

One-to-two transition amplitudes in a finite volume

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Based on:

R. A. Briceño, MTH, A. Walker-Loud
arXiv:1406.5965

Transition amplitudes

(rare B decays)

e.g., $B^0 \rightarrow K^{*0} \ell^+ \ell^-$



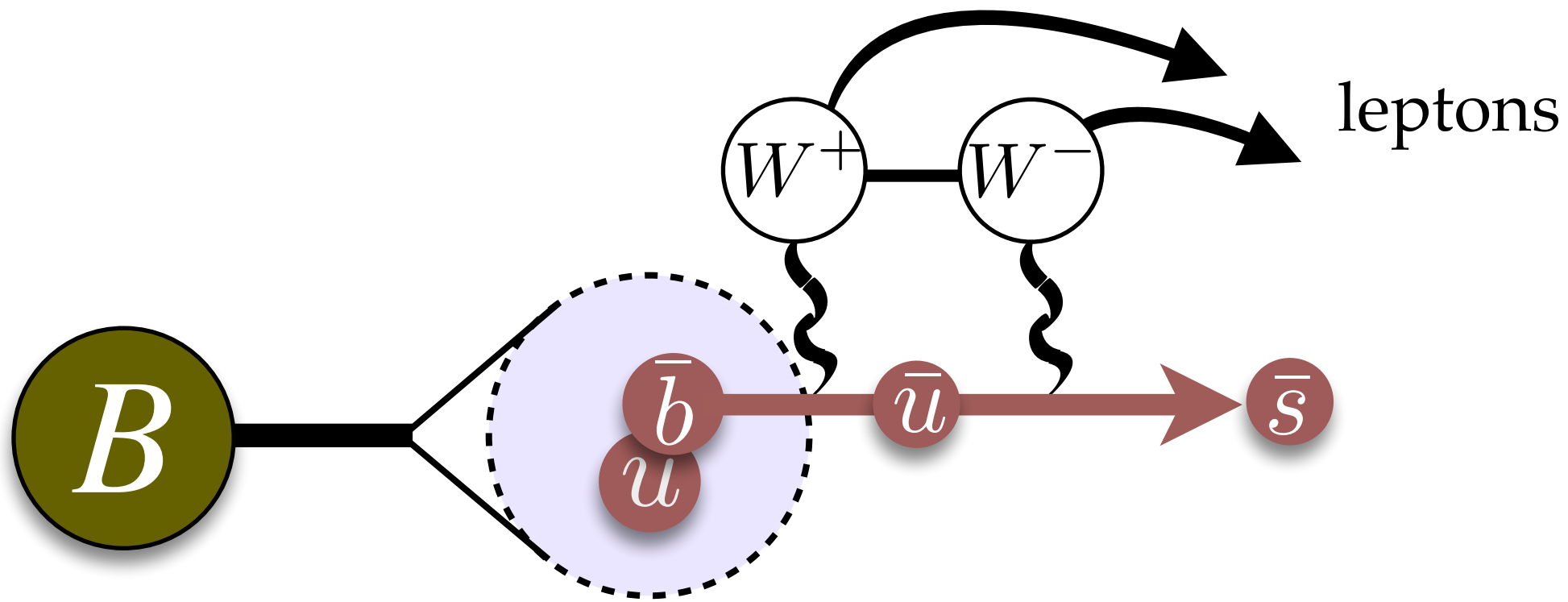
LHCb collaboration (2013)

First unquenched LQCD calculation:
Horgan, Liu, Meinel & Wingate (2013)

Transition amplitudes

(rare B decays)

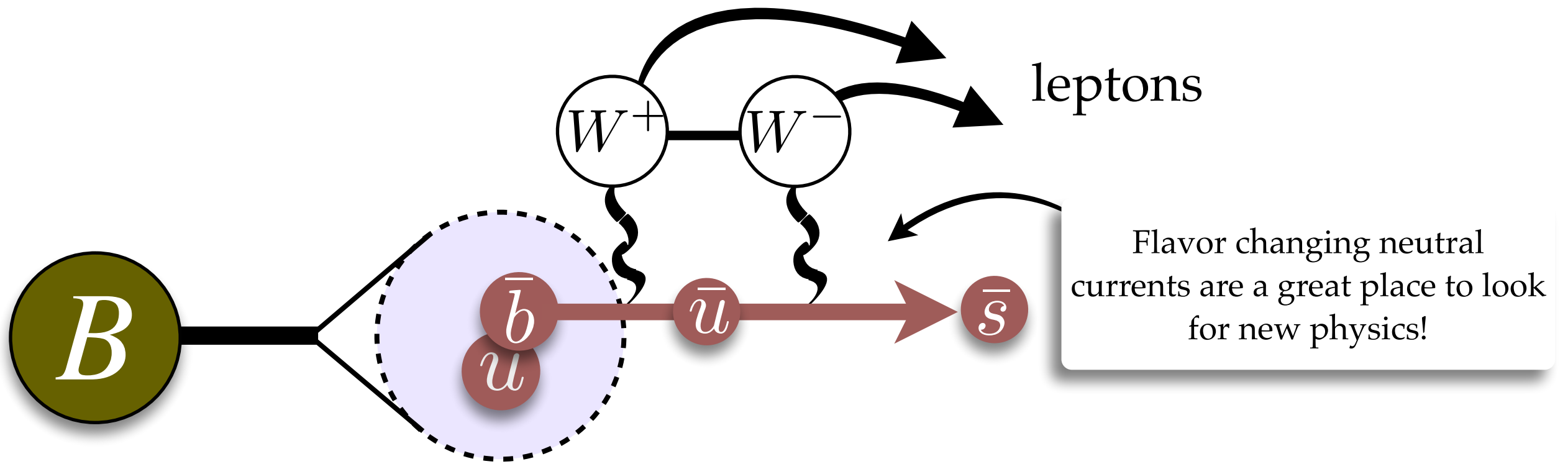
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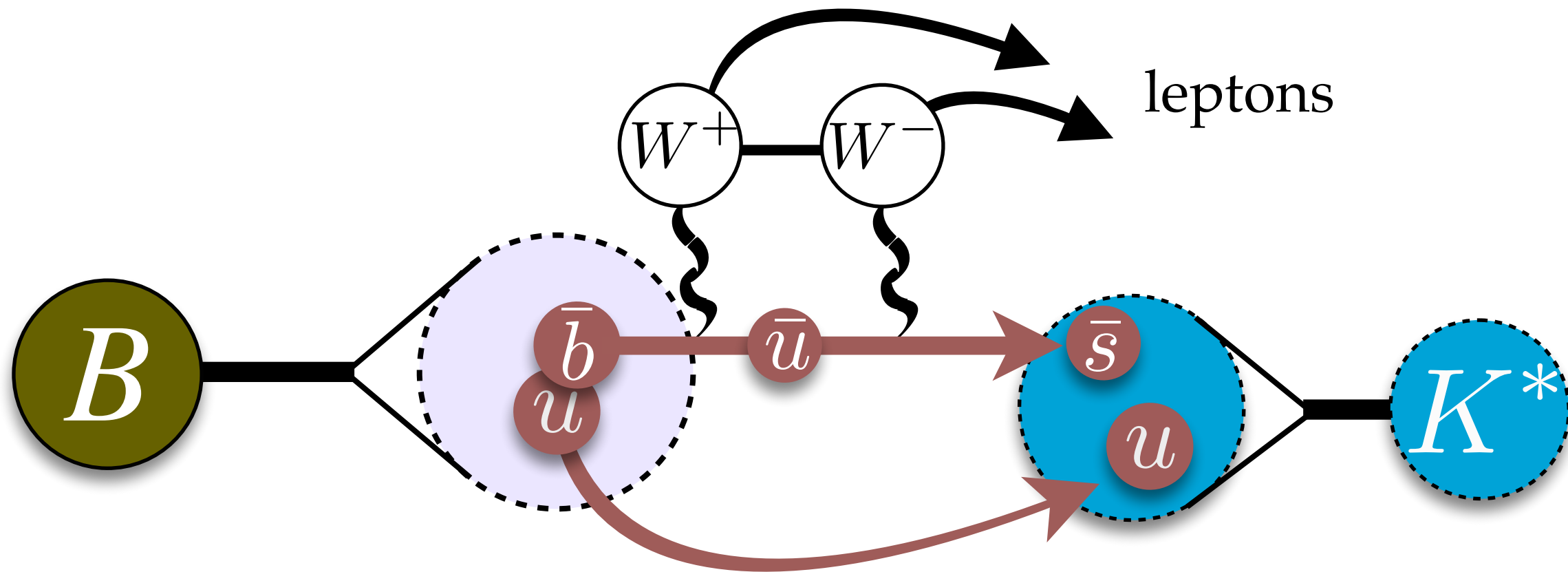
Flavor changing neutral currents are a great place to look for new physics!

Caution: unfortunately, this is not the full story.

Transition amplitudes

(rare B decays)

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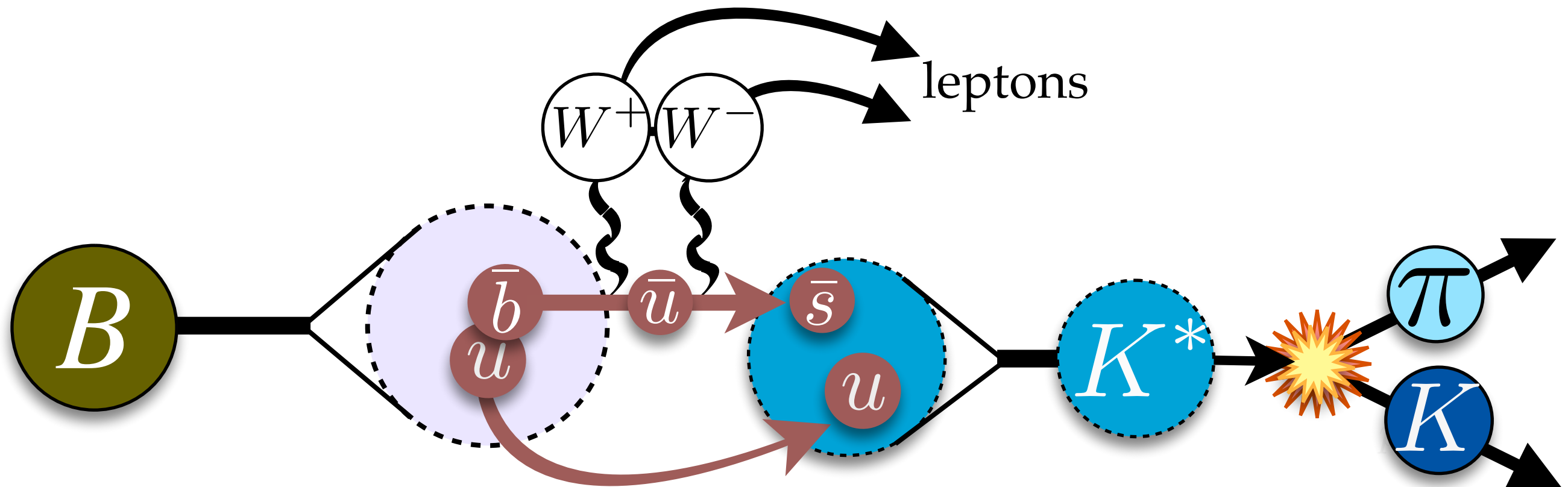
$K^*(892)$:

- $I (J^P) = 1/2 (1^-)$ resonance
- above πK and $\pi\pi K$ thresholds
- just below $K\eta \sim K\pi\pi\pi$ threshold

Transition amplitudes

(rare B decays)

e.g., $B^0 \rightarrow K^{*0} \ell^+ \ell^-$



Must calculate matrix elements of QCD eigenstates

$$\langle \pi K, \text{out} | \mathcal{J}_\mu | B \rangle$$

General Question

How can one use numerical Lattice QCD to determine

$$\langle \phi_1(p), \phi_2(P_f - p), \text{out} | \tilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q}) | \Phi, P_i \rangle ?$$

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$|\Phi, P_i\rangle$ **single scalar particle instate**

$\langle \phi_1(p), \phi_2(P_f - p), \text{out} |$ **two scalar particle outstate**

$\tilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q})$ **generic operator insertion**

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$\tilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q})$ **generic operator insertion**

 **generic Lorentz index**

Could be a scalar $\tilde{\mathcal{H}}$, a vector $\tilde{\mathcal{J}}_\mu$, a tensor $\tilde{\mathcal{T}}_{\mu\nu}$, etc...

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$\tilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q})$ **generic operator insertion**

With arbitrary momentum injection

$$\tilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q}) \equiv \int d^3x e^{-i\mathbf{x}\cdot\mathbf{Q}} J_A(x) \Big|_{x_0=0}$$

General Question

How can one use numerical Lattice QCD to determine

$$\langle \phi_1(p), \phi_2(P_f - p), \text{out} | \tilde{\mathcal{T}}_A(x_0 = 0, \mathbf{Q}) | \Phi, P_i \rangle ?$$

Relevant for many decays

$$K \rightarrow \pi\pi$$

$$\pi\gamma \rightarrow \pi\pi$$

$$B^0 \rightarrow K\pi\ell^+\ell^-$$

General Question

How can one use numerical Lattice QCD to determine

$$\langle \phi_1(p), \phi_2(P_f - p), \text{out} | \tilde{\mathcal{T}}_A(x_0 = 0, \mathbf{Q}) | \Phi, P_i \rangle ?$$

Relevant for many decays

$$K \rightarrow \pi\pi$$

**scalar operator \mathcal{H}
no momentum injection**

$$\pi\gamma \rightarrow \pi\pi$$

Lellouch and Lüscher (2001)
RBC/UKQCD implementing full
error budget calculation

$$B^0 \rightarrow K\pi\ell^+\ell^-$$

**nontrivial Lorentz structure
nonzero momentum injection**

This Work

Agadjanov, et.al., (2014), arXiv:1405.3476.

Outline

Difficulties in extracting matrix elements from LQCD

Introduction to finite vs infinite-volume matrix elements

Two-to-two scattering via LQCD

One-to-two transition amplitudes via LQCD

What can we extract from LQCD?

We are trying to evaluate a difficult integral numerically

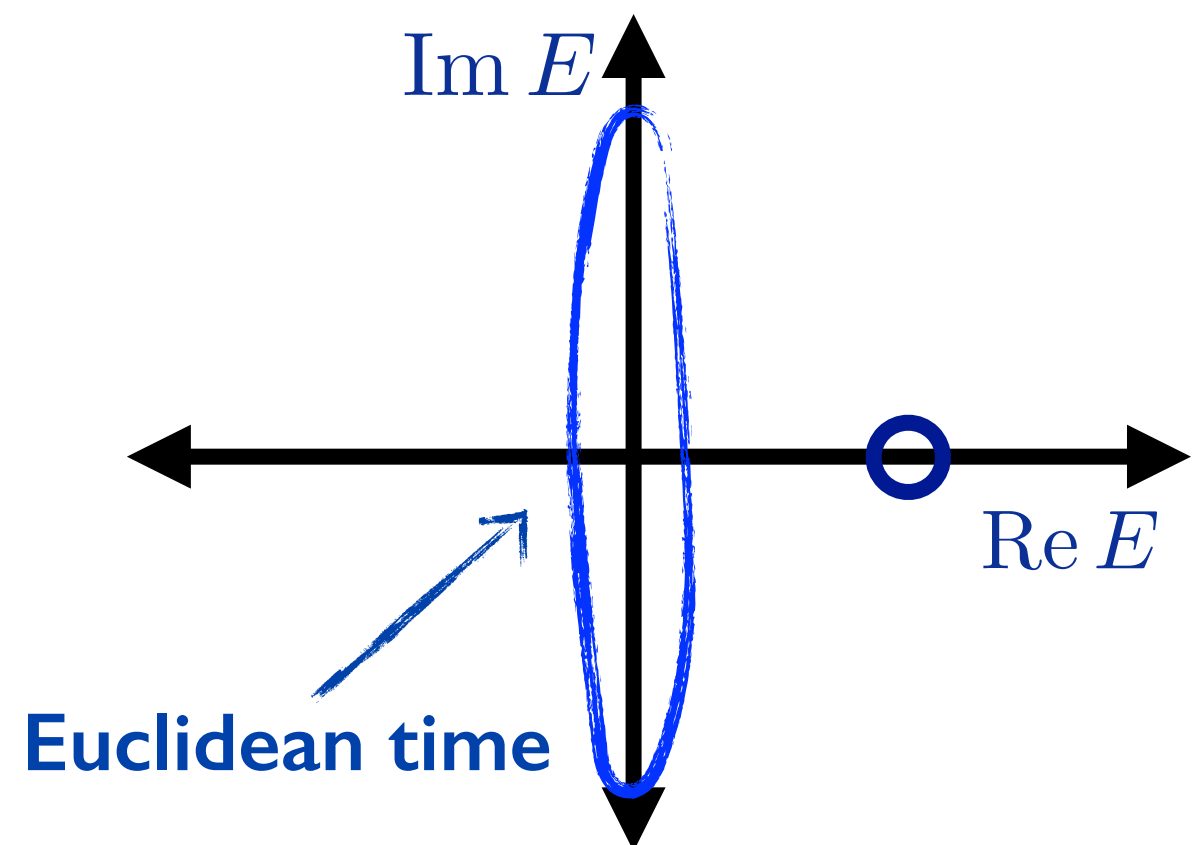
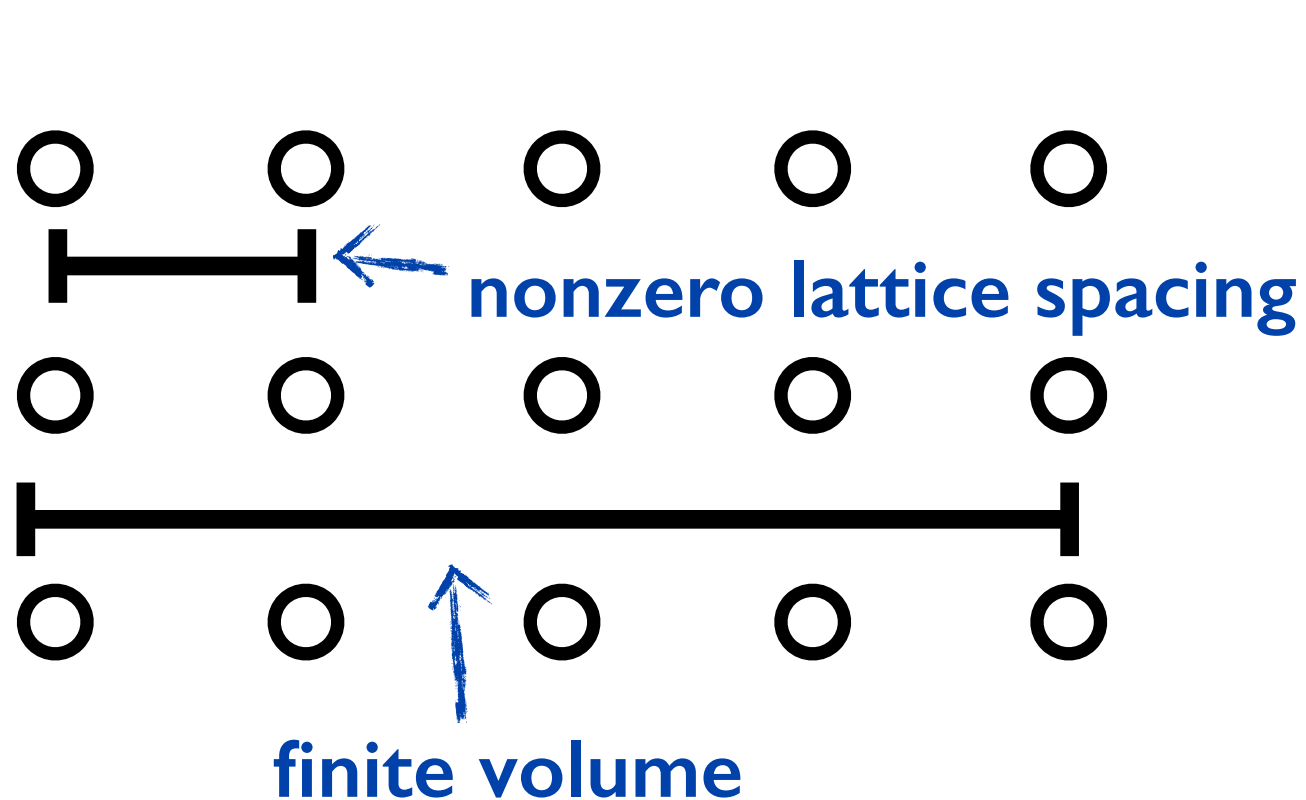
$$\langle T \phi_1 \cdots \phi_n \rangle = \int \mathcal{D}\phi e^{iS} \phi_1 \cdots \phi_n$$

What can we extract from LQCD?

We are trying to evaluate a difficult integral numerically

$$\langle T \phi_1 \cdots \phi_n \rangle_{\text{Euc, latt, fv}} = \int \prod_i^N d\phi_i e^{-S} \phi_1 \cdots \phi_n$$

To do so we have to make three compromises



What can we extract from LQCD?

Not possible to directly calculate

$$\langle \underline{\pi\pi} | \pi\pi \rangle$$

$$\langle \underline{\pi\pi\pi} | \pi\pi\pi \rangle$$

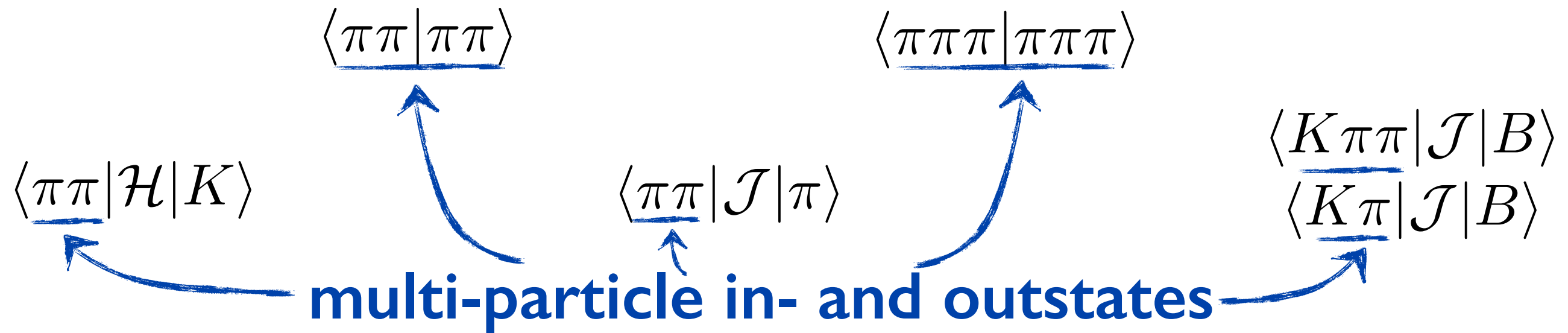
$$\langle \underline{\pi\pi} | \mathcal{H} | K \rangle$$

$$\langle \underline{\pi\pi} | \mathcal{J} | \pi \rangle$$

$$\begin{aligned} &\langle \underline{K\pi\pi} | \mathcal{J} | B \rangle \\ &\langle \underline{K\pi} | \mathcal{J} | B \rangle \end{aligned}$$

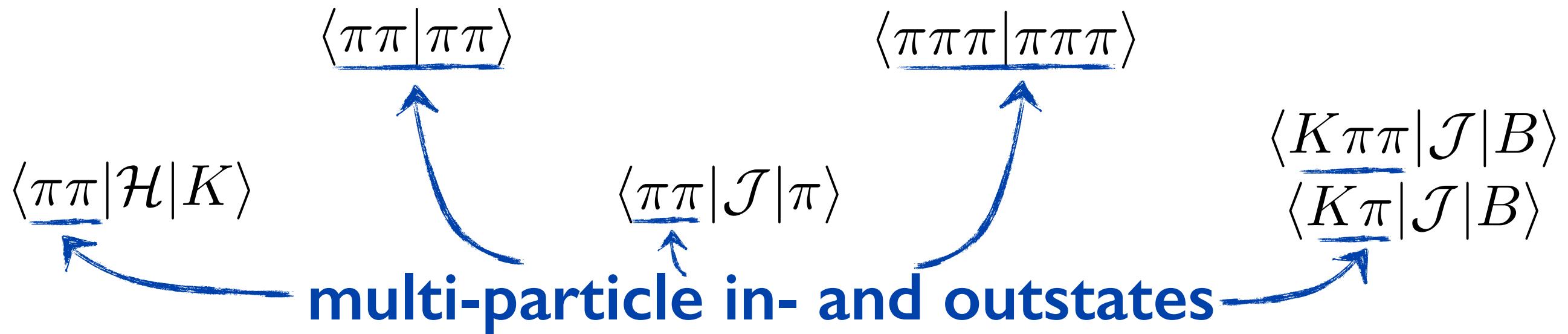
What can we extract from LQCD?

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What can we extract from LQCD?

Not possible to directly calculate



$$\langle \pi\pi | \pi\pi \rangle = \text{Amputate and put on-shell} \\ \langle 0 | \tilde{\pi}(p') \tilde{\pi}(k') \tilde{\pi}(p) \tilde{\pi}(k) | 0 \rangle$$

$$\langle \pi\pi | \mathcal{H} | K \rangle = \text{Amputate and put on-shell} \\ \langle 0 | \tilde{\pi}(p') \tilde{\pi}(k') \mathcal{H}(x) \tilde{K}(P) | 0 \rangle$$

Requires Minkowski momenta and infinite-volume

What can we extract from LQCD?

Instead we can only access

$$H_{\text{QCD}}|n, L\rangle = |n, L\rangle \underline{E_n(L)} \quad \underline{\langle n', L, \text{"}\pi\pi\text{"} | \mathcal{H} | n, L, \text{"}K\text{"} \rangle}$$

finite-volume energies and matrix elements

labels in quotes indicate quantum numbers

What can we extract from LQCD?

Instead we can only access

$$H_{\text{QCD}}|n, L\rangle = |n, L\rangle \underline{E_n(L)} \quad \underline{\langle n', L, \text{"}\pi\pi\text{"} | \mathcal{H} | n, L, \text{"}K\text{"} \rangle}$$

finite-volume energies and matrix elements

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How can we determine

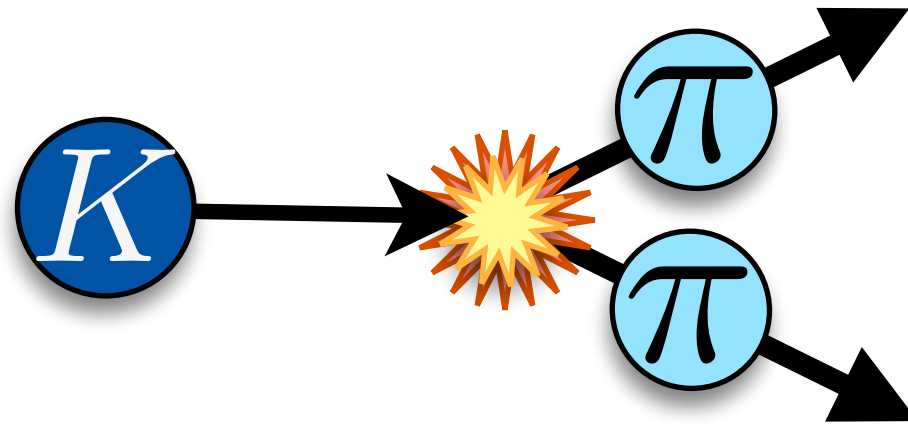
$$\langle \pi(p')\pi(k'), \text{out} | \pi(p)\pi(k), \text{in} \rangle \quad \& \quad \langle \pi(p)\pi(-p), \text{out} | \mathcal{H} | K \rangle$$

from

$$\underline{E_n(L)} \quad \& \quad \underline{\langle n, L, \text{"}\pi\pi\text{"} | \mathcal{H} | n, L, \text{"}K\text{"} \rangle} ?$$

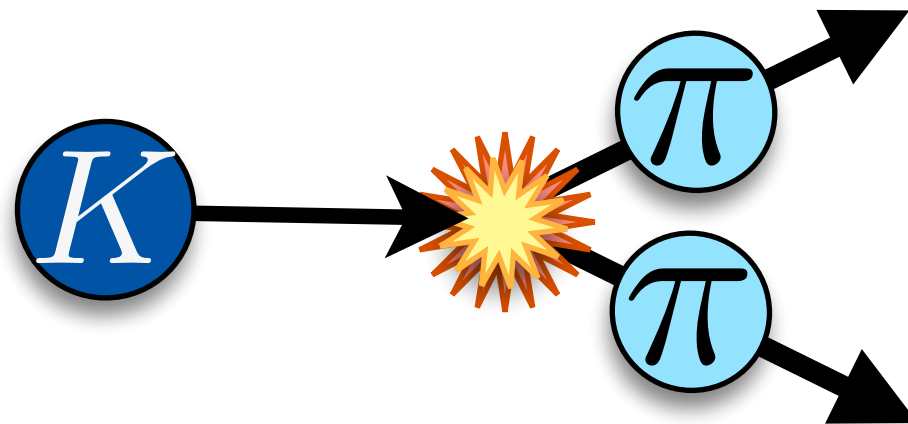
Cannot understand transitions without
first understanding scattering

Lellouch-Lüscher trick is to access transitions like

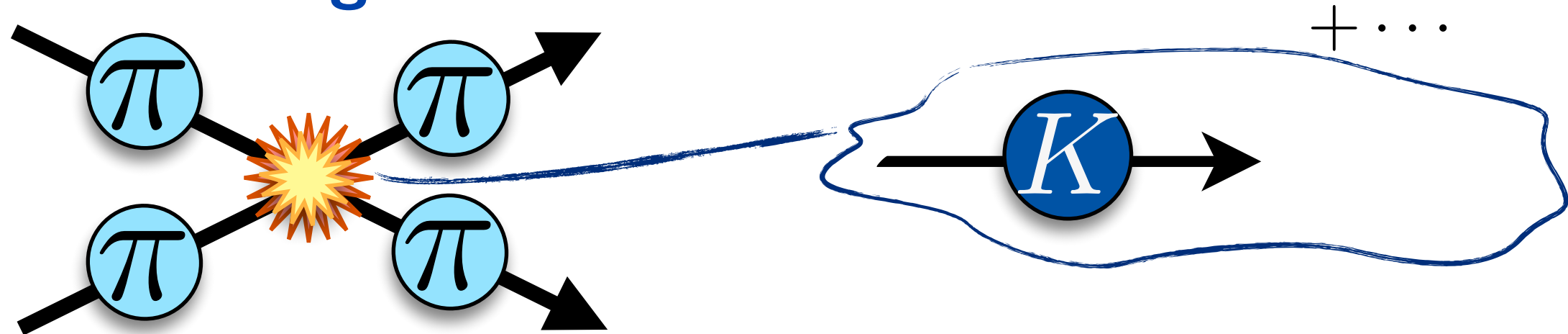


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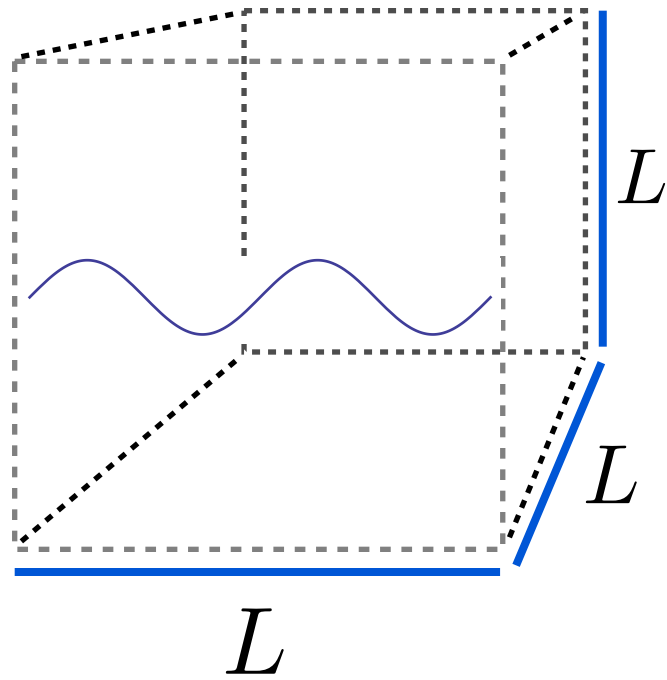


by considering it as an intermediate state in scattering

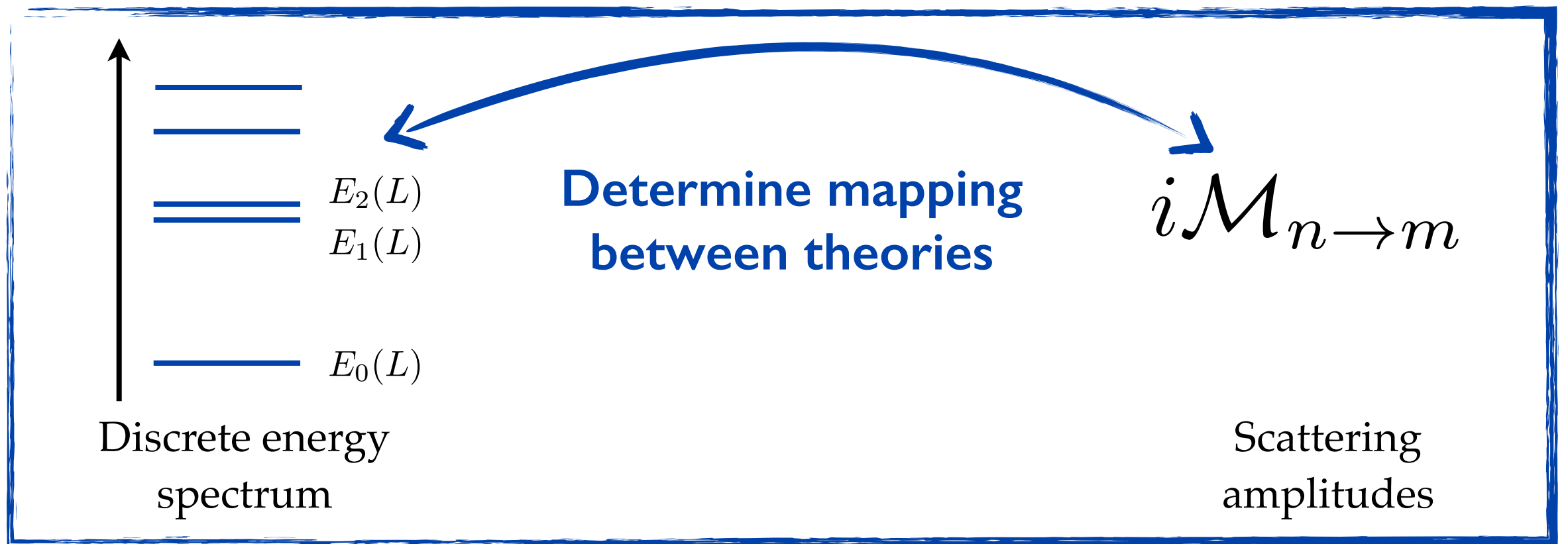
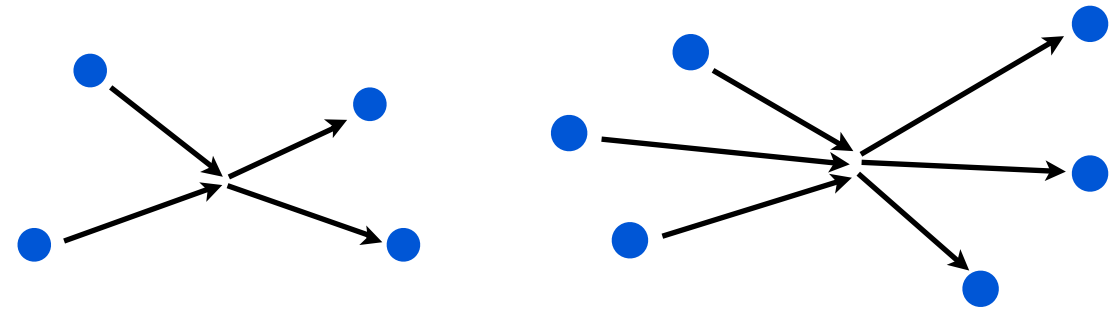


We thus begin by describing how to determine
scattering amplitudes from numerical Lattice QCD

Finite volume



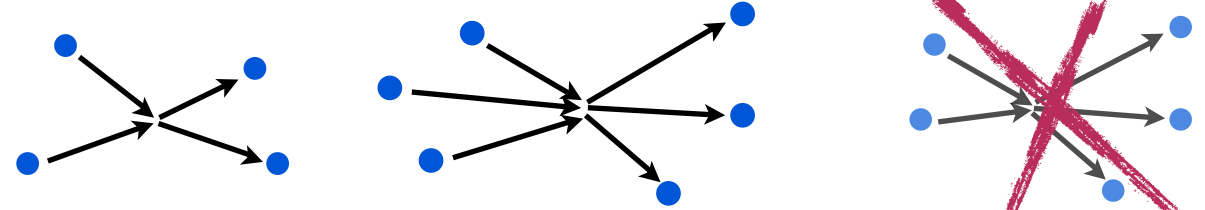
Infinite volume



Basic set-up

Relativistic scalar field theory

\mathbb{Z}_2 symmetry



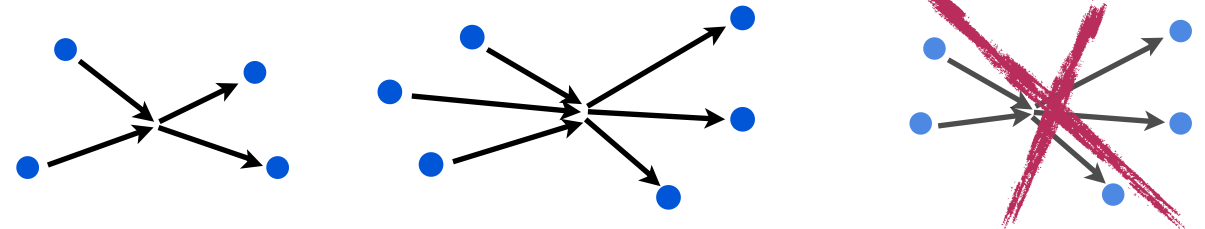
(For pions in QCD this is G-parity)

Include all vertices with even number of legs

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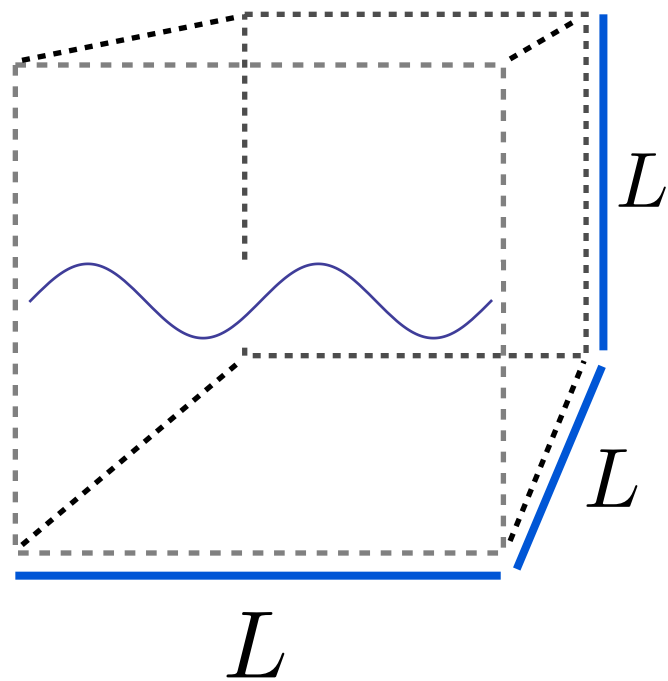
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Include all vertices with even number of legs

Finite volume



cubic, spatial volume (extent L)

periodic boundary conditions $\vec{p} \in (2\pi/L)\mathbb{Z}^3$

time direction **infinite**

Take L large enough to ignore e^{-mL}

Take space to be continuous

dropped throughout!

lattice spacing set to zero

Determine relation using finite-volume correlator

$$C_L(P) \equiv \int_L e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

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Euclidean energy P_0 ,

Momentum $\mathbf{P} = (2\pi/L)\mathbf{n}_P$,

two particle interpolating field

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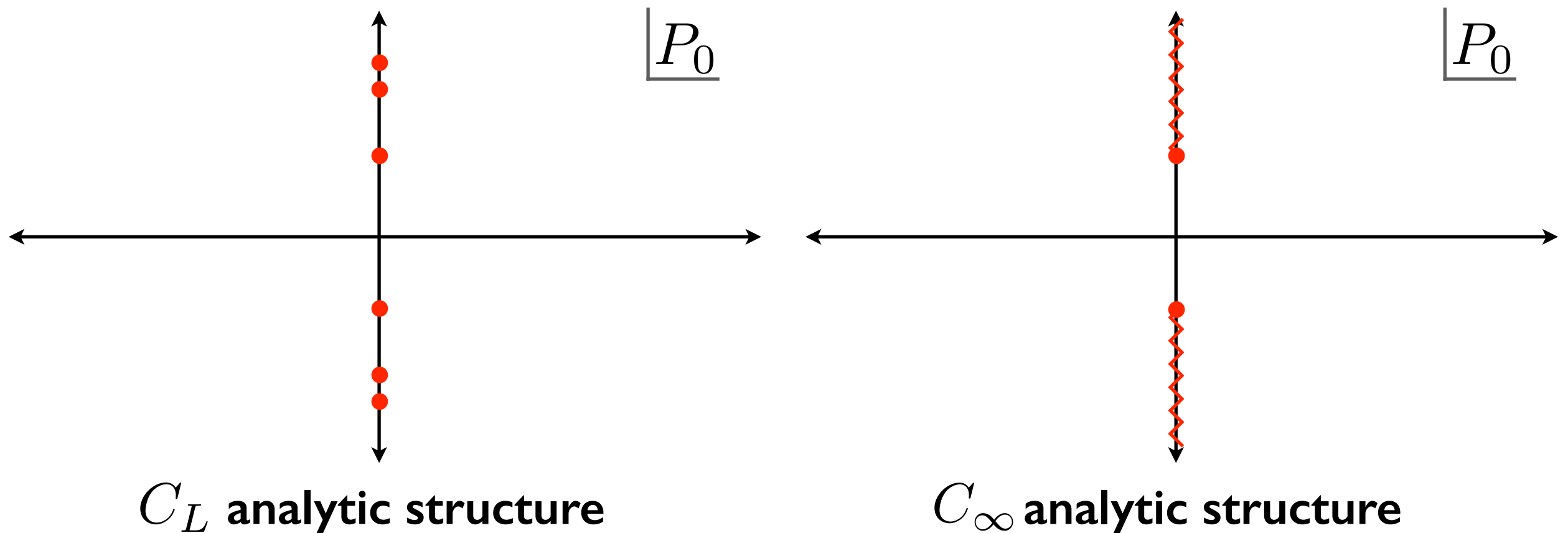
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At fixed L, \vec{P} , poles in C_L give finite-volume spectrum



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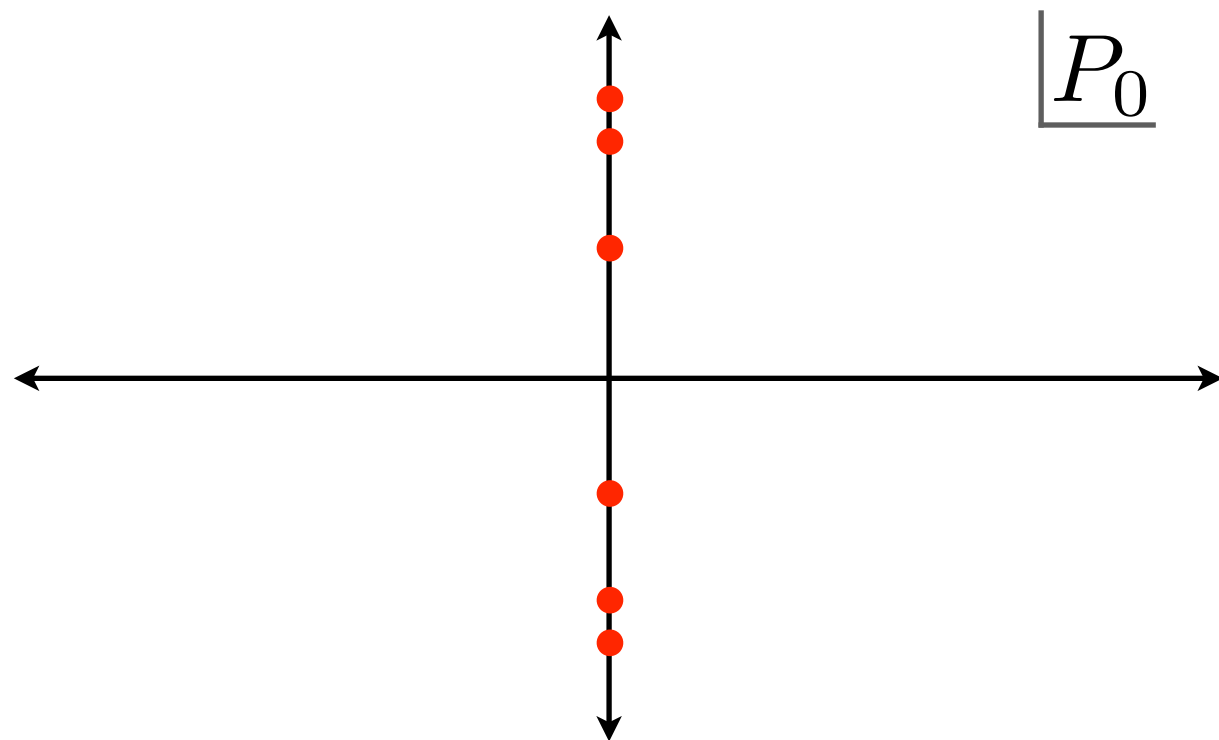
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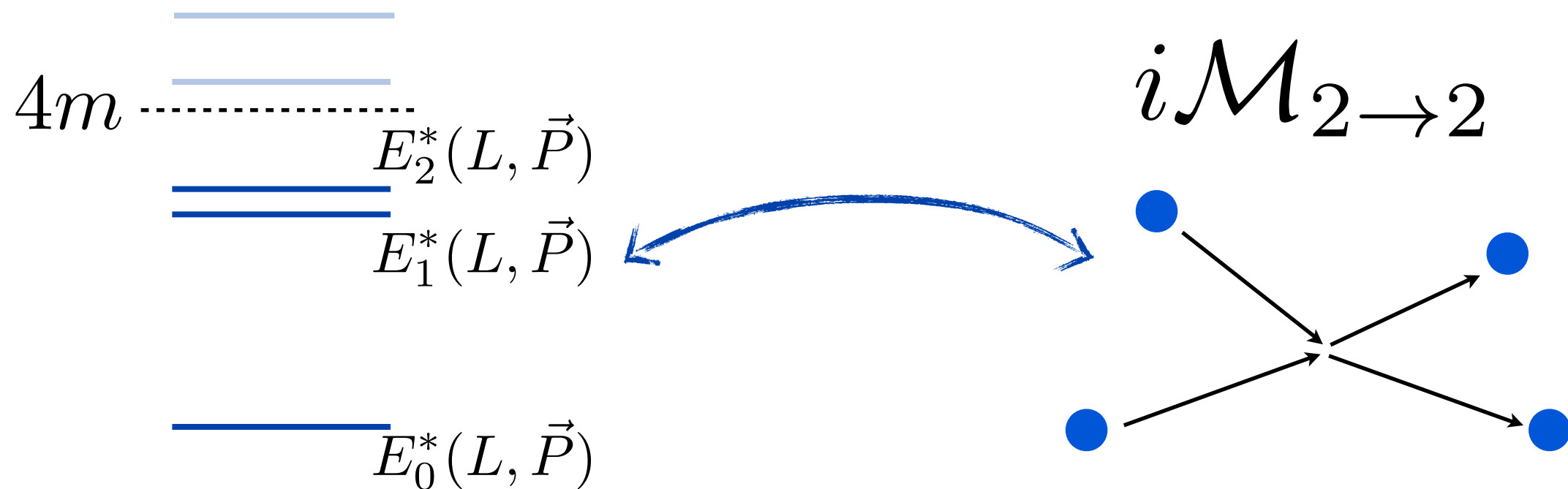
C_L analytic structure

Calculate $C_L(P)$ to all orders in perturbation theory and determine locations of poles.

Determine relation using finite-volume correlator

$$C_L(P) \equiv \int_L e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

Require $E^* < 4m$ where $E^{*2} \equiv -P_0^2 - \mathbf{P}^2$



Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

$$C_L(P) \equiv \text{diagram 1} + \text{diagram 2}$$

$$+ \text{diagram 3} + \text{diagram 4} + \dots$$

fully dressed propagators

infinite set of terms
no assumed suppression

**spatial loop momenta
are summed**

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

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$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

Key observation:

If particles in summed loops cannot all go on shell, then replace

$$\frac{1}{L^3} \sum_{\vec{k}} \longrightarrow \int \frac{d^3 k}{(2\pi)^3}$$

difference is order
 e^{-mL}

$$C_L(P) \equiv \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \dots$$

$$\frac{1}{L^3} \sum_{\vec{k}} \longrightarrow \int_{\vec{k}} \equiv \int \frac{d^3 k}{(2\pi)^3}$$

Since $E^* < 4m$, **only two** particles with total momentum (E, \vec{P}) **can go on-shell**

$$C_L(P) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \dots$$

these loops are now integrated

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$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 \{ \text{diagram}_3 + \text{diagram}_4 + \text{diagram}_5 + \dots \} \text{diagram}_6 + \dots$$

infinite-volume
Bethe-Salpeter kernel

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

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The equation shows a series of Feynman diagrams representing the correlation function $C_L(P)$. The first diagram consists of two vertices, \mathcal{O}^\dagger and \mathcal{O} , connected by two internal lines, with two external lines extending from the vertices. The second diagram is similar but includes a vertex labeled iK between the two internal lines. The third diagram includes two iK vertices. Each internal line in these diagrams is enclosed in a dashed box.

Next we introduce an important identity

$$\frac{1}{L^3} \sum_{\vec{k}} \int_{\vec{k}} \text{diagram}_1 = \text{diagram}_2 + \underbrace{\text{diagram}_3}_F$$

The identity shows the decomposition of a diagram with two internal lines into a diagram with two internal lines and a diagram with a single internal line labeled F . The F diagram is enclosed in a blue bracket.

contains all power-law corrections

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The equation shows a series of Feynman diagrams for $C_L(P)$. The first diagram consists of two vertices, \mathcal{O}^\dagger and \mathcal{O} , connected by two internal lines, each with a black dot. The second diagram is similar but includes a vertex labeled iK between the two internal lines. The third diagram includes two iK vertices. Each internal line in all diagrams has a black dot. Dashed boxes enclose the internal lines in each diagram.

Next we introduce an important identity

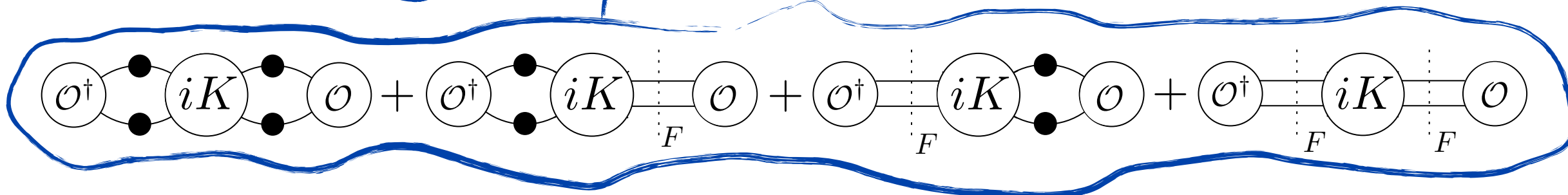
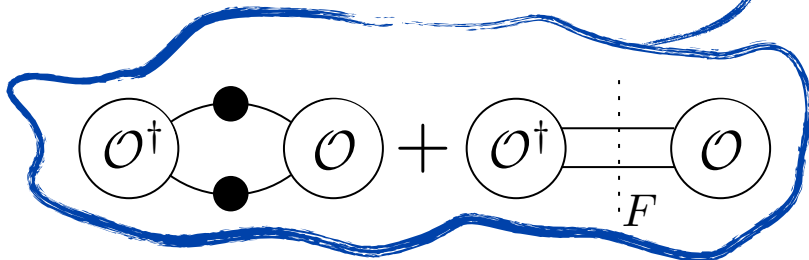
off-shell **on-shell**

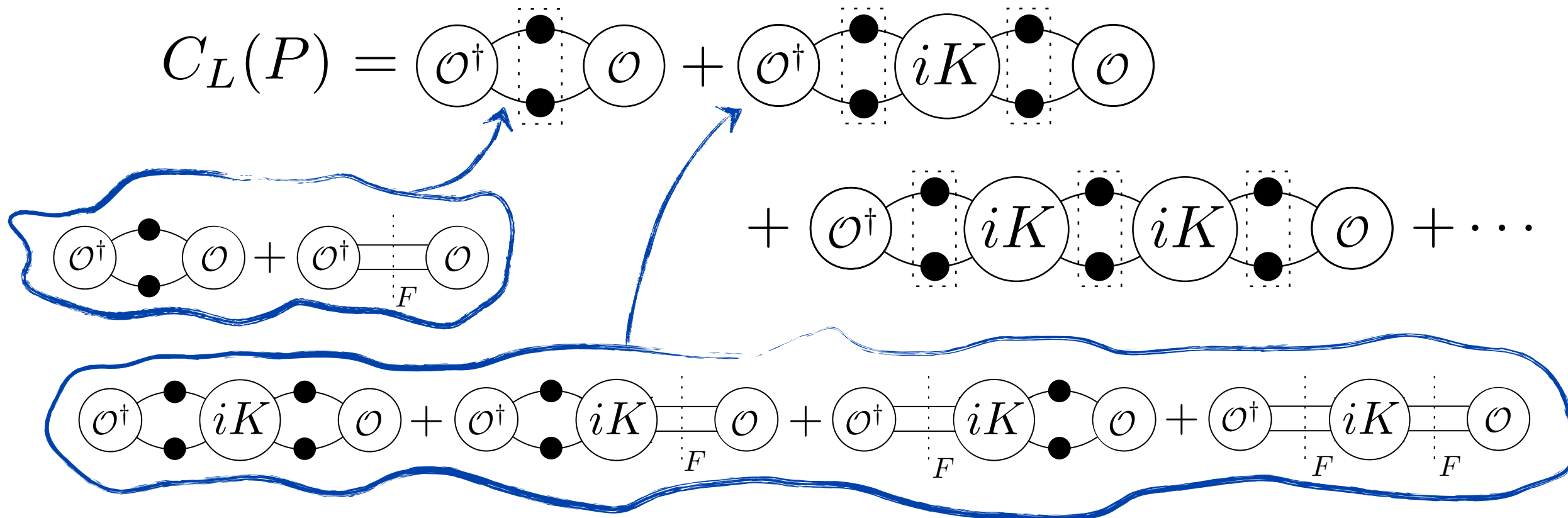
$$\frac{1}{L^3} \sum_{\vec{k}} \int_{\vec{k}} \underbrace{\text{diagram}_F}_{\text{contains all power-law corrections}}$$

The identity states that the off-shell diagram is equal to the on-shell diagram plus a diagram labeled F . The F diagram shows two vertices connected by a single horizontal line, with a vertical dashed line through the center. A blue bracket under the F diagram is labeled "contains all power-law corrections".

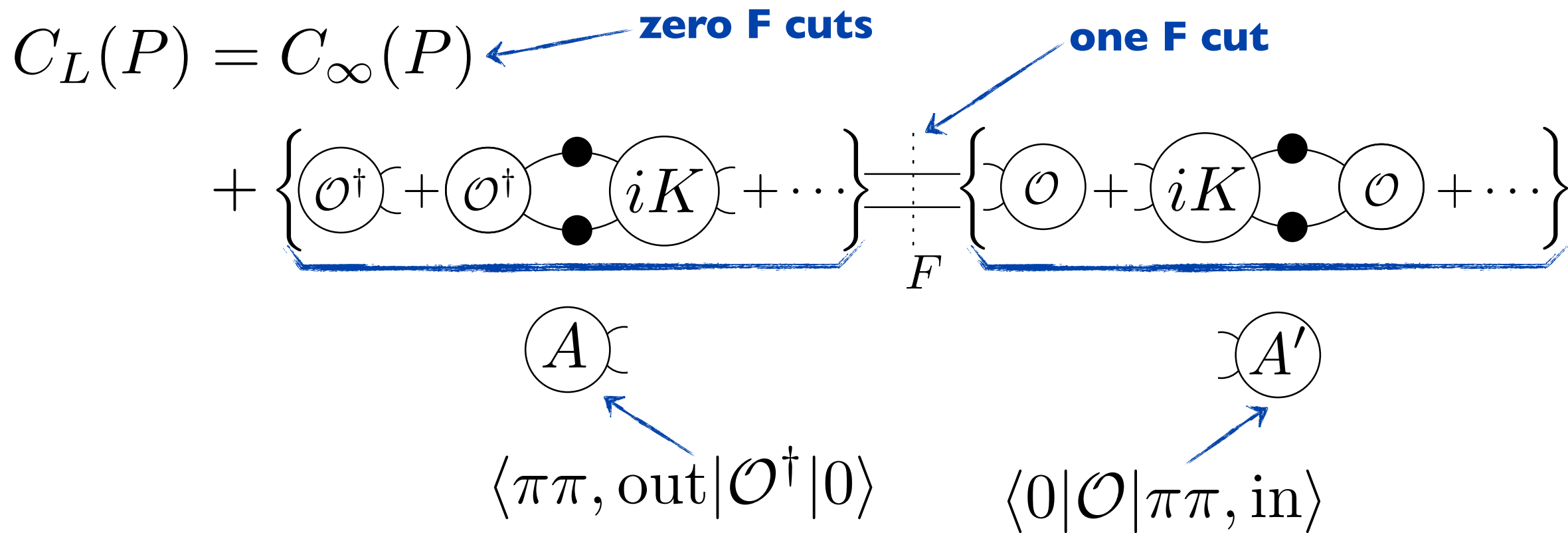
$$C_L(P) = \text{Diagram 1} + \text{Diagram 2}$$

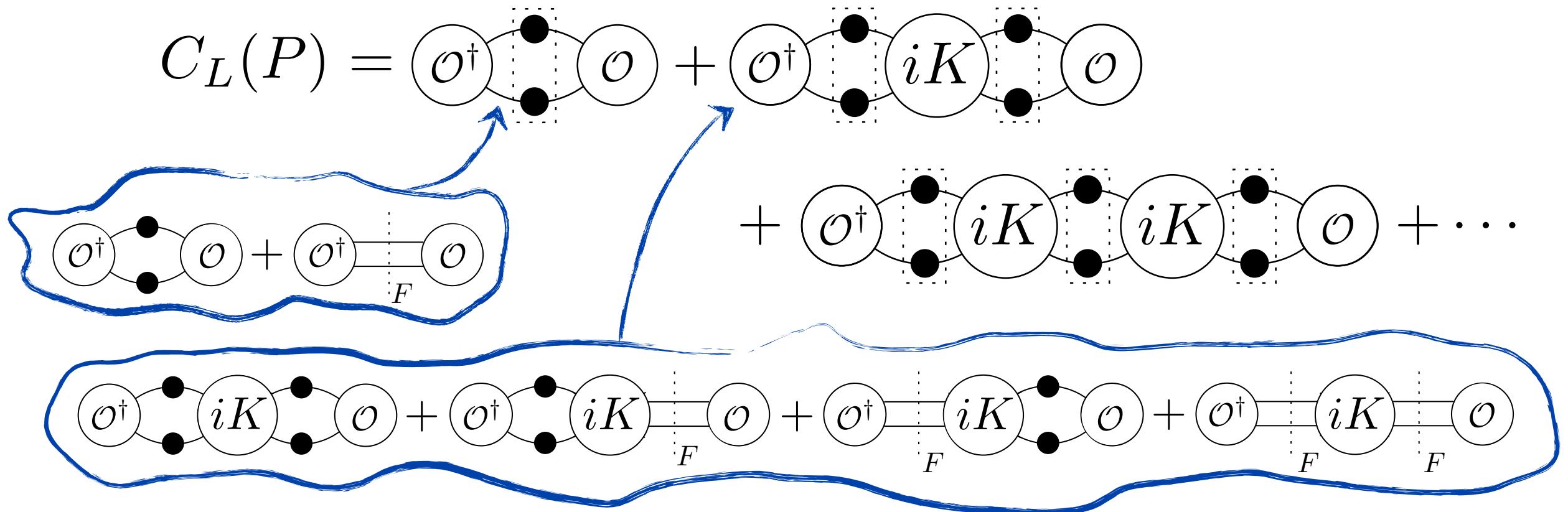
$$+ \text{Diagram 3} + \dots$$





Now regroup by number of F cuts



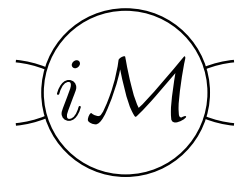


Now regroup by number of F cuts

$$C_L(P) = C_\infty(P) + \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \dots$$

The diagram shows the regrouped expression for $C_L(P)$. The first term is $C_\infty(P)$. The second term is a diagram with two vertices A and A' connected by two lines, with a dashed box labeled F around the cut line. The third term is a diagram with two vertices A and A' connected by two lines, with a dashed box labeled F around the cut line, and a bracketed sum of diagrams in between: iK , iK with two internal lines, iK , and an ellipsis. A blue arrow points from the text "two F cuts" to the first diagram of the third term. A blue box encloses the bracketed sum of diagrams in the third term.

two F cuts




As Promised!

infinite-volume on-shell two-to-two scattering amplitude

$$C_L(P) = C_\infty(P)$$

$$\begin{aligned}
 &+ \begin{array}{c} \textcircled{A} \text{---} \textcircled{A'} \\ \vdots \\ F \end{array} + \begin{array}{c} \textcircled{A} \text{---} \textcircled{i\mathcal{M}} \text{---} \textcircled{A'} \\ \vdots \quad \vdots \quad \vdots \\ F \quad F \quad F \end{array} \\
 &+ \begin{array}{c} \textcircled{A} \text{---} \textcircled{i\mathcal{M}} \text{---} \textcircled{i\mathcal{M}} \text{---} \textcircled{A'} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ F \quad F \quad F \quad F \end{array} + \dots
 \end{aligned}$$

$$C_L(P) = C_\infty(P) + \sum_{n=0}^{\infty} A' i F [i\mathcal{M}_{2 \rightarrow 2} i F]^n A$$


$$C_L(P) = C_\infty(P)$$

$$+ \begin{array}{c} \text{---} \text{---} \text{---} \\ | \quad | \quad | \\ \text{---} \text{---} \text{---} \\ F \quad F \quad F \end{array} \begin{array}{c} \text{---} \text{---} \text{---} \\ | \quad | \quad | \\ \text{---} \text{---} \text{---} \\ F \quad F \quad F \end{array} \begin{array}{c} \text{---} \text{---} \text{---} \\ | \quad | \quad | \\ \text{---} \text{---} \text{---} \\ F \quad F \quad F \end{array} + \dots$$

$$C_L(P) = C_\infty(P) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2 \rightarrow 2} iF]^n A$$

$$C_L(P) = C_\infty(P) + A' iF \frac{1}{1 - i\mathcal{M}_{2 \rightarrow 2} iF} A$$

no poles
no poles
no poles

$$C_L(P) \text{ diverges whenever } iF \frac{1}{1 - i\mathcal{M}_{2 \rightarrow 2} iF} \text{ diverges}$$

Review

1

$$C_L(P) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The diagrammatic equation for $C_L(P)$ is shown. It consists of a sum of terms. The first term is a circle labeled \mathcal{O}^\dagger on the left and a circle labeled \mathcal{O} on the right, connected by two arcs. Two black dots are placed on the upper and lower arcs between the two circles, enclosed in a dashed rectangular box. The second term is similar, but the middle circle is labeled iK . The third term has two middle circles, both labeled iK . The sequence continues with an ellipsis. A blue bracket on the right side of the equation groups the terms from the second term onwards. Inside this bracket, a smaller diagrammatic series is shown, starting with a plus sign, followed by a diagram with three dots on the arcs, then another plus sign, a diagram with two dots, and finally an ellipsis.

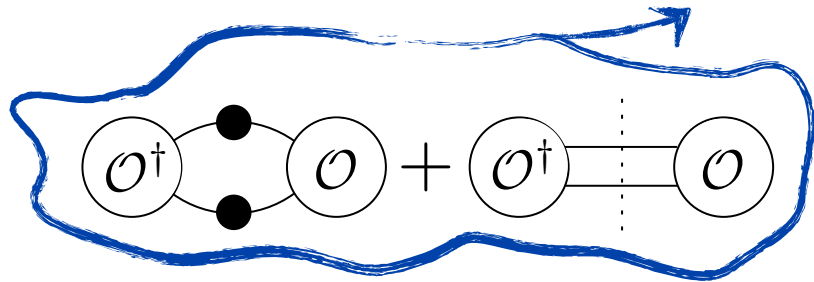
Review

1

$$C_L(P) = \text{diagram} + \text{diagram} + \text{diagram}$$

The first row shows the expansion of $C_L(P)$. It starts with a diagram of two vertices \mathcal{O}^\dagger and \mathcal{O} connected by two arcs, with two black dots in the middle. This is followed by a plus sign and a diagram where the middle vertex is iK . This is followed by another plus sign and a diagram enclosed in a blue hand-drawn cloud. This cloud contains a series of diagrams: a vertex \mathcal{O}^\dagger connected to a vertex \mathcal{O} by two arcs with two dots, followed by a plus sign, a vertex \mathcal{O}^\dagger connected to a vertex \mathcal{O} by a single horizontal line, followed by a plus sign, a diagram with two vertices and four arcs and four dots, followed by a plus sign, a diagram with two vertices and two arcs and two dots, followed by a plus sign and an ellipsis.

2



$$+ \text{diagram} + \text{diagram} + \dots$$

The second row continues the expansion. It starts with a plus sign and a diagram where the middle vertex is iK . This is followed by another plus sign and a diagram where the middle two vertices are iK . This is followed by a plus sign and an ellipsis.

Review

1

$$C_L(P) = \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \mathcal{O} + \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \mathcal{O} + \dots$$

2

$$C_L(P) = \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \mathcal{O} + \mathcal{O}^\dagger \text{---} \mathcal{O} + \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \mathcal{O} + \dots$$

$$C_L(P) = C_\infty(P)$$

$$+ \begin{array}{c} \text{---} \\ | \\ A \end{array} \begin{array}{c} \text{---} \\ | \\ A' \end{array} + \begin{array}{c} \text{---} \\ | \\ A \end{array} \begin{array}{c} \text{---} \\ | \\ i\mathcal{M} \end{array} \begin{array}{c} \text{---} \\ | \\ A' \end{array} + \dots$$

$$+ \begin{array}{c} \text{---} \\ | \\ A \end{array} \begin{array}{c} \text{---} \\ | \\ i\mathcal{M} \end{array} \begin{array}{c} \text{---} \\ | \\ i\mathcal{M} \end{array} \begin{array}{c} \text{---} \\ | \\ A' \end{array} + \dots$$

3

$\langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$
 $\langle 0 | \mathcal{O} | \pi\pi, \text{in} \rangle$

Review

$$C_L(P) = \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \dots$$

1

2

$$+ \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \dots$$

$$C_L(P) = C_\infty(P)$$

$$+ \begin{array}{c} A \\ \vdots \\ F \end{array} \begin{array}{c} A' \\ \vdots \\ F \end{array} + \begin{array}{c} A \\ \vdots \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \vdots \\ F \end{array} \begin{array}{c} A' \\ \vdots \\ F \end{array} + \dots$$

3

$$+ \begin{array}{c} A \\ \vdots \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \vdots \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \vdots \\ F \end{array} \begin{array}{c} A' \\ \vdots \\ F \end{array} + \dots$$

4

$\langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$

$\langle 0 | \mathcal{O} | \pi\pi, \text{in} \rangle$

$$C_L(P) = C_\infty(P) + A' iF \frac{1}{1 - i\mathcal{M}_{2 \rightarrow 2} iF} A$$

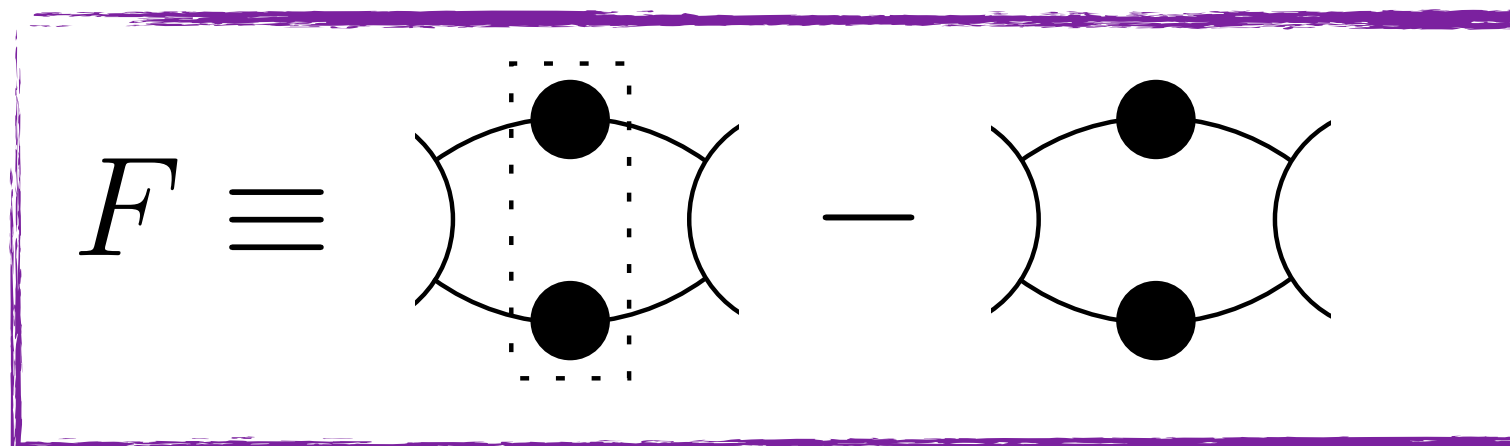
Quantization Condition

At fixed (L, \vec{P}) , finite-volume spectrum
is all solutions to

$$\Delta_{L,P}(E) = \det[1 - \underbrace{i\mathcal{M}_{2 \rightarrow 2}}_{\text{kinematic, not diagonal (related to Lüscher Zeta function)}} \underbrace{iF}_{\text{diagonal matrix in angular momentum space}}] = 0$$

diagonal matrix in
angular momentum space

kinematic, not diagonal
(related to Lüscher
Zeta function)



Quantization Condition

$$\Delta_{L,P}(E) = \det[1 - i\mathcal{M}_{2 \rightarrow 2} iF] = 0$$

...is it useful?

At low energies, s-wave dominates

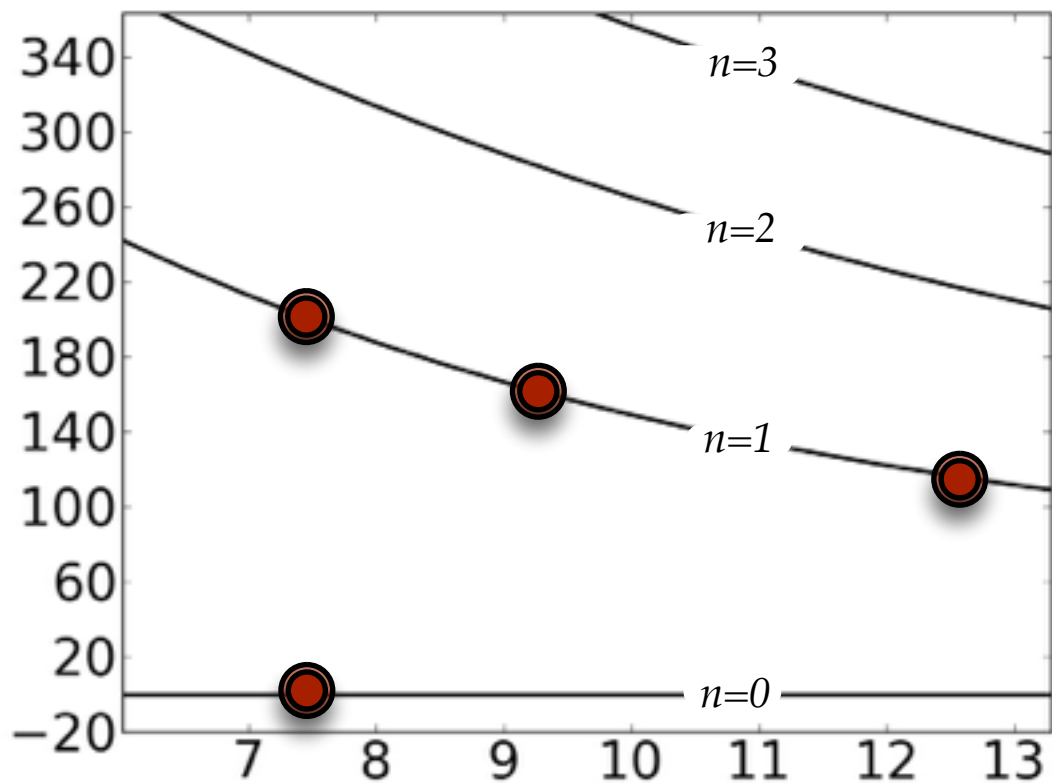
$$[\mathcal{M}_{2 \rightarrow 2}^s(E_n^*)]^{-1} = -F^s(E_n, \vec{P}, L)$$

$$\left(F^s(E, \vec{P}, L) \equiv \frac{1}{2} \left[\frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^3 k}{(2\pi)^3} \right] \frac{1}{2\omega_k 2\omega_{P-k} (E - \omega_k - \omega_{P-k} + i\epsilon)} \right)$$

$$[\mathcal{M}_{2 \rightarrow 2}^s(E_n^*)]^{-1} = -F^s(E_n, \vec{P}, L)$$

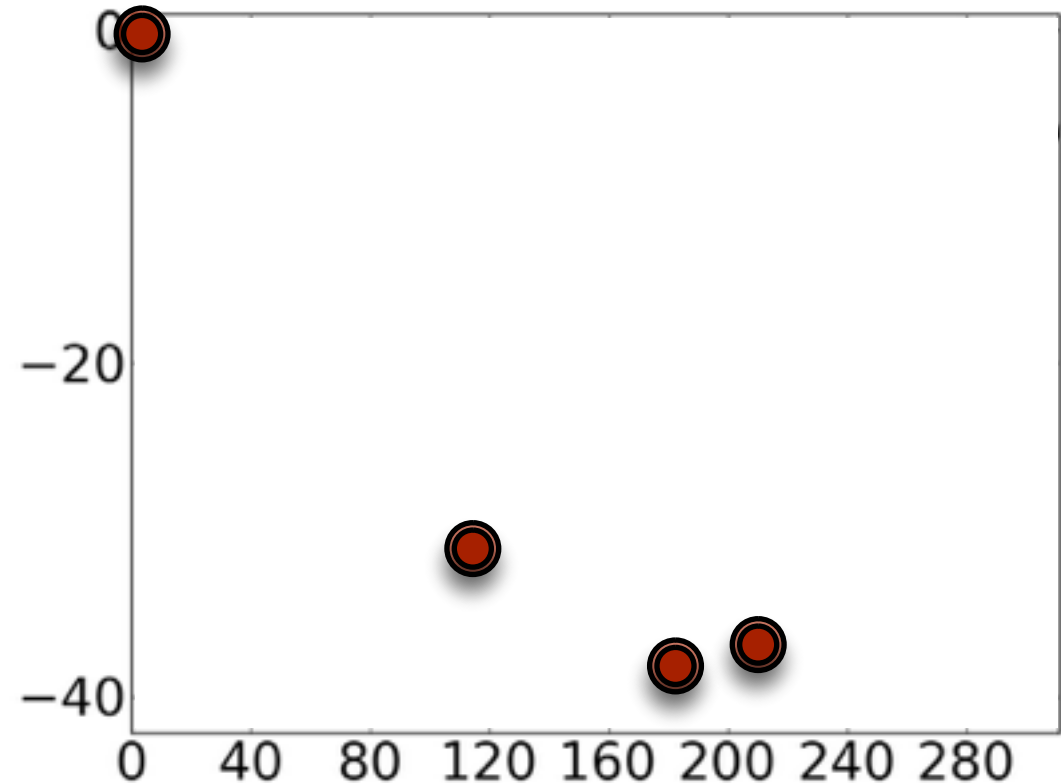
$$\mathcal{M}_{2 \rightarrow 2}^s(E) = \frac{16\pi E}{p \cot \delta(p) - ip}$$

p^* [MeV]



L [fm]

δ [degrees]

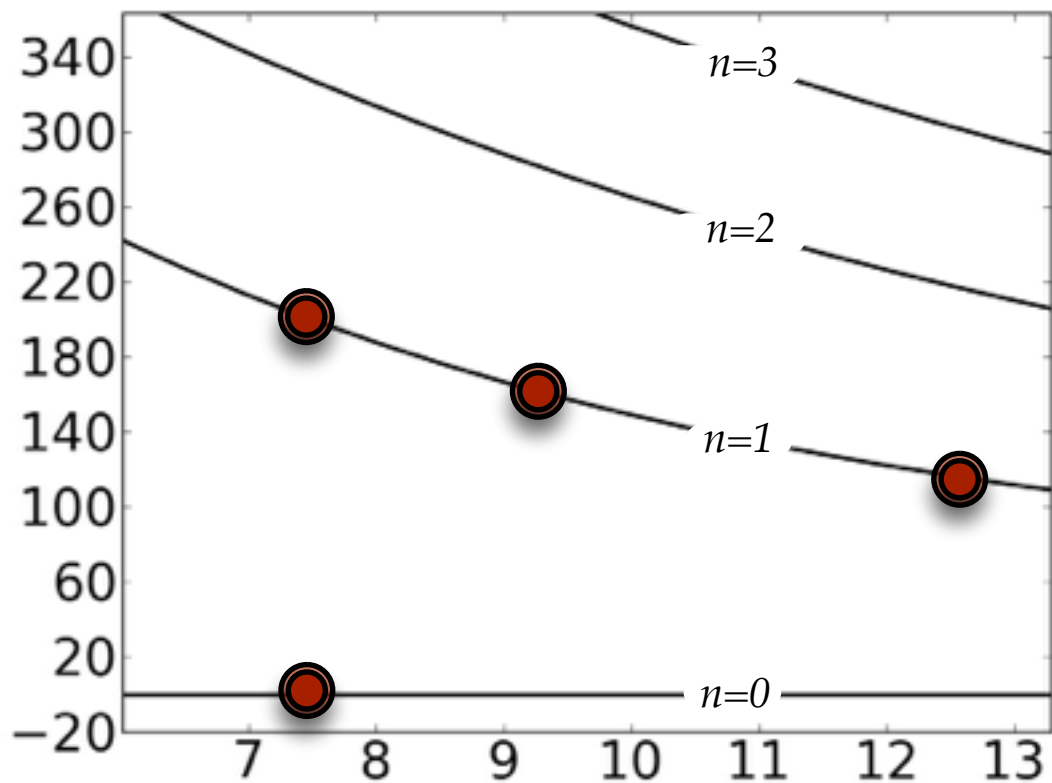


p^* [MeV]

$$[\mathcal{M}_{2 \rightarrow 2}^s(E_n^*)]^{-1} = -F^s(E_n, \vec{P}, L)$$

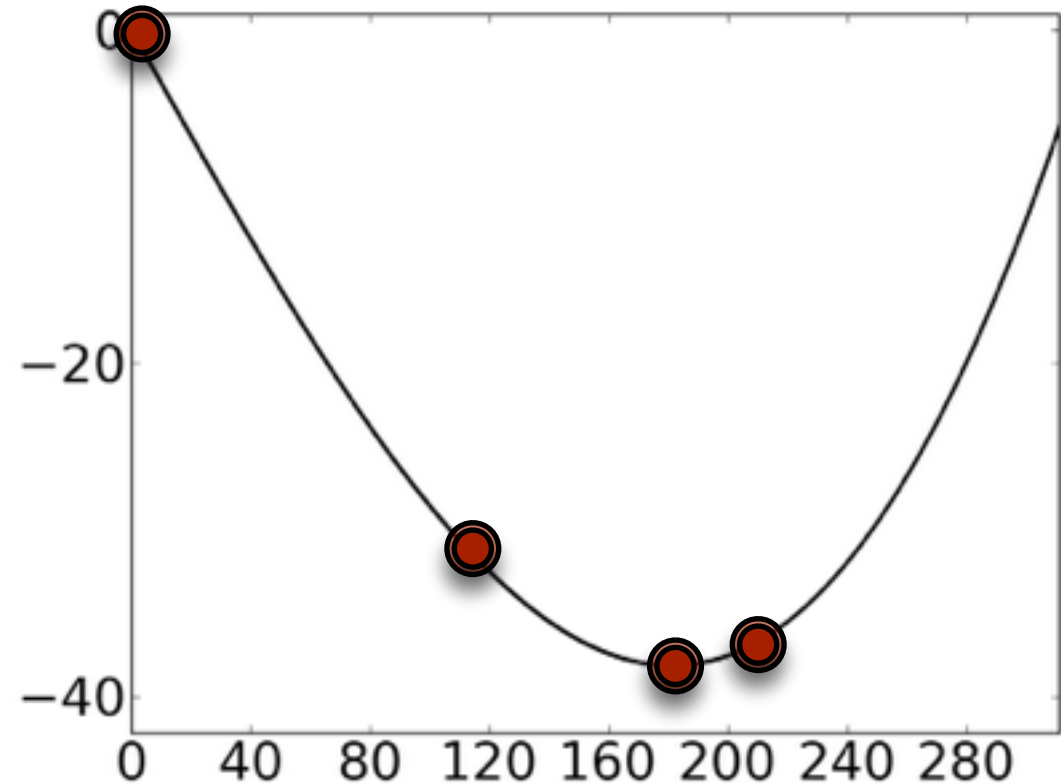
$$\mathcal{M}_{2 \rightarrow 2}^s(E) = \frac{16\pi E}{p \cot \delta(p) - ip}$$

p^* [MeV]



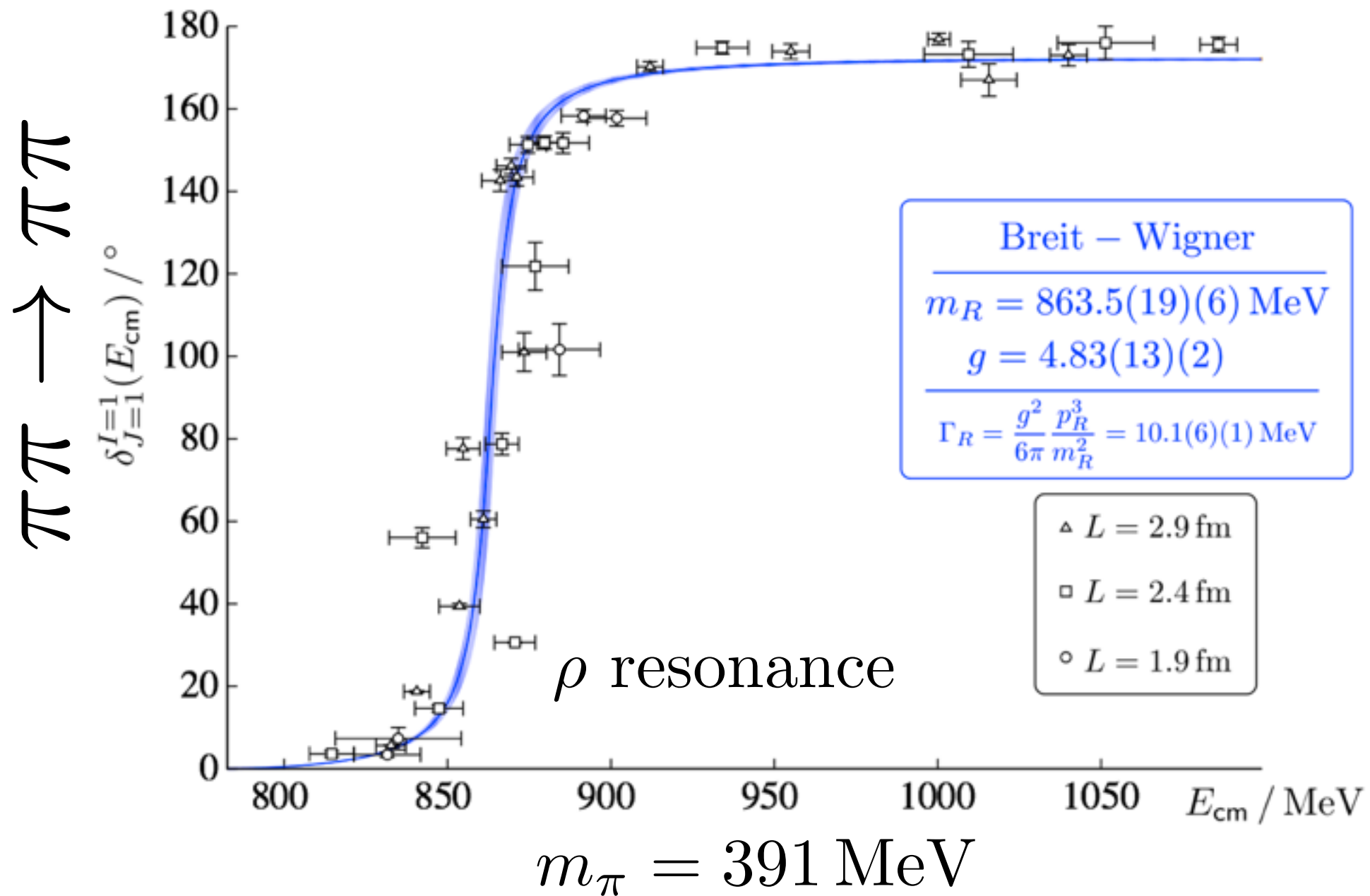
L [fm]

δ [degrees]



p^* [MeV]

$$p_n \cot \delta_{J=1}(p_n) = -16\pi E_n^* \operatorname{Re} F_{10;10}(E_n, \vec{P}, L)$$

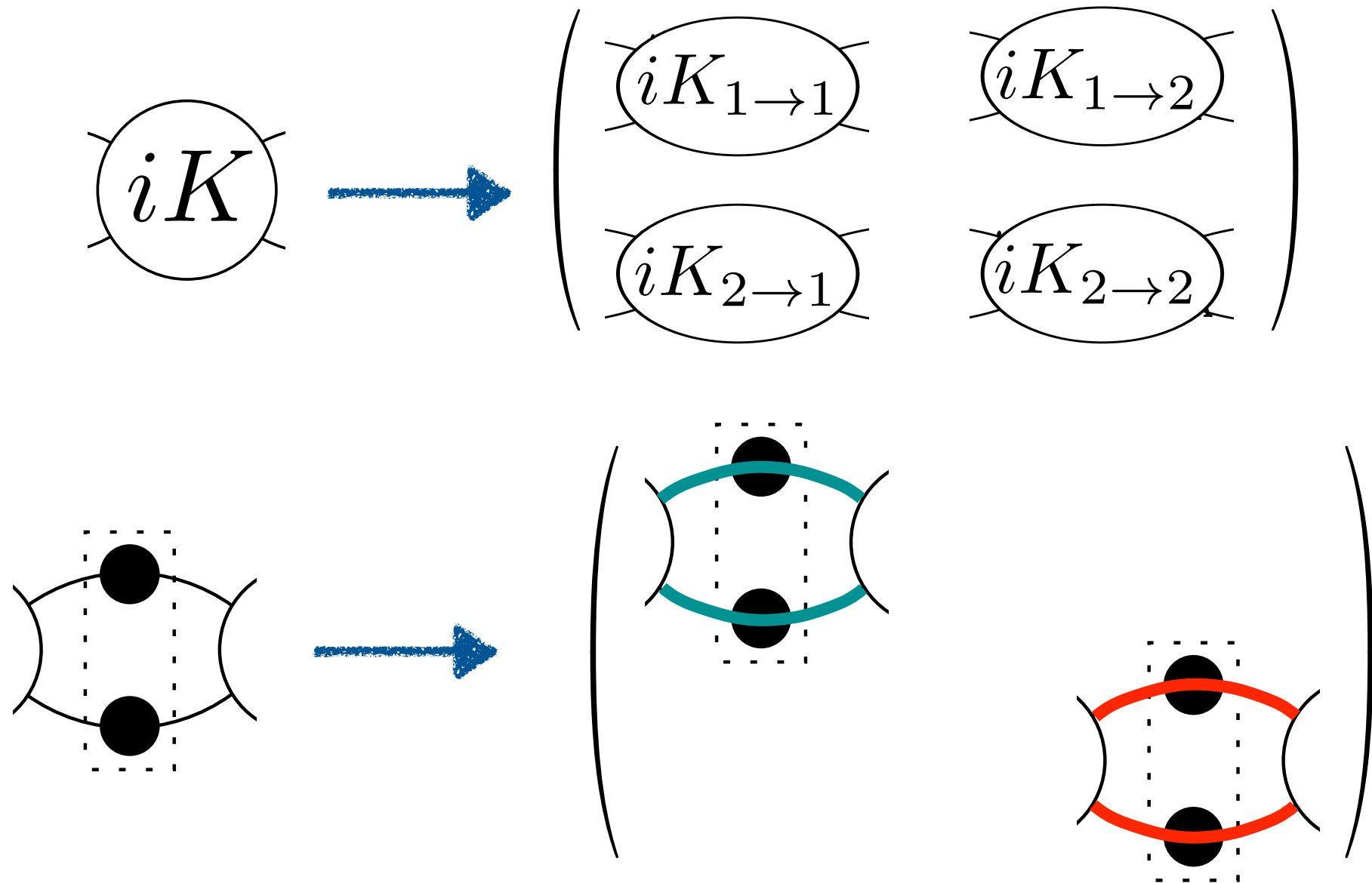


from Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505

Scattering of multiple two-particle channels

$$\pi\pi \rightarrow \bar{K}K \quad \pi K \rightarrow \eta K$$

Make following replacements



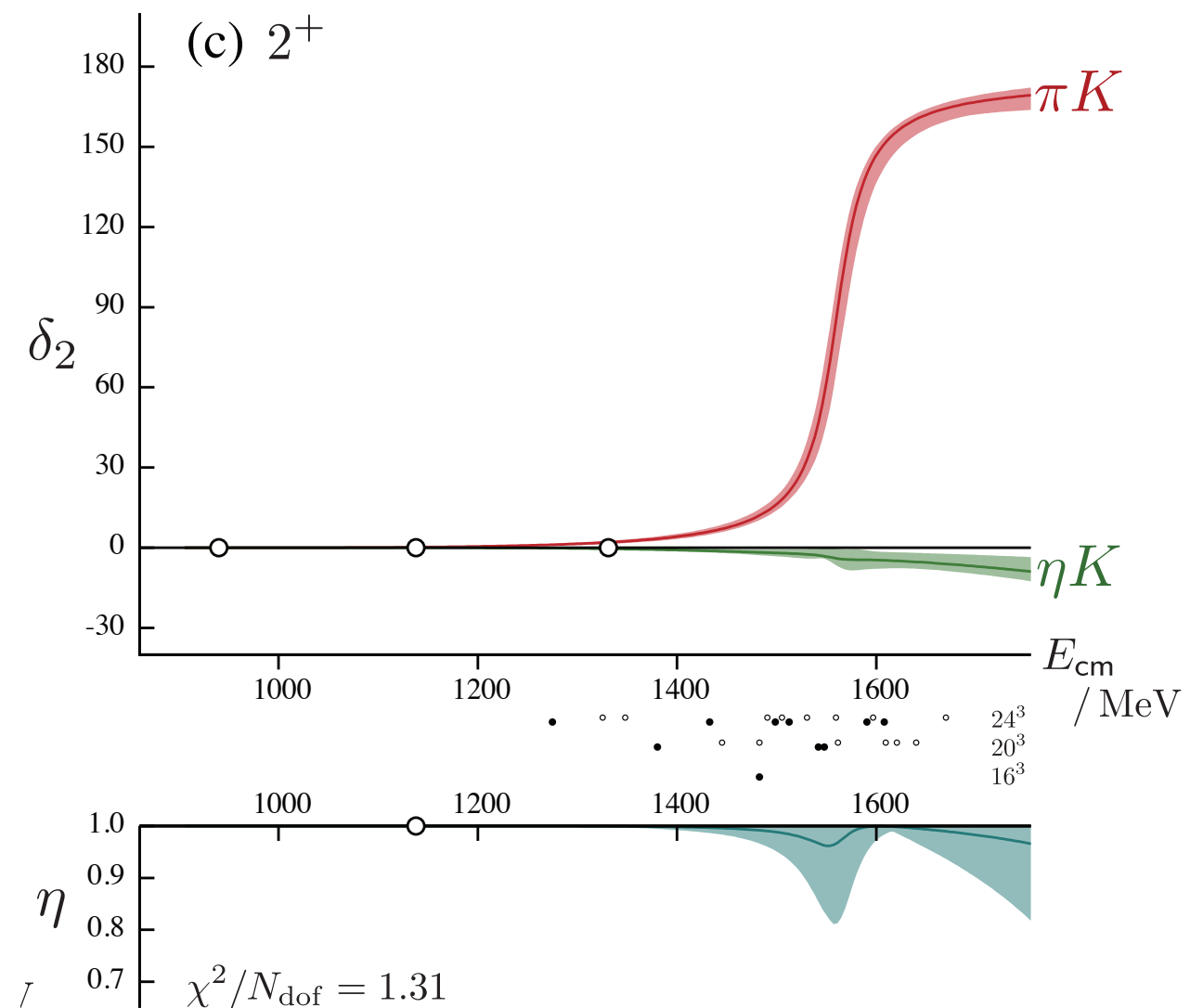
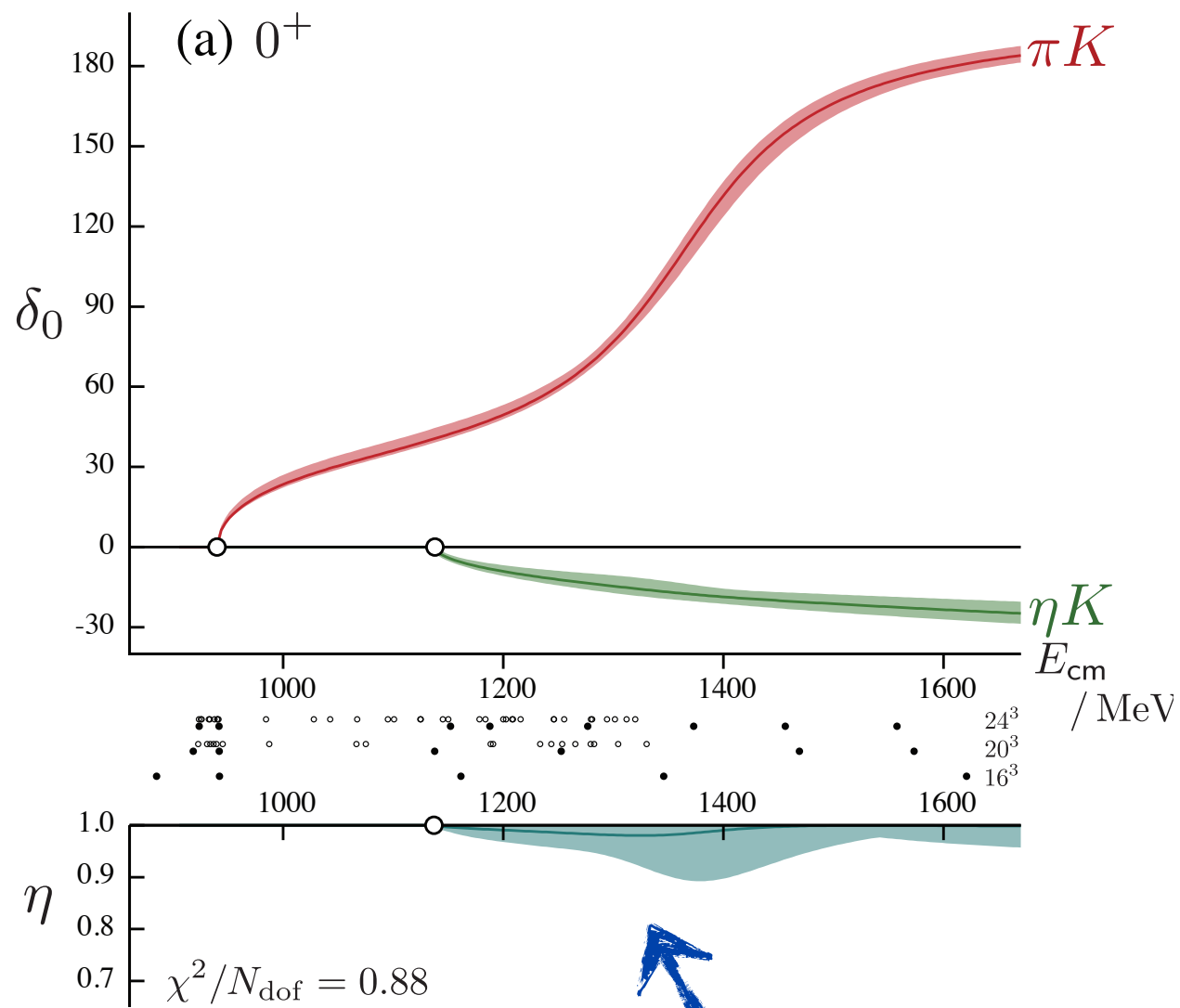
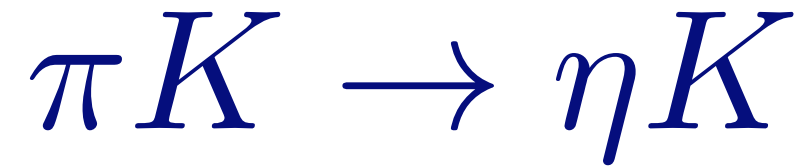
Scattering of multiple two-particle channels

$$\pi\pi \rightarrow \bar{K}K \quad \pi K \rightarrow \eta K$$

One finds

$$\det \left[1 - \begin{pmatrix} i\mathcal{M}_{1\rightarrow 1} & i\mathcal{M}_{1\rightarrow 2} \\ i\mathcal{M}_{2\rightarrow 1} & i\mathcal{M}_{2\rightarrow 2} \end{pmatrix} \begin{pmatrix} iF_1 & 0 \\ 0 & iF_2 \end{pmatrix} \right] = 0$$

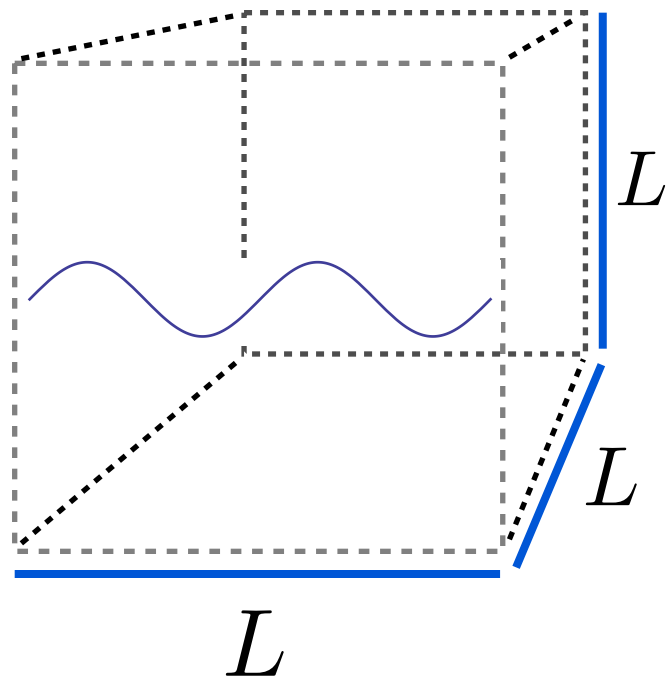
Already implemented in LQCD calculation



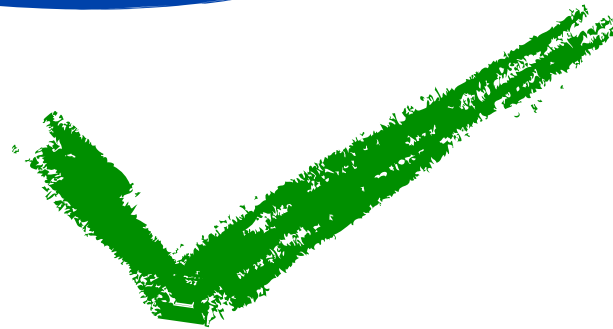
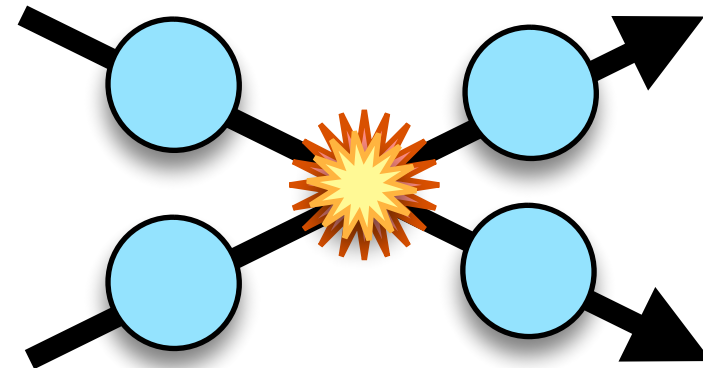
$$\mathcal{M}(\pi K \rightarrow \eta K) \sim \sqrt{1 - \eta^2}$$

from Dudek, Edwards, Thomas, Wilson in arXiv:1406:4158

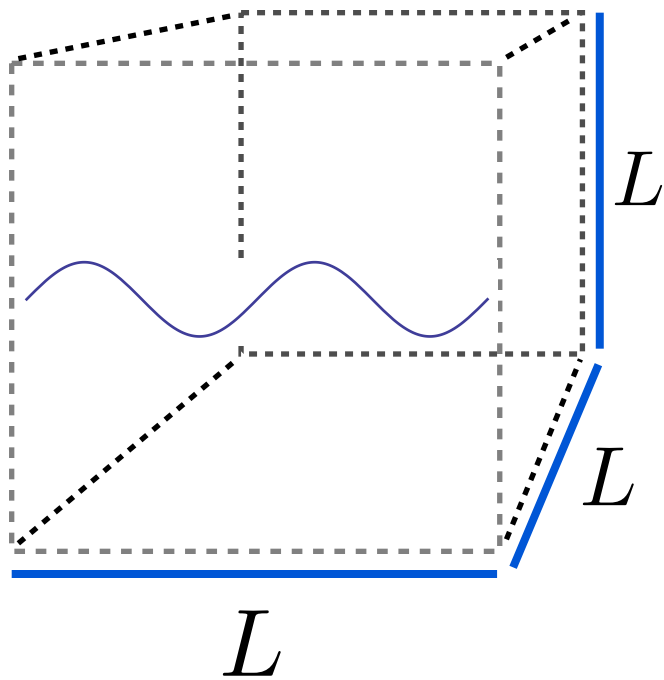
Finite volume



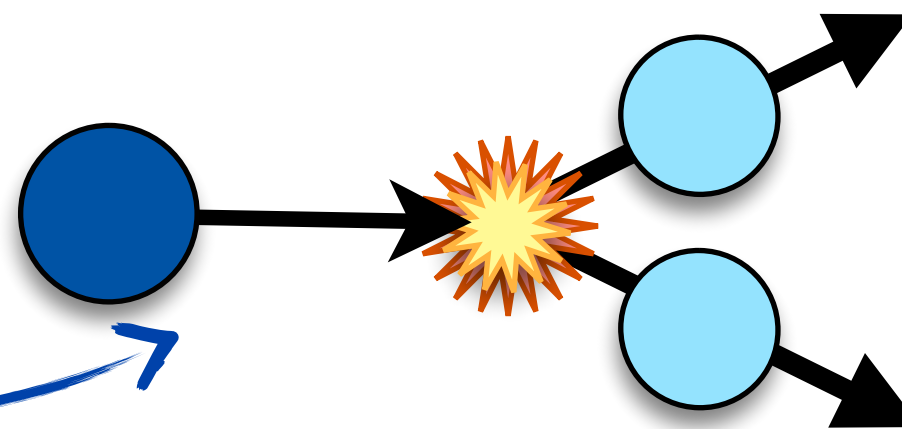
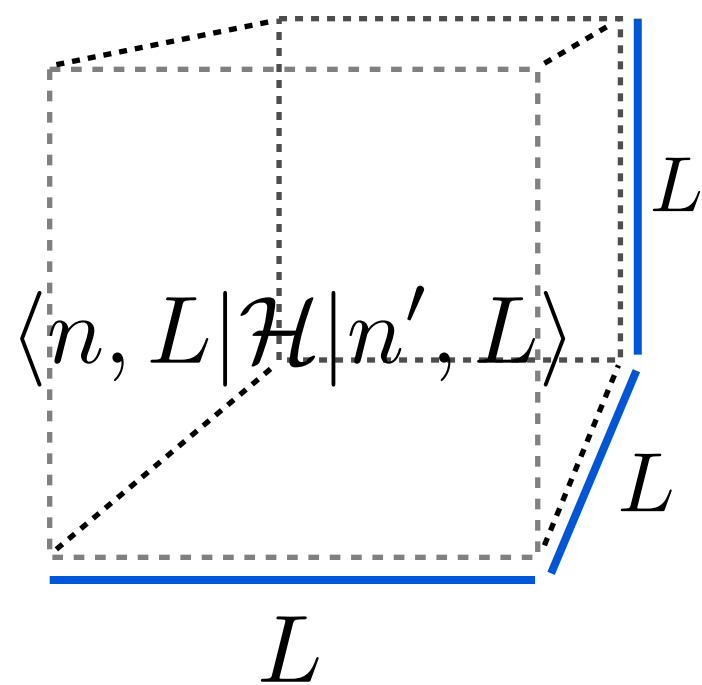
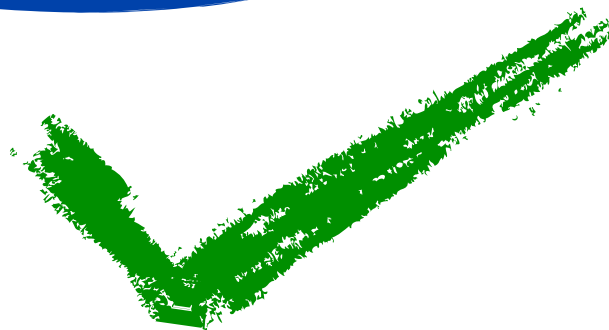
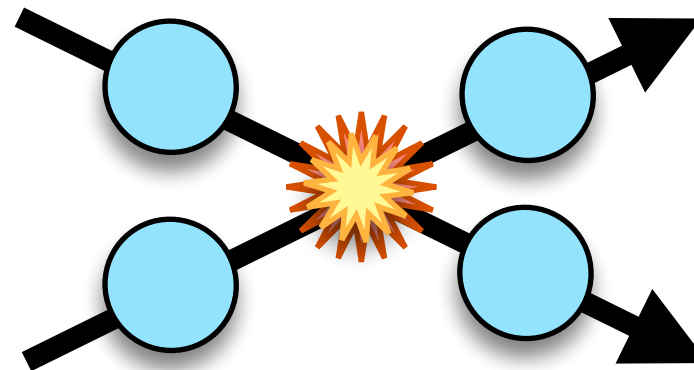
Infinite volume



Finite volume



Infinite volume



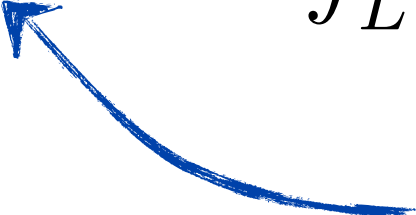
Relate finite- and infinite-volume states

$$C_L(x_0 - y_0, \mathbf{P}) \equiv \int_L d^3x \int_L d^3y e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(y)|0\rangle$$

Relate finite- and infinite-volume states

$$C_L(x_0 - y_0, \mathbf{P}) \equiv \int_L d^3x \int_L d^3y e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(y)|0\rangle$$

before P_0, \mathbf{P}
now $x_0 - y_0, \mathbf{P}$

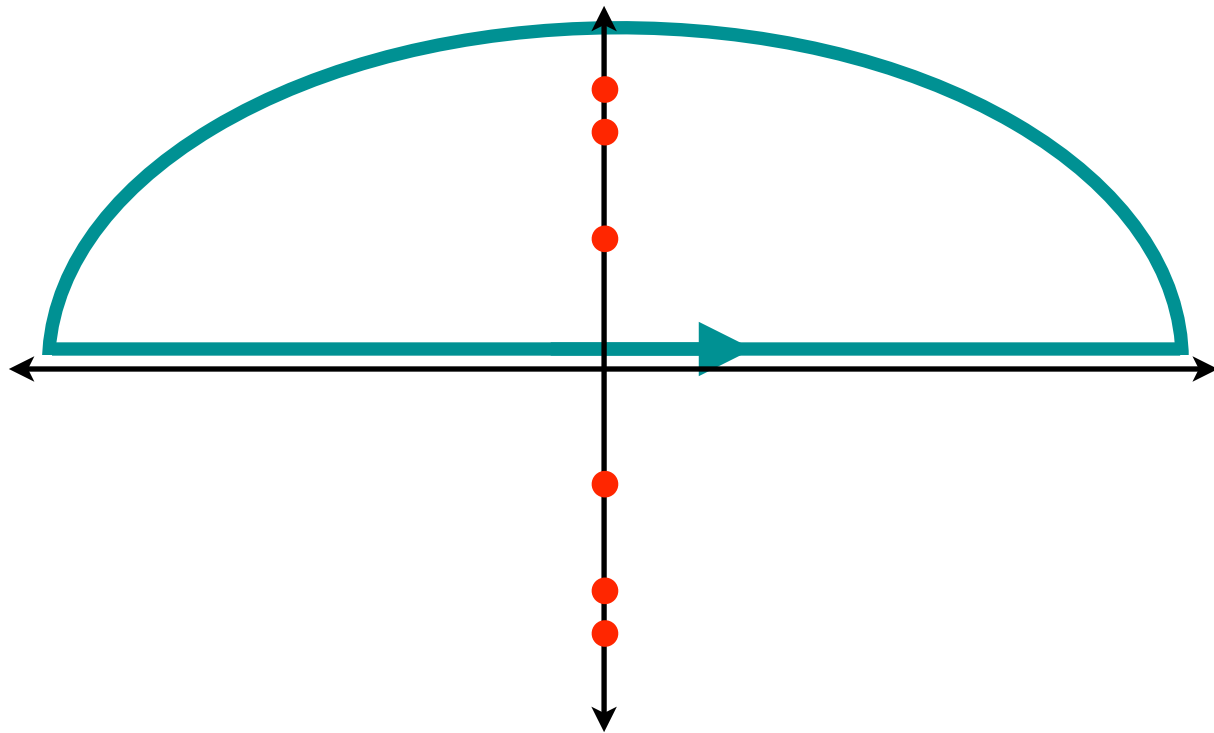


Relate finite- and infinite-volume states

$$C_L(x_0 - y_0, \mathbf{P}) \equiv \int_L d^3x \int_L d^3y e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(y)|0\rangle$$
$$= L^3 \int \frac{dP_0}{2\pi} e^{iP_0(x_0-y_0)} \left[C_\infty(P) + A' \frac{1}{(iF)^{-1} - i\mathcal{M}_{2\rightarrow 2}} A \right]$$

Relate finite- and infinite-volume states

$$C_L(x_0 - y_0, \mathbf{P}) \equiv \int_L d^3x \int_L d^3y e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(y)|0\rangle$$
$$= L^3 \int \frac{dP_0}{2\pi} e^{iP_0(x_0-y_0)} \left[C_\infty(P) + A' \frac{1}{(iF)^{-1} - i\mathcal{M}_{2\rightarrow 2}} A \right]$$



C_L analytic structure

Relate finite- and infinite-volume states

$$\begin{aligned}
 C_L(x_0 - y_0, \mathbf{P}) &\equiv \int_L d^3x \int_L d^3y e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(y)|0\rangle \\
 &= L^3 \int \frac{dP_0}{2\pi} e^{iP_0(x_0-y_0)} \left[C_\infty(P) + A' \frac{1}{(iF)^{-1} - i\mathcal{M}_{2\rightarrow 2}} A \right] \\
 &= \sum_n e^{-E_{n,L}(x_0-y_0)} \overbrace{\langle 0|\mathcal{O}(0, \mathbf{P})|\pi\pi, \text{in}\rangle_n} \mathcal{R}_n \overbrace{\langle \pi\pi, \text{out}|\mathcal{O}^\dagger(0, -\mathbf{P})|0\rangle_n}
 \end{aligned}$$

$$\mathcal{R}_n \equiv \text{Residue of } \frac{1}{(iF)^{-1} - i\mathcal{M}_{2\rightarrow 2}} \text{ at } E_{n,L}$$

Relate finite- and infinite-volume states

$$\begin{aligned}
 C_L(x_0 - y_0, \mathbf{P}) &\equiv \int_L d^3x \int_L d^3y e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(y)|0\rangle \\
 &= L^3 \int \frac{dP_0}{2\pi} e^{iP_0(x_0-y_0)} \left[C_\infty(P) + A' \frac{1}{(iF)^{-1} - i\mathcal{M}_{2\rightarrow 2}} A \right] \\
 &= \sum_n e^{-E_{n,L}(x_0-y_0)} \langle 0|\mathcal{O}(0, \mathbf{P})| \underbrace{\pi\pi, \text{in}}_n \mathcal{R}_n \langle \pi\pi, \text{out}|\mathcal{O}^\dagger(0, -\mathbf{P})|0\rangle_n \\
 &\quad |n, L, \text{"}\pi\pi\text{"}\rangle \langle n, L, \text{"}\pi\pi\text{"}|
 \end{aligned}$$

$$\mathcal{R}_n \equiv \text{Residue of } \frac{1}{(iF)^{-1} - i\mathcal{M}_{2\rightarrow 2}} \text{ at } E_{n,L}$$

Relation on states: Single channel

$$\begin{aligned} |n, L, \text{"}\pi\pi\text{"}\rangle \langle n, L, \text{"}\pi\pi\text{"}| &= \\ \left(|\pi\pi, \text{in}, J=0\rangle \quad |\pi\pi, \text{in}, J=1\rangle \quad \dots \right) & \left(\begin{array}{c} \mathcal{R}_n \end{array} \right) \left(\begin{array}{c} \langle \pi\pi, \text{out}, J=0| \\ \langle \pi\pi, \text{out}, J=1| \\ \vdots \end{array} \right) \end{aligned}$$

$$\mathcal{R}_n \equiv \text{Residue of } \frac{1}{(iF)^{-1} - i\mathcal{M}_{2\rightarrow 2}} \text{ at } E_{n,L}$$

- Matrix in angular momentum space
- Depends on $\mathcal{M}_{2\rightarrow 2}$, $d\mathcal{M}_{2\rightarrow 2}/dE$, L
- Generalization of the Lellouch-Lüscher factor
- Just a normalization factor in case of a single channel

Relation on states: Coupled channels

$$|n, L\rangle \langle n, L| = \begin{pmatrix} |\phi_1\phi_2, J=0\rangle & |\phi_3\phi_4, J=0\rangle & \dots \end{pmatrix} \begin{pmatrix} \mathcal{R}_n \end{pmatrix} \begin{pmatrix} \langle \phi_1\phi_2, J=0| \\ \langle \phi_3\phi_4, J=0| \\ \langle \phi_1\phi_2, J=1| \\ \langle \phi_3\phi_4, J=1| \\ \vdots \end{pmatrix}$$

$\mathcal{R}_n \equiv$ Residue of $\frac{1}{(iF)^{-1} - i\mathcal{M}_{2\rightarrow 2}}$ at $E_{n,L}$

- Matrix in combined

angular momentum and channel space

- Quantifies mixing of angular momentum and channels due to finite volume

Back to...

How can one use numerical Lattice QCD to determine

$$\langle \phi_1(p), \phi_2(P_f - p), \text{out} | \tilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q}) | \Phi, P_i \rangle ?$$


Back to...

How can one use numerical Lattice QCD to determine

$$\langle \phi_1(p), \phi_2(P_f - p), \text{out} | \tilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q}) | \Phi, P_i \rangle ?$$


Contract the relation for two-particle states with $\tilde{\mathcal{J}}(\mathbf{Q}) | \Phi \rangle$

for single particle state, difference between finite and

infinite-volume is exponentially suppressed e^{-mL}



Back to...

How can one use numerical Lattice QCD to determine

$$\langle \phi_1(p), \phi_2(P_f - p), \text{out} | \tilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q}) | \Phi, P_i \rangle ?$$

Contract the relation for two-particle states with $\tilde{\mathcal{J}}(\mathbf{Q})|\Phi\rangle$

$$\langle \Phi | \tilde{\mathcal{J}}^\dagger(\mathbf{Q}) |n, L\rangle \langle n, L| = (|\phi_1\phi_2, J=0\rangle \quad |\phi_3\phi_4, J=0\rangle \quad \dots) \left(\mathcal{R}_n \right) \begin{pmatrix} \langle \phi_1\phi_2, J=0| \\ \langle \phi_3\phi_4, J=0| \\ \langle \phi_1\phi_2, J=1| \\ \langle \phi_3\phi_4, J=1| \\ \vdots \end{pmatrix} \tilde{\mathcal{J}}(\mathbf{Q})|\Phi\rangle$$

Back to...

How can one use numerical Lattice QCD to determine

$$\langle \phi_1(p), \phi_2(P_f - p), \text{out} | \tilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q}) | \Phi, P_i \rangle ?$$

Contract the relation for two-particle states with $\tilde{\mathcal{J}}(\mathbf{Q})|\Phi\rangle$

$$\langle \Phi | \tilde{\mathcal{J}}^\dagger(\mathbf{Q}) |n, L\rangle \langle n, L| = (|\phi_1\phi_2, J=0\rangle \quad |\phi_3\phi_4, J=0\rangle \quad \dots) \left(\mathcal{R}_n \right) \begin{pmatrix} \langle \phi_1\phi_2, J=0| \\ \langle \phi_3\phi_4, J=0| \\ \langle \phi_1\phi_2, J=1| \\ \langle \phi_3\phi_4, J=1| \\ \vdots \end{pmatrix} \tilde{\mathcal{J}}(\mathbf{Q})|\Phi\rangle$$

In the paper we present an alternative derivation in which we explicitly calculate three-point correlators in finite-volume

Master equation

$$|\langle E_{n_f}, \mathbf{P}_f, L | \tilde{\mathcal{J}}_A(0, \mathbf{Q}) | E_i, \mathbf{P}_i, L \rangle| = \frac{1}{\sqrt{2E_i}} \sqrt{\mathcal{A}_{n_f}^\dagger \mathcal{R}_{n_f} \mathcal{A}_{n_f}}$$

$$\mathcal{A}_{n_f} (2\pi)^3 \delta^3(\mathbf{P}_2 + \mathbf{Q} - \mathbf{P}_f) \equiv \begin{pmatrix} \langle \phi_1 \phi_2, J = 0 | \tilde{\mathcal{J}}_A(\mathbf{Q}) | \Phi \rangle \\ \langle \phi_3 \phi_4, J = 0 | \tilde{\mathcal{J}}_A(\mathbf{Q}) | \Phi \rangle \\ \langle \phi_1 \phi_2, J = 1 | \tilde{\mathcal{J}}_A(\mathbf{Q}) | \Phi \rangle \\ \langle \phi_3 \phi_4, J = 1 | \tilde{\mathcal{J}}_A(\mathbf{Q}) | \Phi \rangle \\ \vdots \end{pmatrix}$$

**column vector containing
all transition amplitudes with given quantum numbers**

Master equation

$$|\langle E_{n_f}, \mathbf{P}_f, L | \tilde{\mathcal{J}}_A(0, \mathbf{Q}) | E_i, \mathbf{P}_i, L \rangle| = \frac{1}{\sqrt{2E_i}} \sqrt{\mathcal{A}_{n_f}^\dagger \mathcal{R}_{n_f} \mathcal{A}_{n_f}}$$

Model independent & non-perturbative

Universal: lattice QCD, lattice EFT, etc

Not quite arbitrary quantum numbers:

degenerate or non-degenerate masses, arbitrary momenta and angular momenta... but no intrinsic spin

Asymmetric volumes and boundary conditions:

periodic, anti-periodic, any linear combination and any rectangular prism

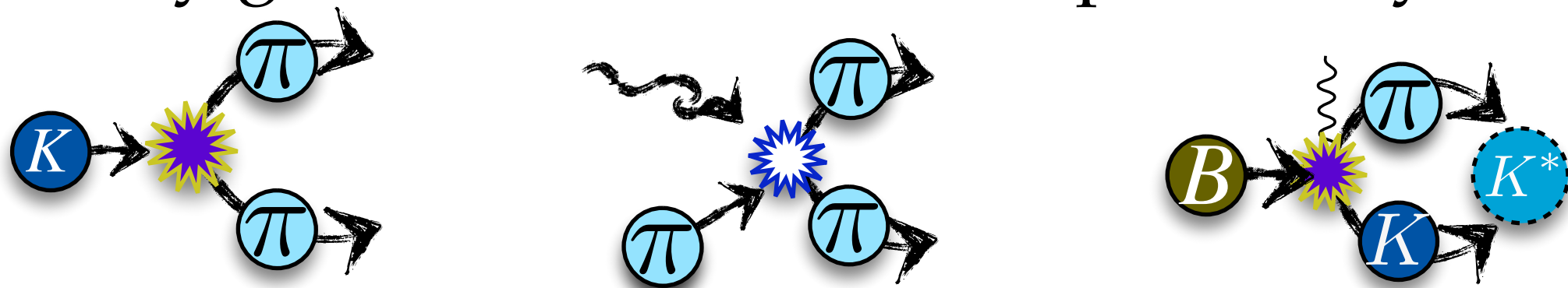
Conclusion

$$\det[1 - i\mathcal{M}_{2\rightarrow 2}iF] = 0$$

$$|\langle E_{n_f}, \mathbf{P}_f, L | \tilde{\mathcal{J}}_A(0, \mathbf{Q}) | E_i, \mathbf{P}_i, L \rangle| = \frac{1}{\sqrt{2E_i}} \sqrt{\mathcal{A}_{n_f}^\dagger \mathcal{R}_{n_f} \mathcal{A}_{n_f}}$$

Presented formalism for extracting
two-to-two scattering and
one-to-two matrix elements using numerical LQCD

Completely general result for scalar particle systems



Next steps... include spin, **three particle states**

see MTH, S. R. Sharpe, arXiv:1408.5933 to appear in *Phys.Rev. D*

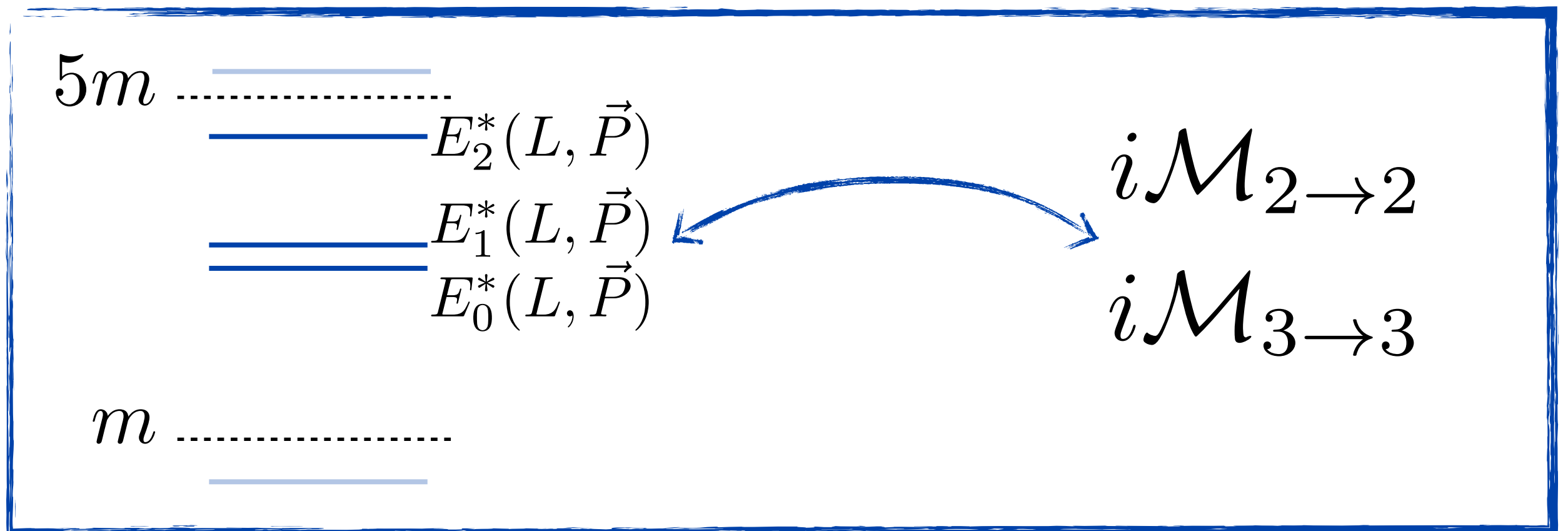


Three-to-three scattering $\pi\pi\pi \rightarrow \pi\pi\pi$

$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | T \sigma(x) \sigma^\dagger(0) | 0 \rangle$$

Require $m < E^* < 5m$

odd-particle quantum numbers



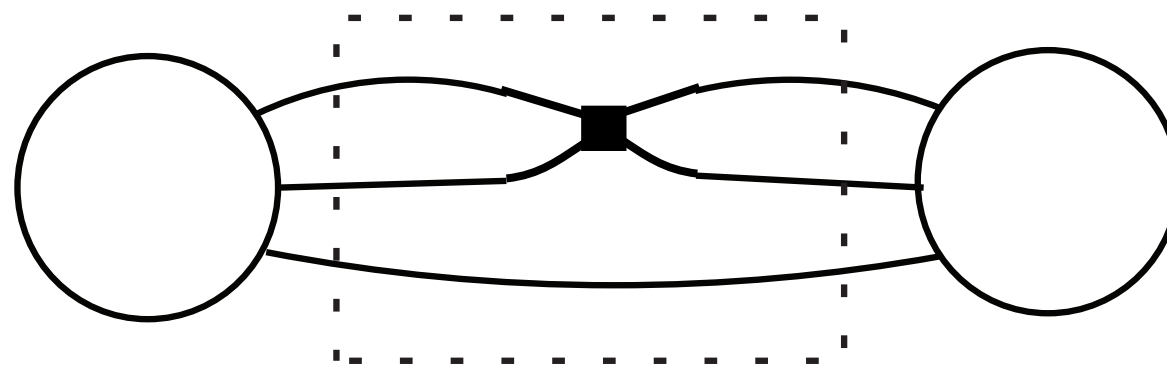
Assume no two-particle bound state

New skeleton expansion

$$C_L(E, \vec{P}) \stackrel{?}{=} \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

(propagators still fully dressed)

No! We also need diagrams like



( **should only contain connected diagrams**)

New skeleton expansion

$$C_L(E, \vec{P}) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

$$+ \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots$$

The diagrams in the expansion are:

- Diagram 1: Two white circles connected by two arcs, enclosed in a dashed box.
- Diagram 2: A white circle, a dashed box containing two arcs, an orange circle, a dashed box containing two arcs, and another white circle.
- Diagram 3: A white circle, a dashed box containing two arcs, an orange circle, a dashed box containing two arcs, another orange circle, a dashed box containing two arcs, and a final white circle.
- Diagram 4: A white circle, a dashed box containing a purple circle and two arcs, and another white circle.
- Diagram 5: A white circle, a dashed box containing two purple circles and two arcs, and another white circle.
- Diagram 6: A white circle, a dashed box containing three purple circles and two arcs, and another white circle.

Kernel definitions:

$$\text{Purple circle} \equiv \text{Diagram A} + \text{Diagram B} + \text{Diagram C} + \dots$$

The diagrams on the right are:

- Diagram A: A vertex with four external lines.
- Diagram B: A vertex with four external lines and two internal arcs.
- Diagram C: A vertex with four external lines and two internal arcs forming a lens shape.

$$\text{Orange circle} \equiv \text{Diagram D} + \text{Diagram E} + \text{Diagram F} + \dots$$

The diagrams on the right are:

- Diagram D: A vertex with four external lines.
- Diagram E: A vertex with four external lines and a horizontal internal line.
- Diagram F: A vertex with four external lines and two internal arcs.

New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots
 \end{aligned}$$

The diagrams in the expansion are as follows:

- Row 1: Two white circles connected by two lines. The first diagram has a dashed box around the first line. The second has a dashed box around the second line. The third has dashed boxes around both lines. The fourth has a dashed box around the entire structure.
- Row 2: Similar to Row 1, but with a purple circle on the first line. The second diagram has purple circles on both lines. The third has purple circles on all three lines.
- Row 3: Similar to Row 2, but with purple circles on both lines. The second diagram has purple circles on all three lines. The third has purple circles on all four lines.
- Row 4: Similar to Row 3, but with purple circles on all three lines. The second diagram has purple circles on all four lines.

Kernel definitions:

$$\begin{aligned}
 \text{Purple circle} & \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 \text{Orange circle} & \equiv \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots
 \end{aligned}$$

The kernel definitions are as follows:

- Purple circle:** Equivalent to a sum of diagrams: a vertex with four external lines, a vertex with four external lines and two internal lines forming a loop, and a vertex with four external lines and two internal lines forming a figure-eight shape.
- Orange circle:** Equivalent to a sum of diagrams: a vertex with four external lines, a vertex with four external lines and two internal lines forming a straight line, and a vertex with four external lines and two internal lines forming a loop.

New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

The diagrams in the expansion are Feynman diagrams with two external white circles. Diagrams 1-3 have orange internal circles, while diagrams 4-11 have purple internal circles. Diagrams 12-13 have orange internal circles. Dashed boxes in the diagrams indicate the skeleton structure.

Kernel definitions:

$$\begin{aligned}
 \text{Purple circle} & \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 \text{Orange circle} & \equiv \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots
 \end{aligned}$$

The kernel definitions show the decomposition of the colored circles into sums of diagrams with external legs. The purple circle is defined by diagrams with two external legs, and the orange circle is defined by diagrams with four external legs.

New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

The diagrams in the expansion are as follows:

- Row 1: Three diagrams showing two white circles connected by two lines. The first diagram has a dashed box around the lines. The second diagram has an orange circle on the top line between the dashed boxes. The third diagram has two orange circles on the top line between the dashed boxes.
- Row 2: Three diagrams showing two white circles connected by two lines. The first diagram has a purple circle on the top line between the dashed boxes. The second diagram has two purple circles on the top line between the dashed boxes. The third diagram has three purple circles on the top line between the dashed boxes.
- Row 3: Three diagrams showing two white circles connected by two lines. The first diagram has two purple circles on the top line between the dashed boxes. The second diagram has three purple circles on the top line between the dashed boxes. The third diagram has four purple circles on the top line between the dashed boxes.
- Row 4: Two diagrams showing two white circles connected by two lines. The first diagram has three purple circles on the top line between the dashed boxes. The second diagram has four purple circles on the top line between the dashed boxes.
- Row 5: Two diagrams showing two white circles connected by two lines. The first diagram has a purple circle on the top line between the dashed boxes and an orange circle on the top line between the dashed boxes. The second diagram has an orange circle on the top line between the dashed boxes and a purple circle on the top line between the dashed boxes.

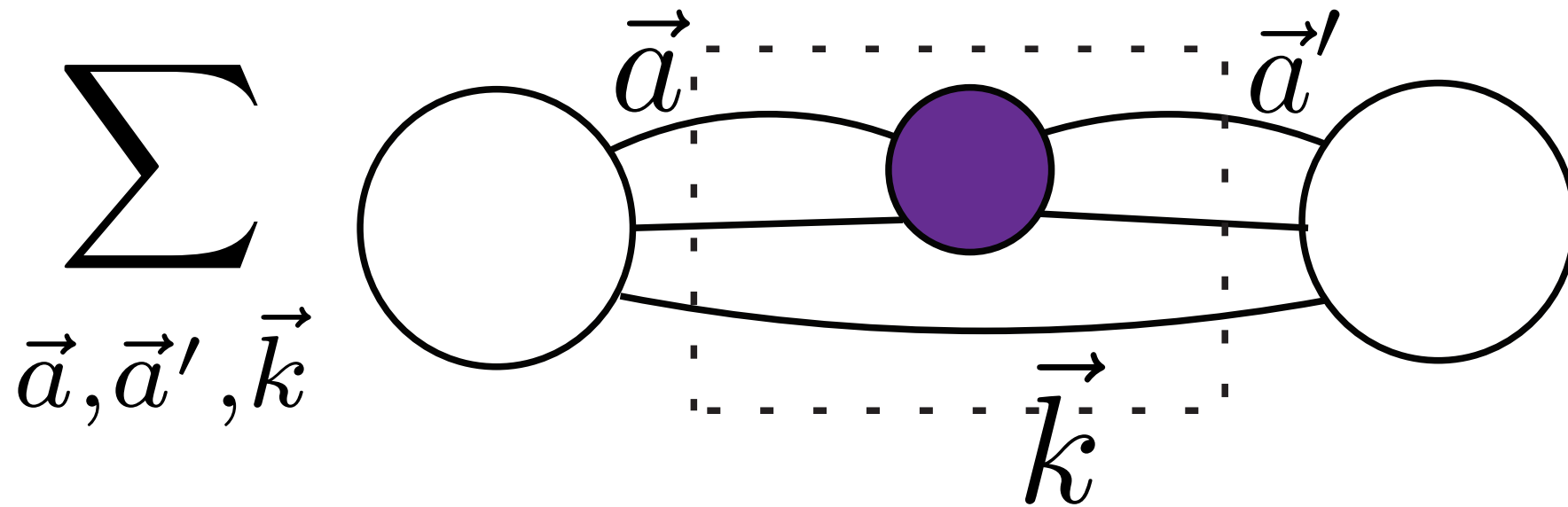
Compare to two-particle skeleton expansion

$$C_L(E, \vec{P}) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

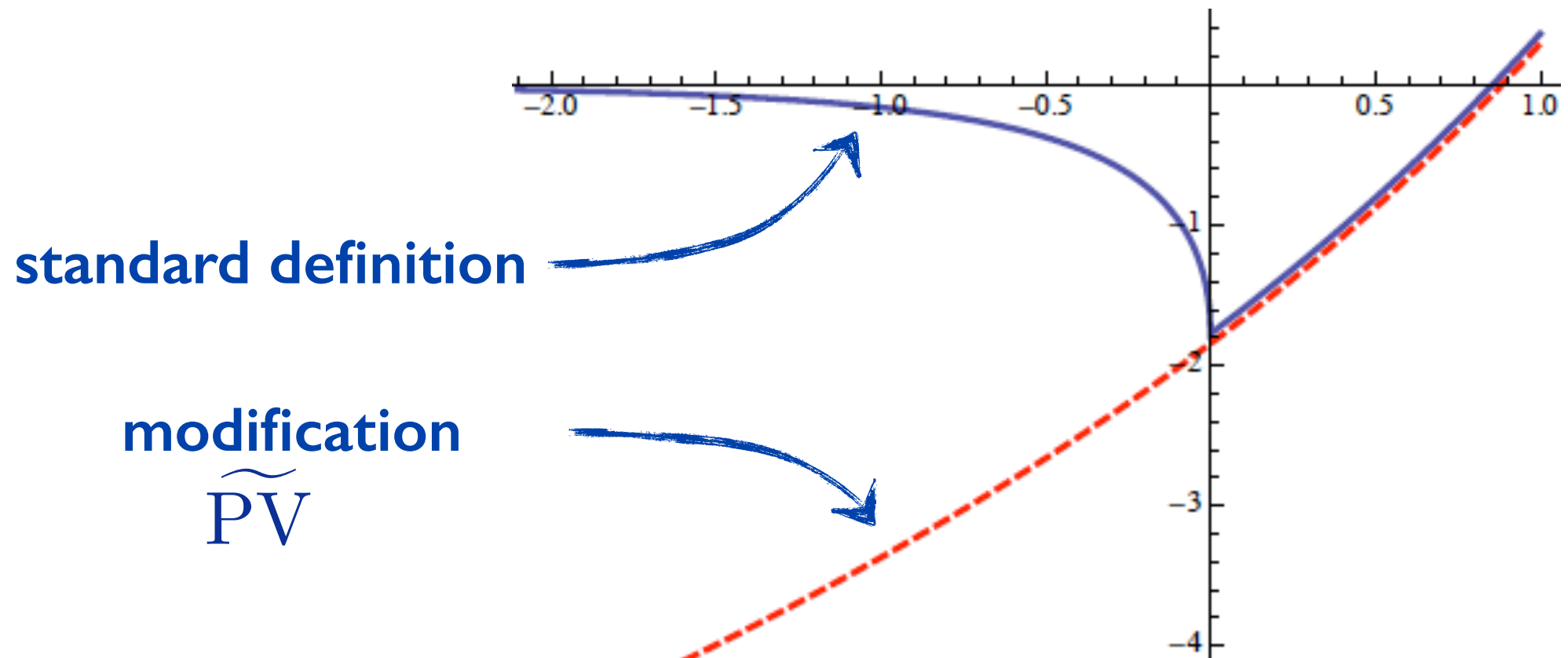
The diagrams in the two-particle skeleton expansion are as follows:

- Diagram 1: Two white circles connected by two lines, with a dashed box around the lines.
- Diagram 2: Two white circles connected by two lines, with a purple circle on the top line between the dashed boxes.
- Diagram 3: Two white circles connected by two lines, with two purple circles on the top line between the dashed boxes.

Cusps



Relate \vec{a} sum to an integral, gives cusp in \vec{k}



Leads to new infinite-volume quantities

has a cusp

$$i\mathcal{M}_{2\rightarrow 2} = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$
The equation shows a series of Feynman diagrams for the process $i\mathcal{M}_{2\rightarrow 2}$. The first diagram is a single purple circle with four external dashed lines. The second diagram is two purple circles connected by two internal lines, with four external dashed lines. The third diagram is three purple circles connected by two internal lines, with four external dashed lines. The series continues with an ellipsis.

$$i\tilde{\mathcal{K}}_{2\rightarrow 2} = \text{diagram 4} = \text{diagram 5}$$
The equation shows a diagrammatic representation of $i\tilde{\mathcal{K}}_{2\rightarrow 2}$. The first part is a purple circle with four external dashed lines. This is equal to a purple circle with four external solid black lines. This is equal to another purple circle with four external dashed lines.

$$\text{diagram 6} + \text{diagram 7} + \text{diagram 8} + \dots$$
The equation shows a series of Feynman diagrams for the process $i\tilde{\mathcal{K}}_{2\rightarrow 2}$. The first diagram is a single purple circle with four external dashed lines. The second diagram is two purple circles connected by two internal lines, with the internal lines labeled 'PV' in purple. The third diagram is three purple circles connected by two internal lines, with the internal lines labeled 'PV' in purple. The series continues with an ellipsis.

has no cusp

Three-particle result

At fixed (L, \vec{P}) , finite-volume spectrum
is all solutions to

$$\Delta_{L,P}(E) = \det \left[1 - \underbrace{i\tilde{\mathcal{K}}_{df,3 \rightarrow 3}}_{\text{matrix in } \vec{k}, \ell, m \text{ space}} \underbrace{iF_3}_{\text{depends on kinematics and two-particle scattering}} \right] = 0$$

matrix in \vec{k}, ℓ, m
space

depends on kinematics
and two-particle
scattering

MTH, S. R. Sharpe, arXiv:1408.5933 to appear in *Phys.Rev. D*

K. Polejaeva, A. Rusetsky *Eur. Phys. J. A*48 (2012) 67

R.A. Briceño, Z. Davoudi, *Phys. Rev. D*87 (2013) 094507

Conclusion

$$\det[1 - i\mathcal{M}_{2\rightarrow 2}iF] = 0$$

$$|\langle E_{n_f}, \mathbf{P}_f, L | \tilde{\mathcal{J}}_A(0, \mathbf{Q}) | E_i, \mathbf{P}_i, L \rangle| = \frac{1}{\sqrt{2E_i}} \sqrt{\mathcal{A}_{n_f}^\dagger \mathcal{R}_{n_f} \mathcal{A}_{n_f}}$$

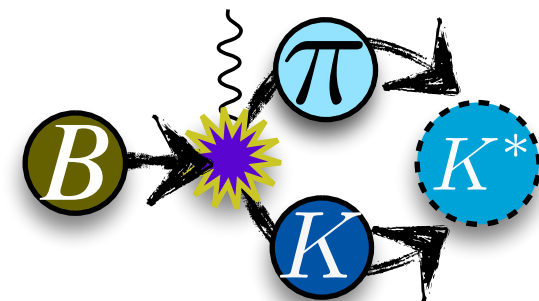
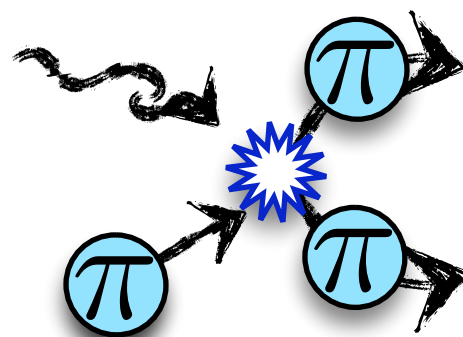
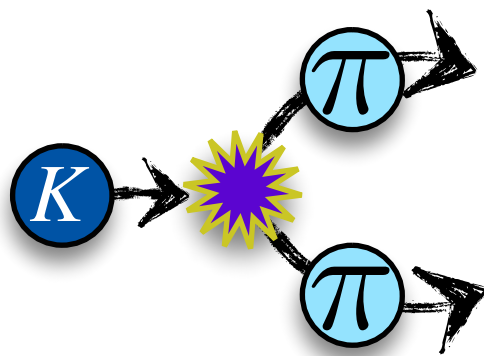
Presented formalism for extracting

two-to-two scattering

one-to-two matrix elements

three-to-three scattering using numerical LQCD

Completely general result for scalar particle systems



Next steps... include spin, one to three transitions

Finite vs infinite volume matrix elements

$$\langle \pi(p)\pi(-p), \text{out} | \mathcal{H} | K \rangle \stackrel{?}{\approx} \langle n, L, \text{“}\pi\pi\text{”} | \mathcal{H} | 0, L, \text{“}K\text{”} \rangle$$

Finite vs infinite volume matrix elements

$$\langle \pi(p)\pi(-p), \text{out} | \mathcal{H} | K \rangle \stackrel{?}{\approx} \langle n, L, \text{“}\pi\pi\text{”} | \mathcal{H} | 0, L, \text{“}K\text{”} \rangle$$

complex number

real number

Finite vs infinite volume matrix elements

$$\langle \pi(p)\pi(-p), \text{out} | \mathcal{H} | K \rangle \stackrel{?}{\approx} \langle n, L, \text{"}\pi\pi\text{"} | \mathcal{H} | 0, L, \text{"}K\text{"} \rangle$$

complex number

real number

$$|\langle \pi(p)\pi(-p), \text{out} | \mathcal{H} | K \rangle| \stackrel{?}{\approx} |\langle n, L, \text{"}\pi\pi\text{"} | \mathcal{H} | 0, L, \text{"}K\text{"} \rangle|$$

Finite vs infinite volume matrix elements

~~$$\langle \pi(p)\pi(-p), \text{out} | \mathcal{H} | K \rangle \stackrel{?}{\approx} \langle n, L, \text{"}\pi\pi\text{"} | \mathcal{H} | 0, L, \text{"}K\text{"} \rangle$$~~

complex number

real number

$$|\langle \pi(p)\pi(-p), \text{out} | \mathcal{H} | K \rangle| \stackrel{?}{\approx} |\langle n, L, \text{"}\pi\pi\text{"} | \mathcal{H} | 0, L, \text{"}K\text{"} \rangle|$$

units do not match

$$\langle \pi(p') | \pi(p) \rangle = 2\omega_p (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \quad \omega_p = \sqrt{\mathbf{p}^2 + m_\pi^2}$$

$$\langle n', L | n, L \rangle = \delta_{n,n'}$$

Finite vs infinite volume matrix elements

~~$$\langle \pi(p)\pi(-p), \text{out} | \mathcal{H} | K \rangle \stackrel{?}{\sim} \langle n, L, \text{"}\pi\pi\text{"} | \mathcal{H} | 0, L, \text{"}K\text{"} \rangle$$~~

complex number

real number

~~$$|\langle \pi(p)\pi(-p), \text{out} | \mathcal{H} | K \rangle| \stackrel{?}{\sim} |\langle n, L, \text{"}\pi\pi\text{"} | \mathcal{H} | 0, L, \text{"}K\text{"} \rangle|$$~~

units do not match

$$\langle \pi(p') | \pi(p) \rangle = 2\omega_p (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \quad \omega_p = \sqrt{\mathbf{p}^2 + m_\pi^2}$$

$$\langle n', L | n, L \rangle = \delta_{n,n'}$$

At the very least, we must account for
different normalizations

Warm up... non-interacting pions

$$|0, L, \text{“}K\text{”}\rangle = \sqrt{\frac{1}{2M_K L^3}} |\mathbf{0}, L, K\rangle_{\text{rel}}$$

 **unit normalization**

 **relativistic
normalization**

Warm up... non-interacting pions

$$|0, L, \text{“}K\text{”}\rangle = \sqrt{\frac{1}{2M_K L^3}} |0, L, K\rangle_{\text{rel}}$$

unit normalization

**relativistic
normalization**

$$\langle \mathbf{p}, \infty, K | \mathbf{k}, \infty, K \rangle_{\text{rel}} = 2\omega_k (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{k})$$

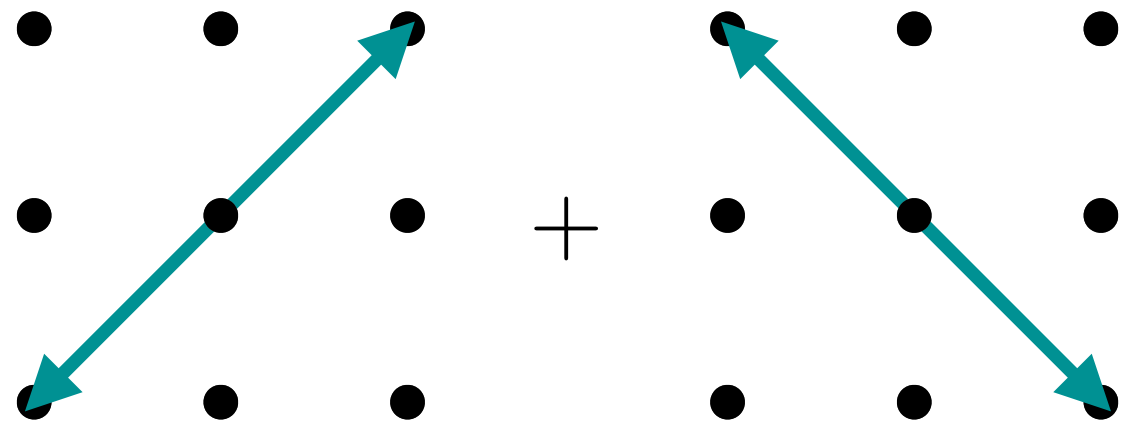
becomes

$$\langle \mathbf{p}, L, K | \mathbf{k}, L, K \rangle_{\text{rel}} = 2\omega_k L^3 \delta_{kp}$$

Warm up... non-interacting pions

$$|0, L, \text{“}K\text{”}\rangle = \sqrt{\frac{1}{2M_K L^3}} |0, L, K\rangle_{\text{rel}}$$

$$\langle n, L, \text{“}\pi\pi\text{”} | = \sqrt{\frac{2}{\nu_n} \frac{1}{M_K^2 L^6}} \left[\langle n00, L, \pi | \langle -n00, L, \pi |_{\text{rel}} + \dots \right]$$

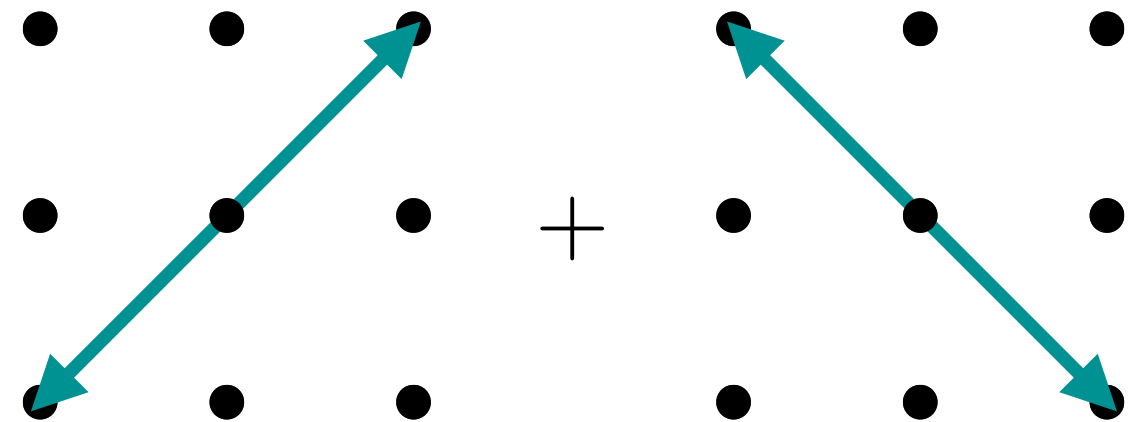


Warm up... non-interacting pions

$$|0, L, \text{“}K\text{”}\rangle = \sqrt{\frac{1}{2M_K L^3}} |0, L, K\rangle_{\text{rel}}$$

$$\langle n, L, \text{“}\pi\pi\text{”} | = \sqrt{\frac{2}{\nu_n} \frac{1}{M_K^2 L^6}} \left[\langle n00, L, \pi | \langle -n00, L, \pi |_{\text{rel}} + \dots \right]$$

number of integer vectors \mathbf{z}
such that $\mathbf{z}^2 = n$.

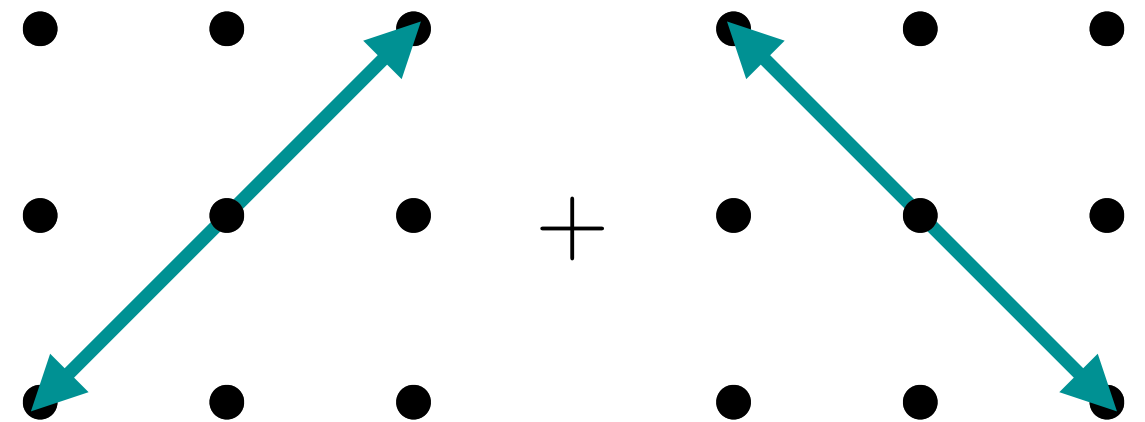


Warm up... non-interacting pions

$$|0, L, \text{"K"}\rangle = \sqrt{\frac{1}{2M_K L^3}} |0, L, K\rangle_{\text{rel}}$$

$$\langle n, L, \text{"}\pi\pi\text{"} | = \sqrt{\frac{2}{\nu_n} \frac{1}{M_K^2 L^6}} \left[\langle n00, L, \pi | \langle -n00, L, \pi |_{\text{rel}} + \dots \right]$$

number of integer vectors \mathbf{z}
such that $\mathbf{z}^2 = n$.



$$|\langle n, L, \text{"}\pi\pi\text{"} | \mathcal{H} | 0, L, \text{"K"} \rangle|^2 = \frac{\nu_n}{4M_K^3 L^9} |\langle \pi(p) \pi(-p), \text{out} | \mathcal{H} | K \rangle|^2$$

Notation

$$|\langle n, L, \text{“}\pi\pi\text{”} | \mathcal{H} | 0, L, \text{“}K\text{”} \rangle|^2 = \frac{\nu_n}{4M_K^3 L^9} |\langle \pi(p)\pi(-p), \text{out} | \mathcal{H} | K \rangle|^2$$

Notation

$$|\langle n, L, \text{“}\pi\pi\text{”} | \mathcal{H} | 0, L, \text{“}K\text{”} \rangle|^2 = \frac{\nu_n}{4M_K^3 L^9} |\langle \pi(p)\pi(-p), \text{out} | \mathcal{H} | K \rangle|^2$$

can be re-expressed as...

$$|\langle n, L, \text{“}\pi\pi\text{”} | \tilde{\mathcal{H}}(\mathbf{Q} = \mathbf{0}) | 0, L, \text{“}K\text{”} \rangle| = \frac{1}{\sqrt{2M_K}} \sqrt{[\mathcal{A}_{K \rightarrow \pi\pi}^\dagger \mathcal{R} \mathcal{A}_{K \rightarrow \pi\pi}]}$$

$$\tilde{\mathcal{H}}(\mathbf{Q}) = \int d\mathbf{x} e^{-i\mathbf{Q}\cdot\mathbf{x}} \mathcal{H}(x_0 = 0, \mathbf{x})$$

$$\mathcal{A}_{K \rightarrow \pi\pi} (2\pi)^3 \delta^3(\mathbf{p} + \mathbf{k}) \equiv \langle \pi(p)\pi(k) | \tilde{\mathcal{H}}(\mathbf{Q} = \mathbf{0}) | K \rangle$$

$$\mathcal{R} = \frac{\nu_n}{2M_K^2 L^3}$$