t\to\infty$$

the latter is very small because $\int_{4m_\pi^2}^{\infty} \frac{dt}{t^2}\, \mathrm{Re}\Pi(t)\, \mathrm{Im}\Pi(t) = 0$ (Greynat, de Rafael '12).

- Lattice Method :







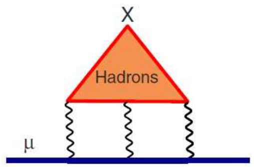
$$\sim \int_0^\infty dQ^2 \underbrace{\widetilde{f}(Q^2)}_{known} \left[\Pi(0) - \Pi(Q^2)\right]$$

$$\sim \int_0^\infty dQ^2 \underbrace{f(Q^2)}_{known} \left[\Pi(0) - \Pi(Q^2)\right]^2$$

$$\sim \int_0^\infty dQ^2 \underbrace{f(Q^2)}_{known} \left[\widetilde{\Pi}(0) - \widetilde{\Pi}(Q^2)\right]$$, [$f(Q^2)$ is the same as LO-HVP.]

HLbL

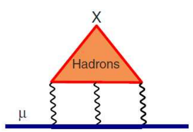
- <u>Standard Method</u> (Knecht, Nyffeler ’02; Prades, de Rafael, Vainshtein ’09; Nyffeler ’09):



$= 10.5(2.6) \times 10^{-10}$

A <u>model</u>, matching (as much as possible) long and short distances of the LbL subdiagram.

- <u>Lattice Method</u> (Chowdhury, Blum, Izubuchi, Hayakawa, Yamada and Yamazaki ’08):



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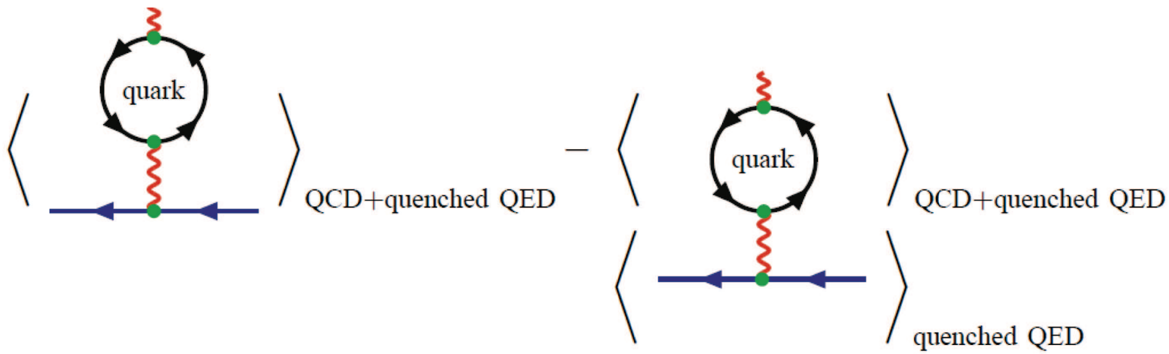


Looking forward to some results soon.

Conclusions

Conclusions

- You can take a shortcut: believe all results but HLxL. Then compute HLxL on the lattice with $\sim 1 \times 10^{-10}$ error. May model estimates for HLxL be wrong by a factor of ~ 3 ?

- Or, you can take the high road: Compute everything, LO-HVP + handbag + HO-HVP+ HLxL with $\sim 1 \times 10^{-10}$ error. This is more painful (but we'll learn a lot). In this case, keep on reading...

- LO-HVP: VMD-type fits turn out to be not reliable for an accuracy in $(g-2)_\mu$ of few per cent $\sim 20 \times 10^{-10}$ error .

- LO-HVP: Do not necessarily trust the χ^2 of your fit for assessing the accuracy in $(g-2)_\mu$, if you don't have very good data in the region of curvature of the integrand.

- LO-HVP: Blow up the region of the integrand around $Q^2 \sim m_\mu^2$. Showing plots of $\Pi(Q^2)$ for large Q^2 ranges is very misleading.

- LO-HVP: Benchmark your fitting method with a model. Should try our exercise on your Q^2 values and Cov. matrix to get a good check on your systematic error. (If you are interested, we can try to help.)

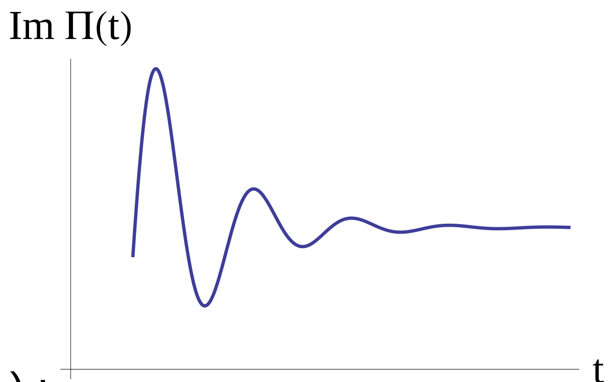
Conclusions(II)

- LO-HVP: Twisting may help. Take into account that the Vac. Pol. tensor is not transverse. This should be numerically studied thoroughly.
- For HO-HVP + handbag + HLxL: putting QED on the lattice is the next step.
- Getting $(g-2)_\mu$ with accuracy of $\sim 1 \times 10^{-10}$ won't be a rose garden, but it is important to try.

BACK-UP SLIDES

Duality Violations(I)

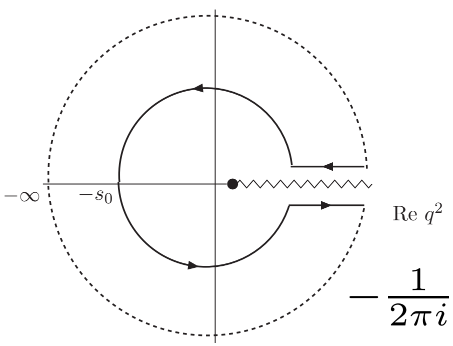
- OPE valid in euclidean, but not in minkowski. We know that spectrum $\neq$ OPE



- We expect (@ large t) :

$$\mathrm{Im}\Pi_{DV} \sim \mathrm{Im}(\Pi - \Pi_{OPE}) \sim \kappa \underbrace{e^{-\gamma t}}_{\text{OPE asympt.}} \underbrace{\sin(\alpha + \beta t)}_{\text{Regge}}$$

- $\Pi_{DV}(s) \to 0$ as $|s| \to \infty$. Then:



(Cata-Golterman-S.P. '05)

$$-\frac{1}{2\pi i}\oint_{|z|=s_0} dz\, w(z)\, \Pi_{DV}(z) = -\underbrace{\int_{s_0}^{\infty} ds}_{\text{extrapolation!}}\; w(s)\, \frac{1}{\pi}\mathrm{Im}\Pi_{DV}(s)$$

D. Violations(II)

Blok-Shifman-Zhang '98; Cata-Golterman-SP '05'08; Jamin '11

Explicit realization only models, no theory. Take $\Lambda_{QCD} = 1$; $F \sim 0.1$, decay constant.

- 1 resonance ($M \to M + i\Gamma/2$):

$$\frac{F^2}{q^2 - n} \longrightarrow \frac{F^2}{q^2 - n - i\sqrt{n}\,\Gamma}$$

- Regge-like tower: $n = 1, 2, 3, \ldots$

$$\Pi(q^2) \sim \sum_n^\infty \frac{F^2}{z+n} \quad , \quad z = \underbrace{(-q^2)^\zeta}_{\text{cut},\ q^2>0} \quad , \quad \zeta \simeq 1 - \mathcal{O}(\frac{1}{N_c})$$

$$\sim \quad \psi(z) = \frac{d \log \Gamma(z)}{dz}$$

- For $q^2 < 0 \longrightarrow \Pi(q^2) \sim \log z + \sum \frac{c_n}{z^n}$

- For $q^2 > 0 \longrightarrow \psi(z) = \psi(-z) - \frac{1}{z} - \pi \cot(\pi z)$,

$$\mathrm{Im}\Pi(q^2) \sim \mathrm{Im}(\log z) + \mathrm{Im} \sum \frac{c_n}{z^n} + \underline{\underline{F^2 \ e^{\frac{-q^2}{N_c}} \ \sin(\alpha + \beta\, q^2)}} \qquad F \sim 0.1 \quad ; \quad \alpha, \beta \sim 1$$

LO-HVP: Twisted bc's

Defining the currents (naive quarks; for staggered replace $\gamma_\mu \to \eta_\mu(x)$):

$$j^+_\mu(x) = \frac{1}{2}\Big(\overline{q}(x)\gamma_\mu U_\mu(x) q_t(x+\widehat{\mu}) + \overline{q}(x+\widehat{\mu})\gamma_\mu U^\dagger_\mu(x) q_t(x)\Big)$$

$$j^-_\mu(x) = \frac{1}{2}\Big(\overline{q}_t(x)\gamma_\mu U_\mu(x) q(x+\widehat{\mu}) + \overline{q}_t(x+\widehat{\mu})\gamma_\mu U^\dagger_\mu(x) q(x)\Big)$$

one gets the following Ward Id:

$$0 = \sum_\mu \partial^-_\mu \left\langle j^+_\mu(x) j^-_\nu(y)\right\rangle$$

$$+\frac{1}{2}\,\delta(x-y)\Big\langle \overline{q}_t(y+\widehat{\nu})\gamma_\nu U^\dagger_\nu(y) q_t(y) - \overline{q}(y)\gamma_\nu U_\nu(y) q(y+\widehat{\nu})\Big\rangle$$

$$-\frac{1}{2}\,\delta(x-\widehat{\nu}-y)\Big\langle \overline{q}(y+\widehat{\nu})\gamma_\nu U^\dagger_\nu(y) q(y) - \overline{q}_t(y)\gamma_\nu U_\nu(y) q_t(y+\widehat{\nu})\Big\rangle$$

and the Vac. Pol. tensor cannot be made transverse (unless at $\theta = 0$, since the last two terms are a total derivative). (Aubin, Blum, Golterman, SP '13)

LO-HVP: Twisted bc's

To check this, define

$$\Pi_{\mu\nu}(x-y) = \langle j^+_\mu(x) j^-_\nu(y)\rangle$$

$$-\frac{\delta_{\mu\nu}}{4}\delta(x-y)\Big(\langle\, \overline{q}(y)\gamma_\nu U_\nu(y) q(y+\widehat{\nu}) - \overline{q}(y+\widehat{\nu})\gamma_\nu U^\dagger_\nu(y) q(y)$$

$$+\overline{q}_t(y)\gamma_\nu U_\nu(y) q_t(y+\widehat{\nu}) - \overline{q}_t(y+\widehat{\nu})\gamma_\nu U^\dagger_\nu(y) q_t(y)\,\rangle\Big)$$

This means , $(\widehat{p}_\mu = \frac{2}{a}\sin\frac{ap_\mu}{2}$)

$$\Pi_{\mu\nu}(\widehat{p}) = (\widehat{p}^2\delta_{\mu\nu} - \widehat{p}_\mu\widehat{p}_\nu)\,\Pi(\widehat{p}^2) \;+\; \frac{\delta_{\mu\nu}}{a^2}\,X_\nu(\widehat{p}) \quad\Rightarrow\quad \sum_\mu \widehat{p}_\mu \Pi_{\mu\nu}(\widehat{p}) = \frac{\widehat{p}_\nu}{a^2} X_\nu(\widehat{p})$$

with

$$X_\nu(\widehat{p}) = \frac{i}{2}\,a^3\,\cot(\frac{ap_\nu}{2})\,\big\langle j^t_\nu(0) - j_\nu(0)\big\rangle \quad\sim\quad \cot(\frac{ap_\nu}{2})\,a\,\frac{\theta_\nu}{L_\nu}\Big(1+\mathcal{O}(\frac{\theta^2}{L^2})\Big) \qquad \xrightarrow{L_\nu\to\infty} \quad 0$$

$$j^t_\nu(x) = \tfrac{1}{2}\Big(\overline{q}_t(x)\gamma_\nu U_\nu(x) q_t(x+\widehat{\nu}) + \overline{q}_t(x+\widehat{\nu})\gamma_\nu U^\dagger_\nu(x) q_t(x)\Big)$$