One-to-two transition amplitudes in a finite volume

Maxwell T. Hansen

Institut für Kernphysik and Helmholz Institute Mainz
Johannes Gutenberg-Universität Mainz

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Based on:
R. A. Briceño, MTH, A. Walker-Loud
arXiv:1406.5965

Transition amplitudes

(rare B decays)

e.g., $B^0 \to K^{*0}\ell^+\ell^-$

LHCb collaboration (2013)

First unquenched LQCD calculation:
Horgan, Liu, Meinel & Wingate (2013)

Transition amplitudes

(rare B decays)

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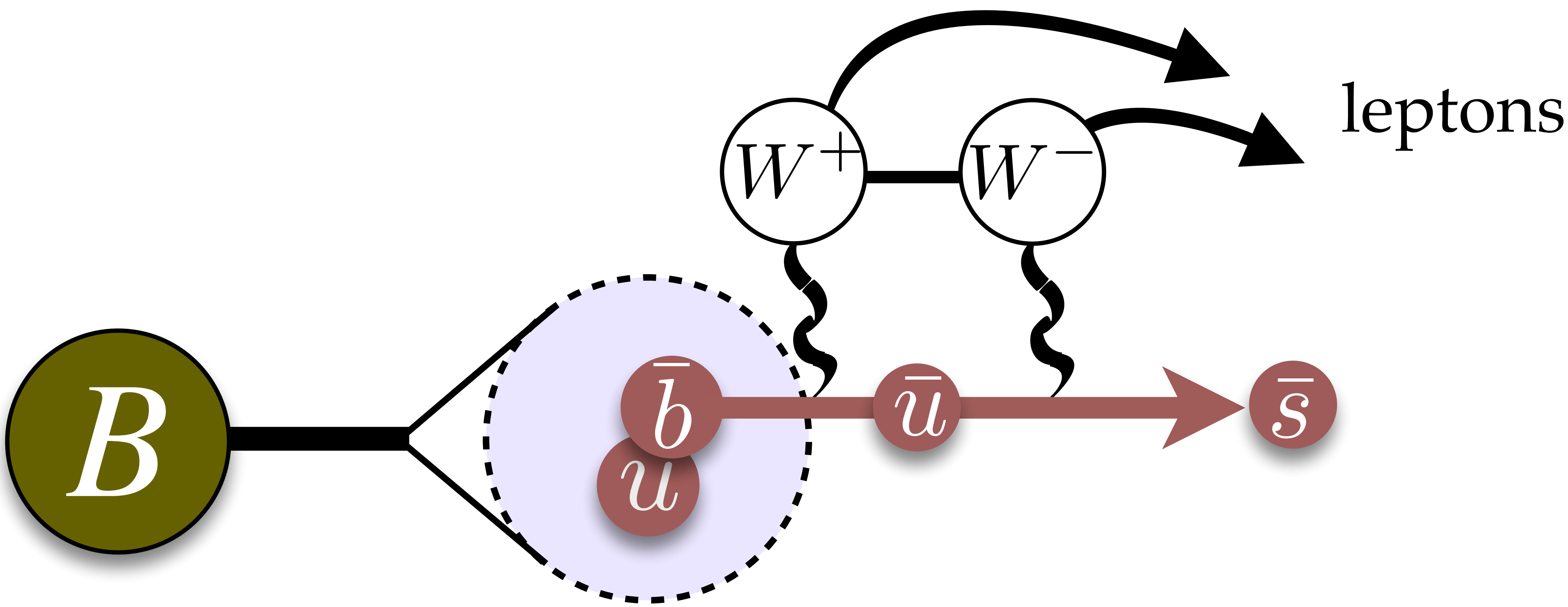

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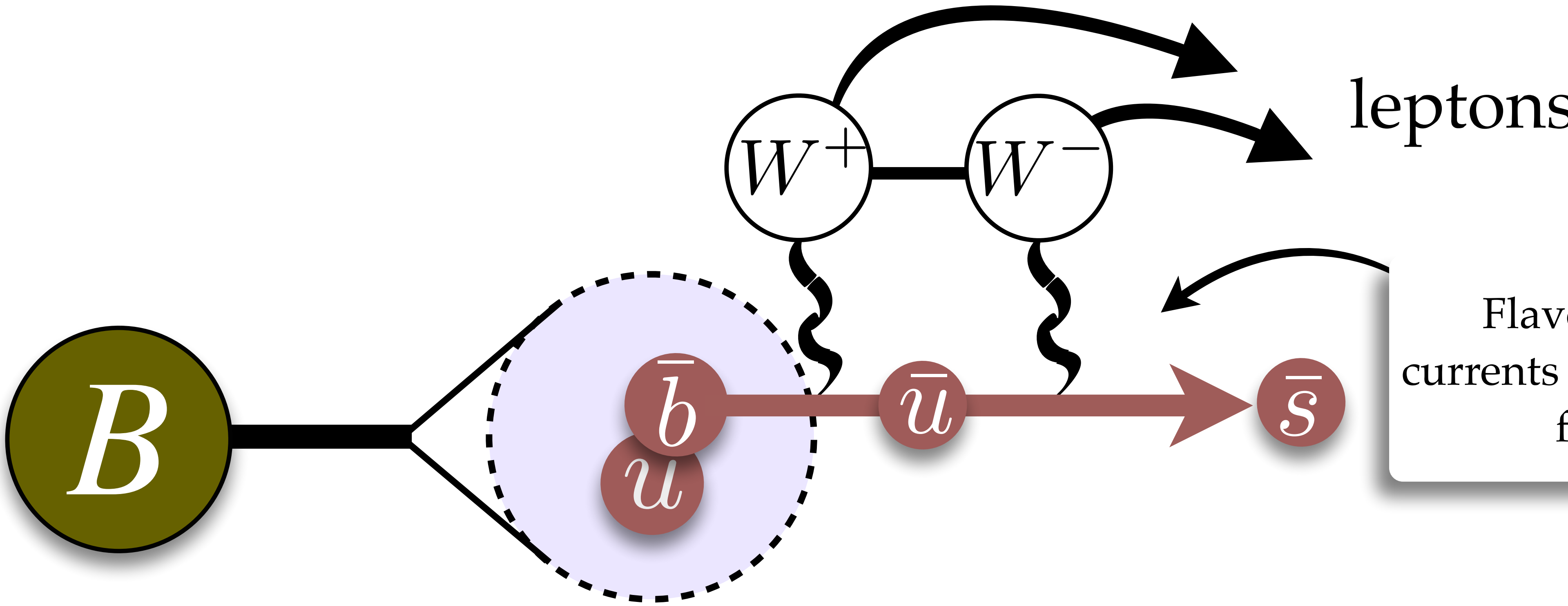


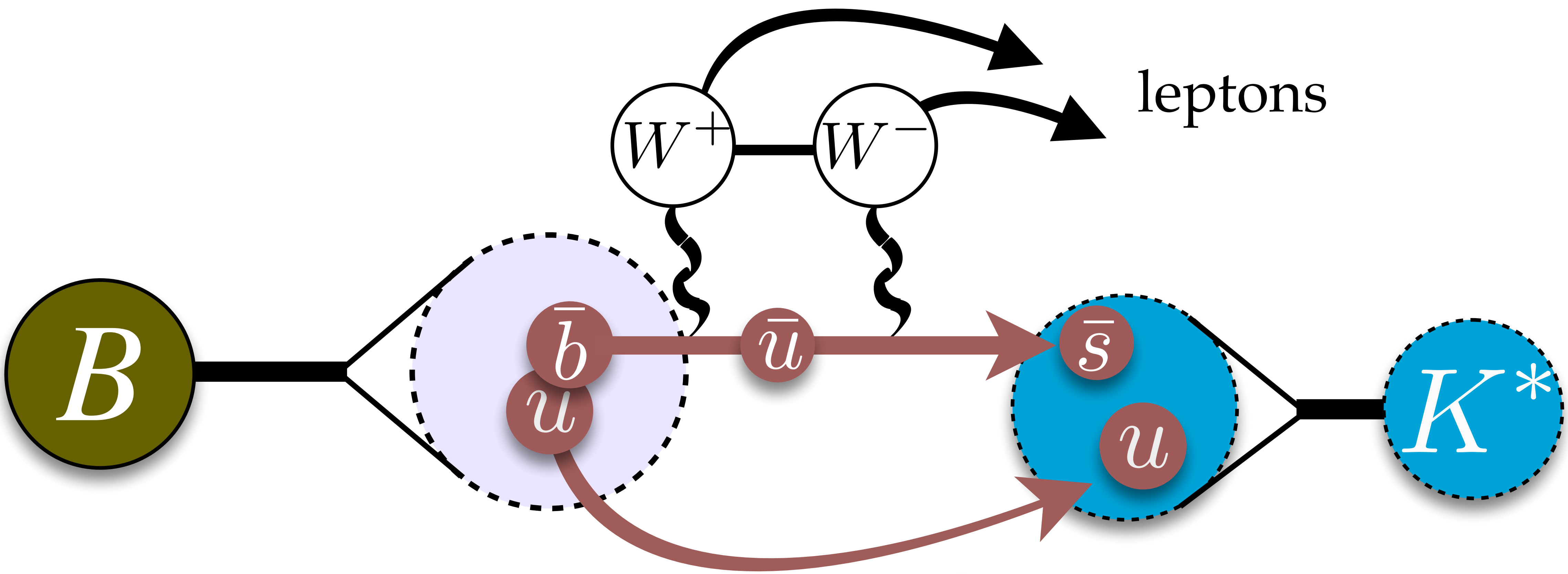
Flavor changing neutral currents are a great place to look for new physics!

Caution: unfortunately, this is not the full story.

Transition amplitudes

(rare B decays)

e.g., $B^0 \to K^{*0}\ell^+\ell^-$



K^*(892):

- $I\,(J^P) = ½\,(1^-)$ resonance
- above πK and $\pi\pi$K thresholds
- just below Kη~K$\pi\pi\pi$ threshold

Transition amplitudes

(rare B decays)

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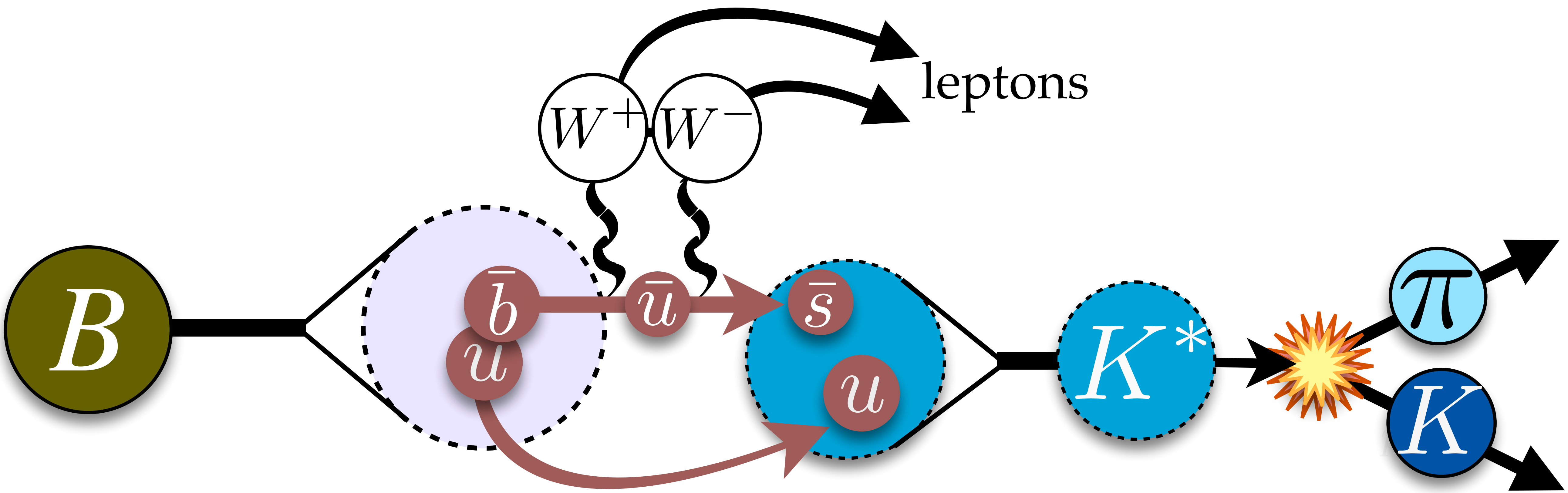

Must calculate matrix elements of QCD eigenstates

$$\langle \pi K, \mathrm{out} | \mathcal{J}_\mu | B \rangle$$

General Question

How can one use numerical Lattice QCD to determine

$$\langle \phi_1(p), \phi_2(P_f - p), \mathrm{out} | \widetilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q}) | \Phi, P_i \rangle \textbf{ ?}$$

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$|\Phi, P_i\rangle$ **single scalar particle instate**

$\langle \phi_1(p), \phi_2(P_f - p), \text{out}|$ **two scalar particle outstate**

$\widetilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q})$ **generic operator insertion**

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$\widetilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q})$ **generic operator insertion**

generic Lorentz index

Could be a scalar $\widetilde{\mathcal{H}}$**, a vector** $\widetilde{\mathcal{J}}_\mu$**, a tensor** $\widetilde{\mathcal{T}}_{\mu\nu}$**, etc…**

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$\widetilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q})$ **generic operator insertion**

With arbitrary momentum injection

$$\widetilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q}) \equiv \int d^3x\, e^{-i\mathbf{x}\cdot\mathbf{Q}}\, J_A(x)\Big|_{x_0=0}$$

General Question

How can one use numerical Lattice QCD to determine

$$\langle \phi_1(p), \phi_2(P_f - p), \mathrm{out} | \widetilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q}) | \Phi, P_i \rangle \, ?$$

Relevant for many decays

$$K \to \pi\pi$$

$$\pi\gamma \to \pi\pi$$

$$B^0 \to K\pi\ell^+\ell^-$$

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How can one use numerical Lattice QCD to determine

$$\langle \phi_1(p), \phi_2(P_f - p), \text{out} | \widetilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q}) | \Phi, P_i \rangle \,?$$

Relevant for many decays

$$K \to \pi\pi$$

$$\pi\gamma \to \pi\pi$$

$$B^0 \to K\pi\ell^+\ell^-$$

scalar operator $\mathcal{H}$

no momentum injection

Lellouch and Lüscher (2001)

RBC/UKQCD implementing full error budget calculation

nontrivial Lorentz structure

nonzero momentum injection

This Work

Agadjanov, et.al., (2014), arXiv:1405.3476.

Outline

Difficulties in extracting matrix elements from LQCD

Introduction to finite vs infinite-volume matrix elements

Two-to-two scattering via LQCD

One-to-two transition amplitudes via LQCD

What can we extract from LQCD?

We are trying to evaluate a difficult integral numerically

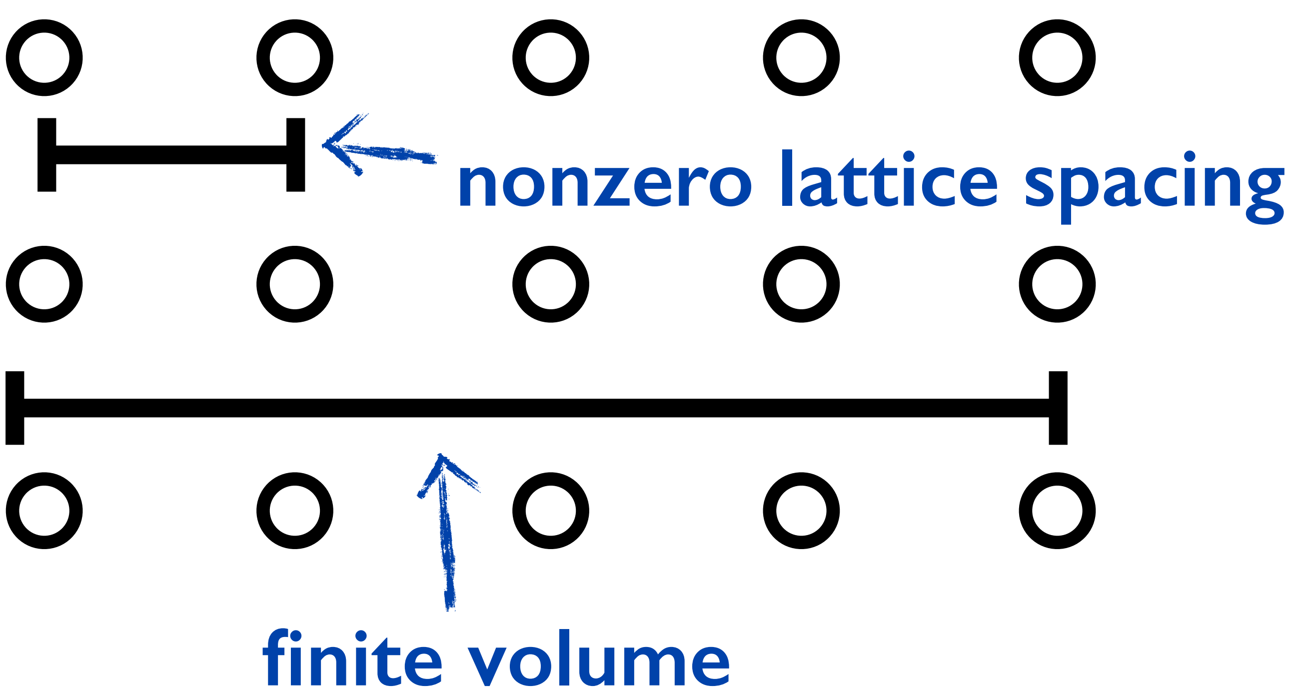
$$\langle T\phi_1 \cdots \phi_n \rangle = \int \mathcal{D}\phi \, e^{iS} \, \phi_1 \cdots \phi_n$$

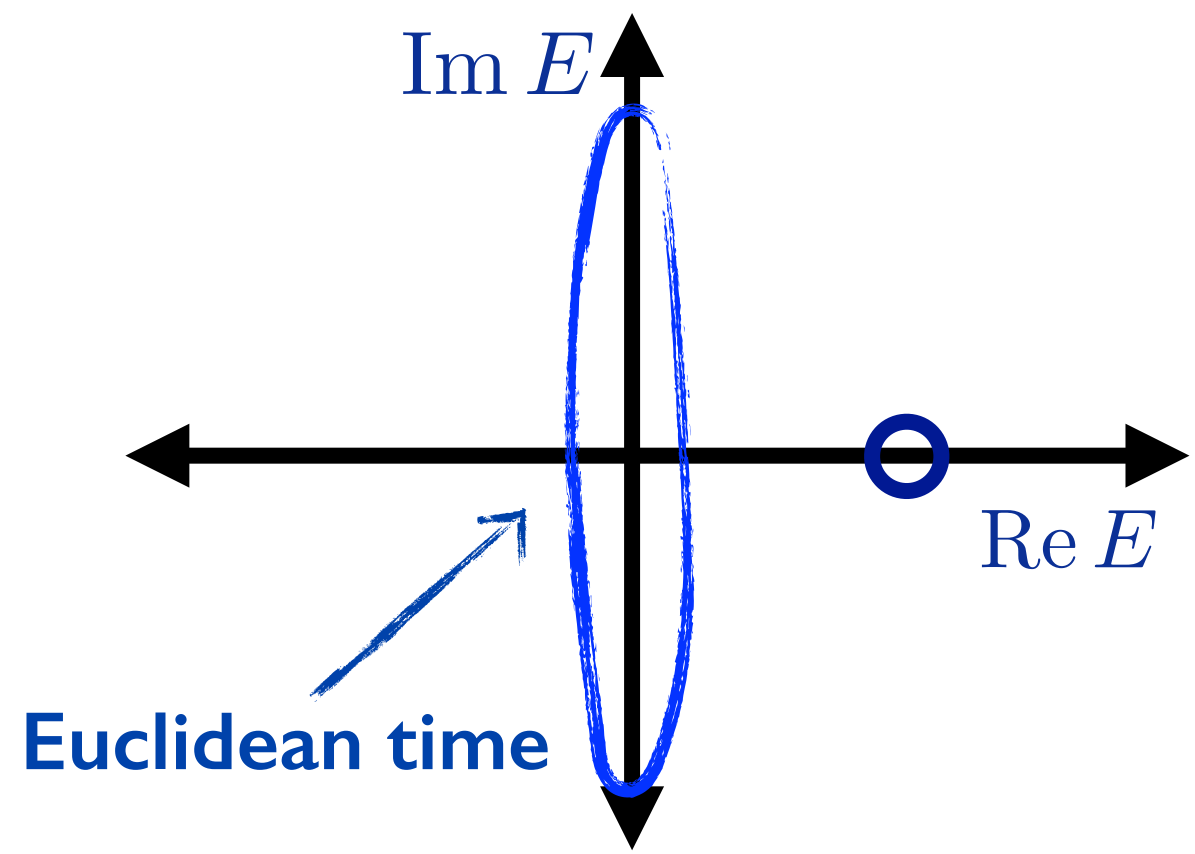
What can we extract from LQCD?

We are trying to evaluate a difficult integral numerically

$$\langle T\phi_1 \cdots \phi_n \rangle_{\text{Euc, latt, fv}} = \int \prod_i^N d\phi_i \, e^{-S} \, \phi_1 \cdots \phi_n$$

To do so we have to make three compromises





What can we extract from LQCD?

Not possible to directly calculate

$\langle \pi\pi | \pi\pi \rangle$

$\langle \pi\pi\pi | \pi\pi\pi \rangle$

$\langle K\pi\pi | \mathcal{J} | B \rangle$

$\langle \pi\pi | \mathcal{H} | K \rangle$

$\langle \pi\pi | \mathcal{J} | \pi \rangle$

$\langle K\pi | \mathcal{J} | B \rangle$

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multi-particle in- and outstates

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multi-particle in- and outstates

$$\langle \pi\pi | \pi\pi \rangle = \text{Amputate and put on-shell } \langle 0 | \underline{\tilde{\pi}(p')\tilde{\pi}(k')\ \tilde{\pi}(p)\tilde{\pi}(k)} | 0 \rangle$$

$$\langle \pi\pi | \mathcal{H} | K \rangle = \text{Amputate and put on-shell } \langle 0 | \underline{\tilde{\pi}(p')\tilde{\pi}(k')\ \mathcal{H}(x)\ \tilde{K}(P)} | 0 \rangle$$

Requires Minkowski momenta and infinite-volume

What can we extract from LQCD?

Instead we can only access

$$H_{\rm QCD}|n,L\rangle = |n,L\rangle \underline{E_n(L)} \qquad \underline{\langle n', L, \text{“}\pi\pi\text{”}|\mathcal{H}|n, L, \text{“}K\text{”}\rangle}$$

finite-volume energies and matrix elements

labels in quotes indicate quantum numbers

What can we extract from LQCD?

Instead we can only access

$H_{\mathrm{QCD}}|n,L\rangle = |n,L\rangle E_n(L)$ $\langle n', L, \text{“}\pi\pi\text{”}|\mathcal{H}|n, L, \text{“}K\text{”}\rangle$

finite-volume energies and matrix elements

labels in quotes indicate quantum numbers

How can we determine

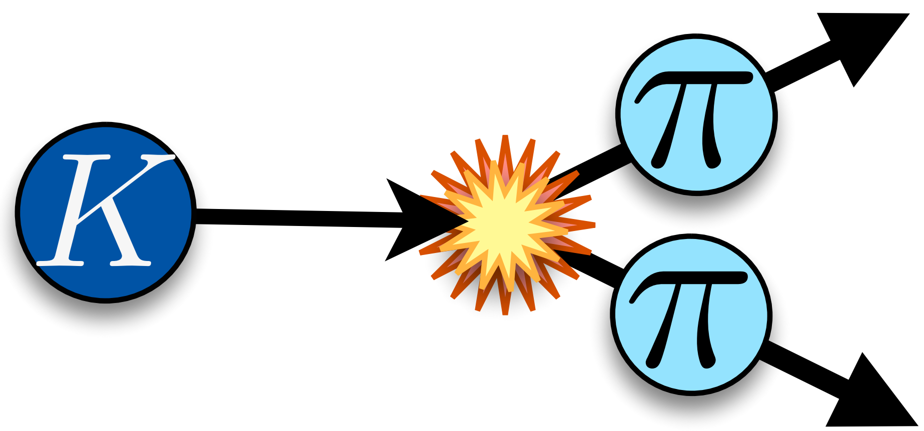
$\langle \pi(p')\pi(k'), \mathrm{out}|\pi(p)\pi(k), \mathrm{in}\rangle$ & $\langle \pi(p)\pi(-p), \mathrm{out}|\mathcal{H}|K\rangle$

from

$E_n(L)$ & $\langle n, L, \text{“}\pi\pi\text{”}|\mathcal{H}|n, L, \text{“}K\text{”}\rangle$?

Cannot understand transitions without first understanding scattering

Lellouch-Lüscher trick is to access transitions like



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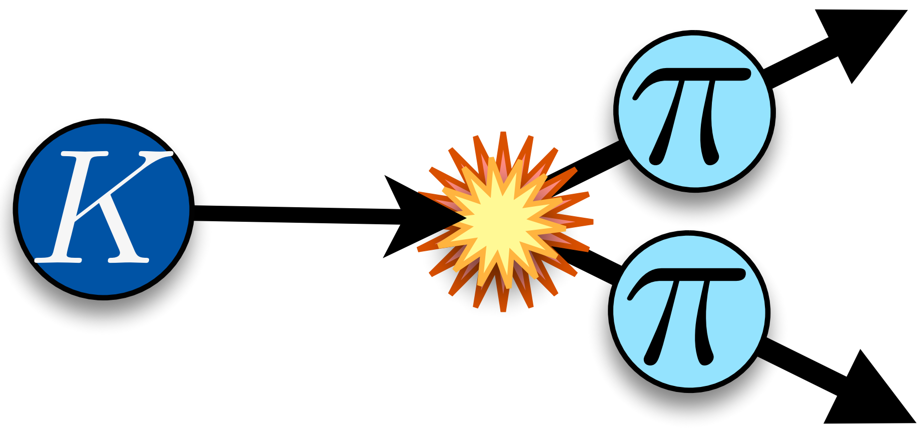
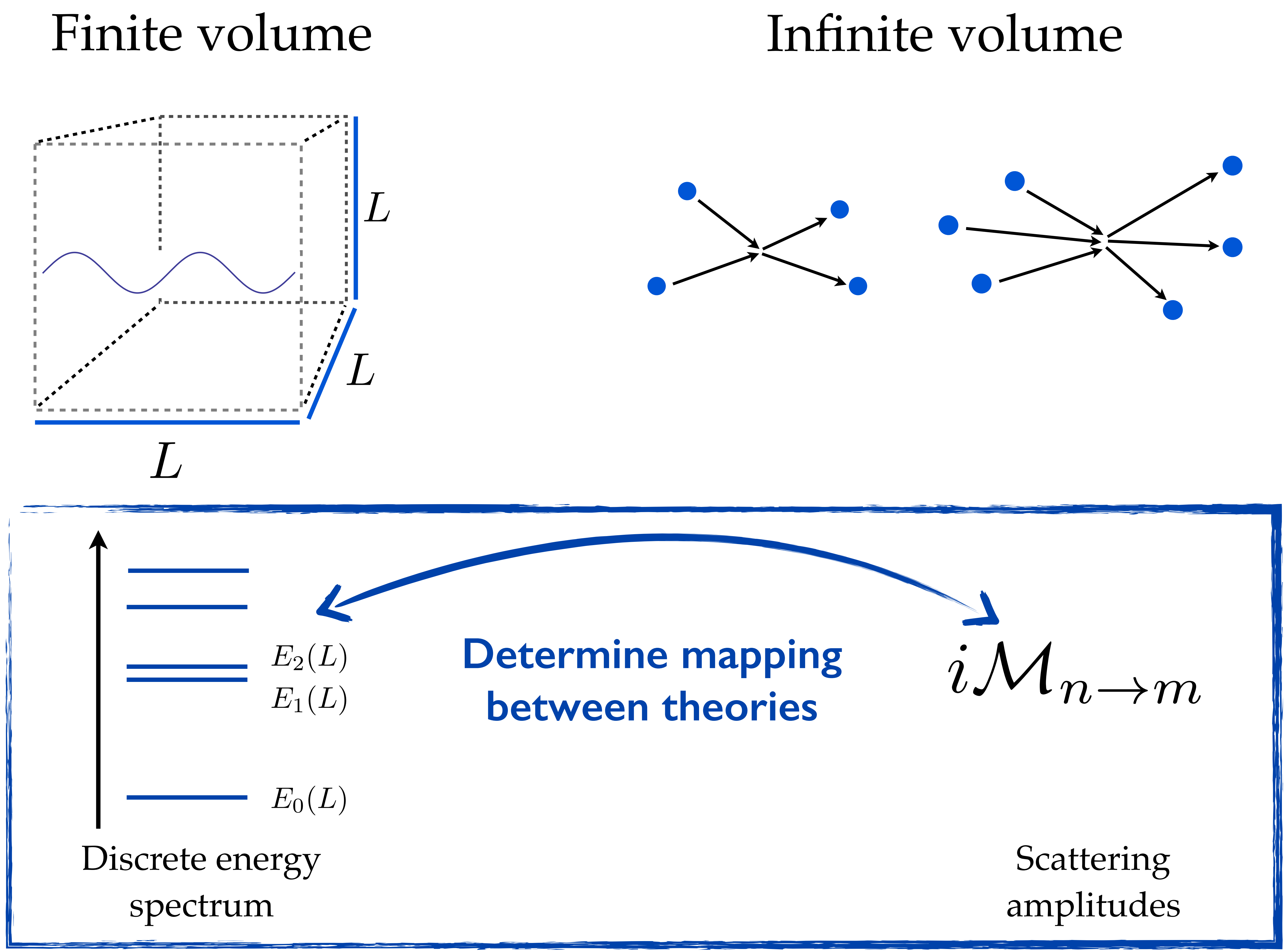


by considering it as an intermediate state in scattering

We thus begin by describing how to determine scattering amplitudes from numerical Lattice QCD

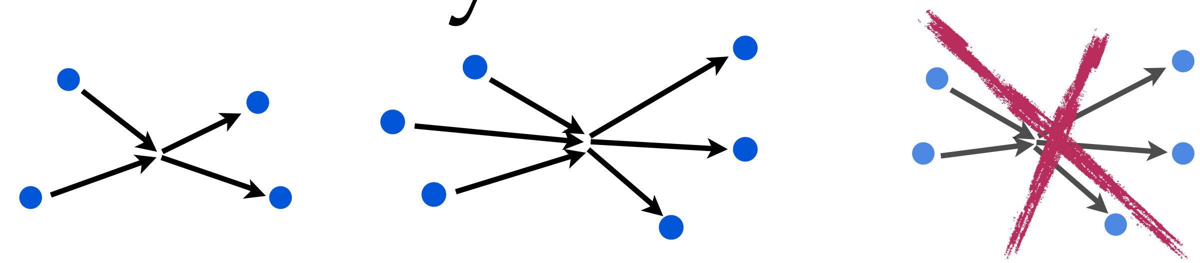


Finite volume
Infinite volume
L
L
L
$E_2(L)$
$E_1(L)$
$E_0(L)$
Discrete energy spectrum
Determine mapping between theories
$i\mathcal{M}_{n\to m}$
Scattering amplitudes

Basic set-up

Relativistic scalar field theory

$\mathbb{Z}_2$ **symmetry**



(For pions in QCD this is G-parity)

Include all vertices with even number of legs

Basic set-up

Relativistic scalar field theory

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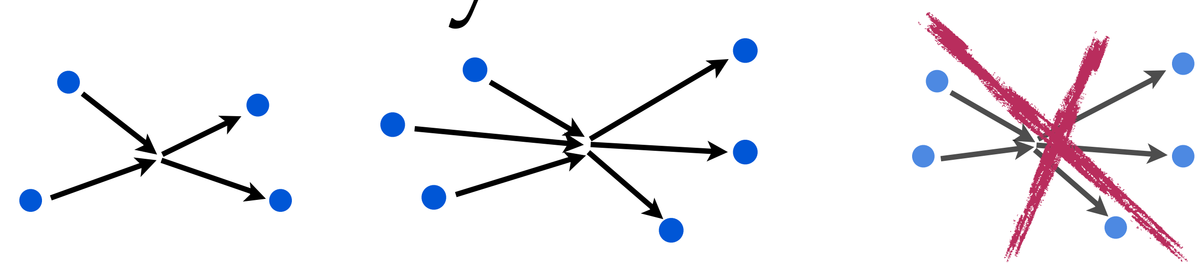
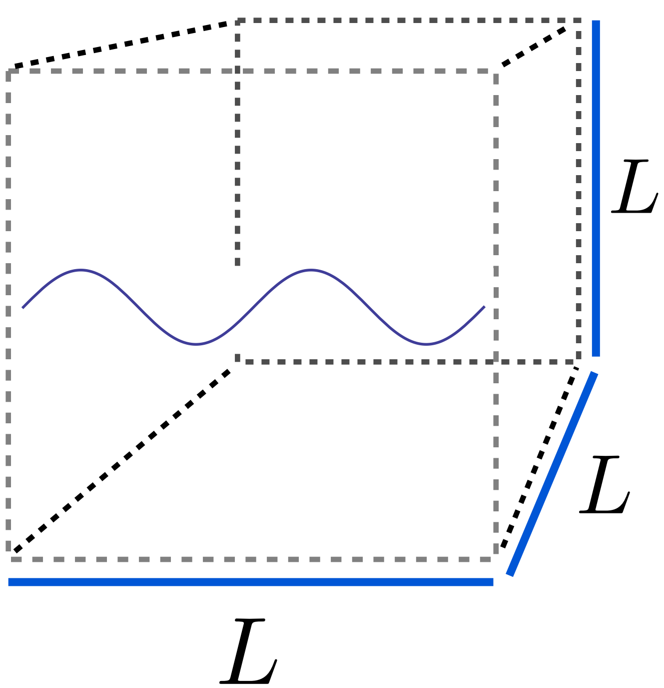


(For pions in QCD this is G-parity)

Include all vertices with even number of legs

Finite volume



cubic, spatial volume (extent L)

periodic boundary conditions $\vec{p} \in (2\pi/L)\mathbb{Z}^3$

time direction **infinite**

Take L large enough to ignore e^{-mL}

dropped throughout!

Take space to be continuous

lattice spacing set to zero

Determine relation using finite-volume correlator

$$C_L(P) \equiv \int_L e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

Determine relation using finite-volume correlator

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two particle interpolating field

Euclidean energy P_0,

Momentum $\mathbf{P} = (2\pi/L)\mathbf{n_P}$,

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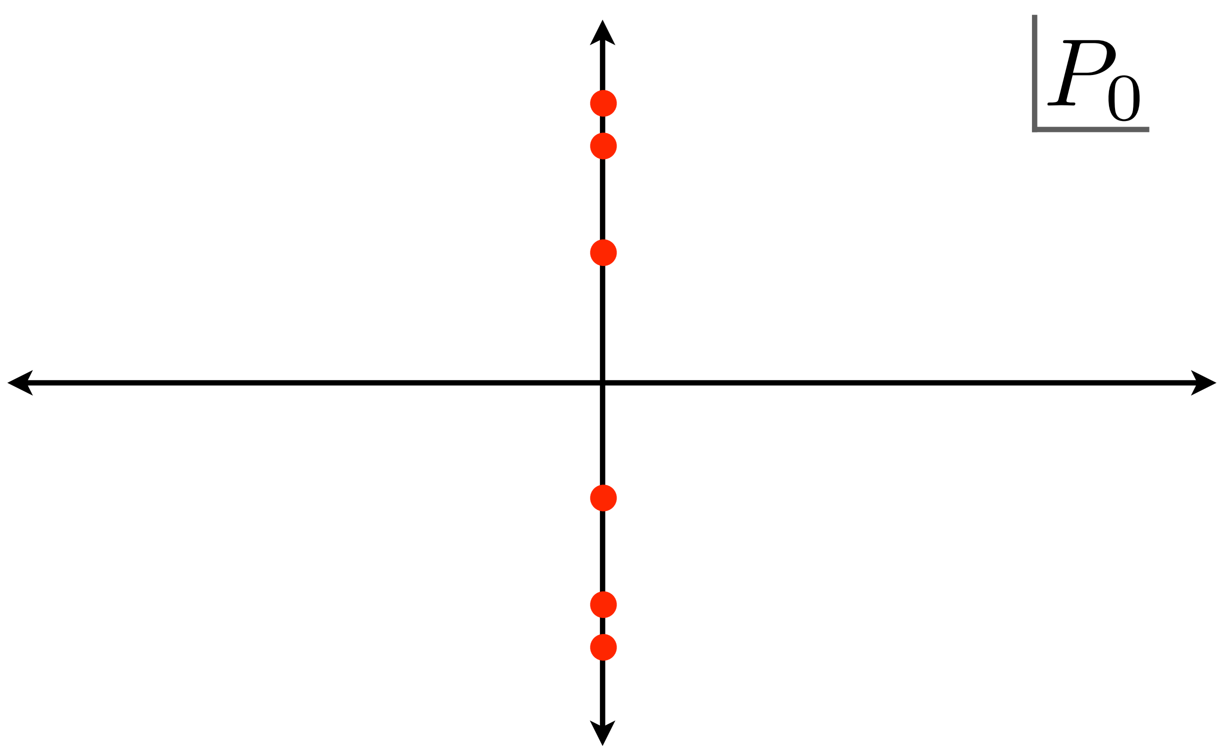
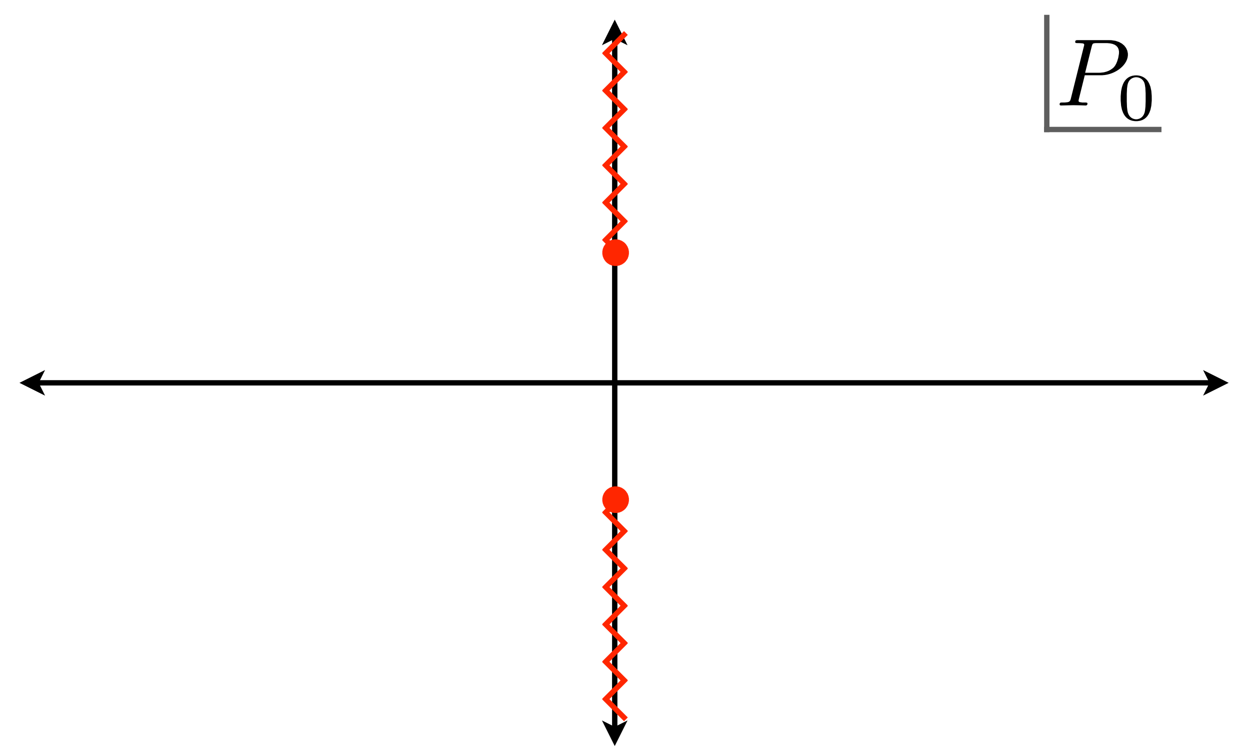
At fixed L, $\vec{P}$, poles in C_L give finite-volume spectrum



C_L **analytic structure**



C_∞ **analytic structure**

Determine relation using finite-volume correlator

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Euclidean energy P_0,

Momentum $\mathbf{P} = (2\pi/L)\mathbf{n_P}$,

two particle interpolating field

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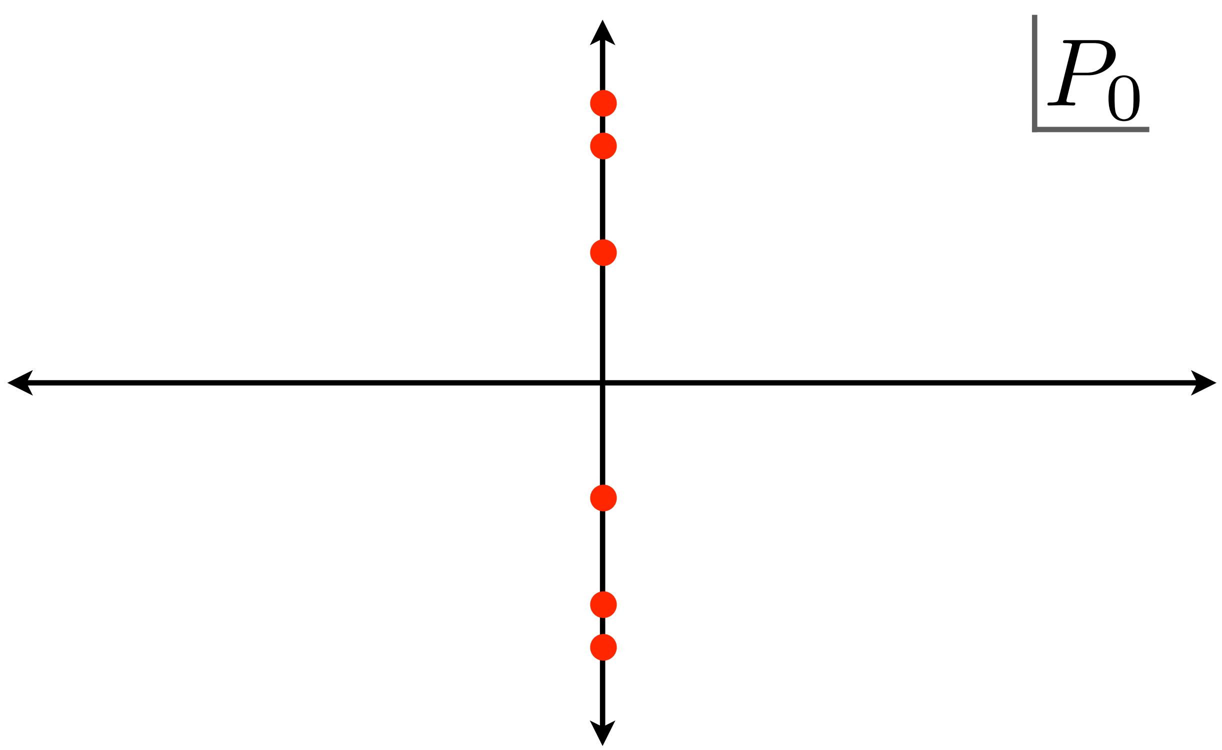


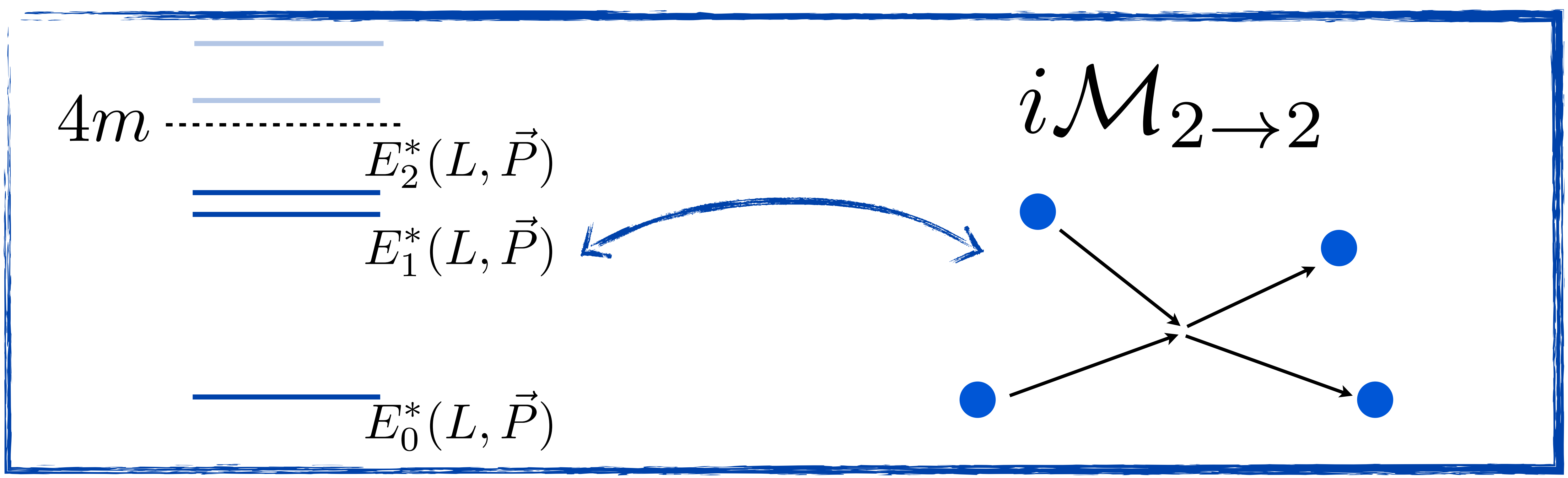
C_L analytic structure

Calculate $C_L(P)$ to all orders in perturbation theory and determine locations of poles.

Determine relation using finite-volume correlator

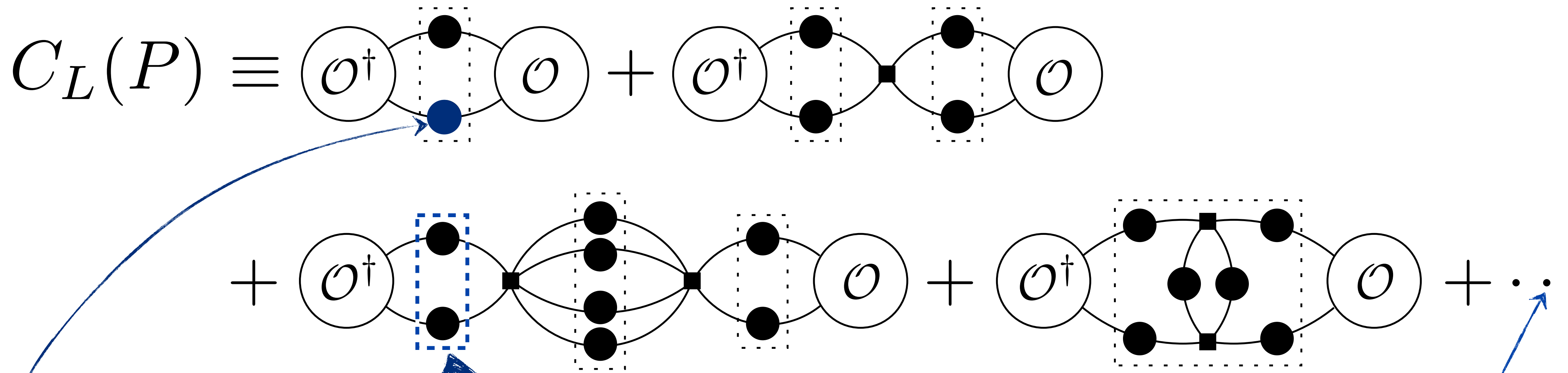
$$C_L(P) \equiv \int_L e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

Require $E^* < 4m$ **where** $E^{*2} \equiv -P_0^2 - \mathbf{P}^2$



Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

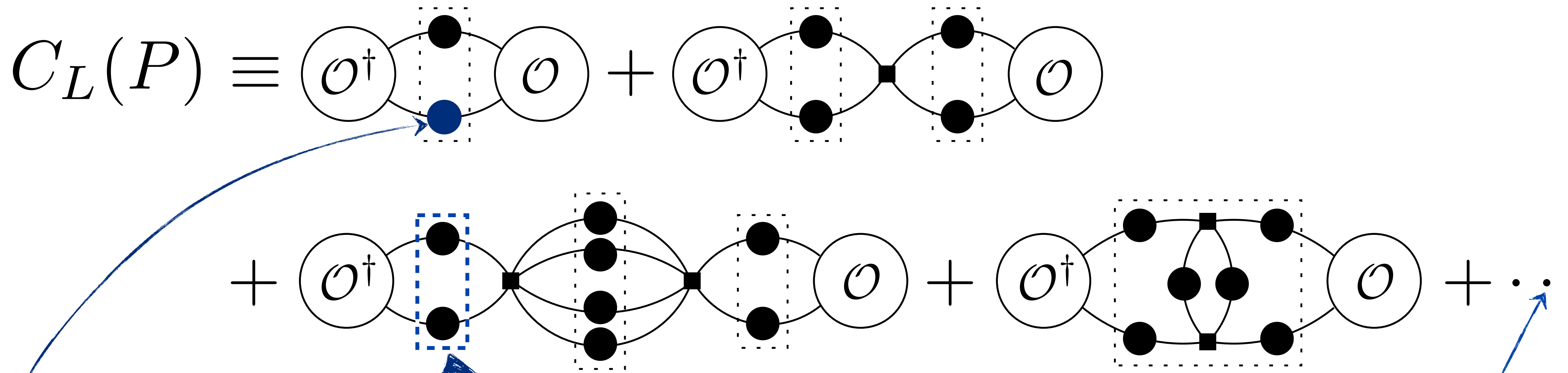


fully dressed propagators

infinite set of terms
no assumed suppression

spatial loop momenta are summed

$$\frac{1}{L^3}\sum_{\vec{k}\in(2\pi/L)\mathbb{Z}^3}\int\frac{dk^0}{2\pi}$$



spatial loop momenta are summed

$$\frac{1}{L^3}\sum_{\vec{k}\in(2\pi/L)\mathbb{Z}^3}\int\frac{dk^0}{2\pi}$$

Key observation:

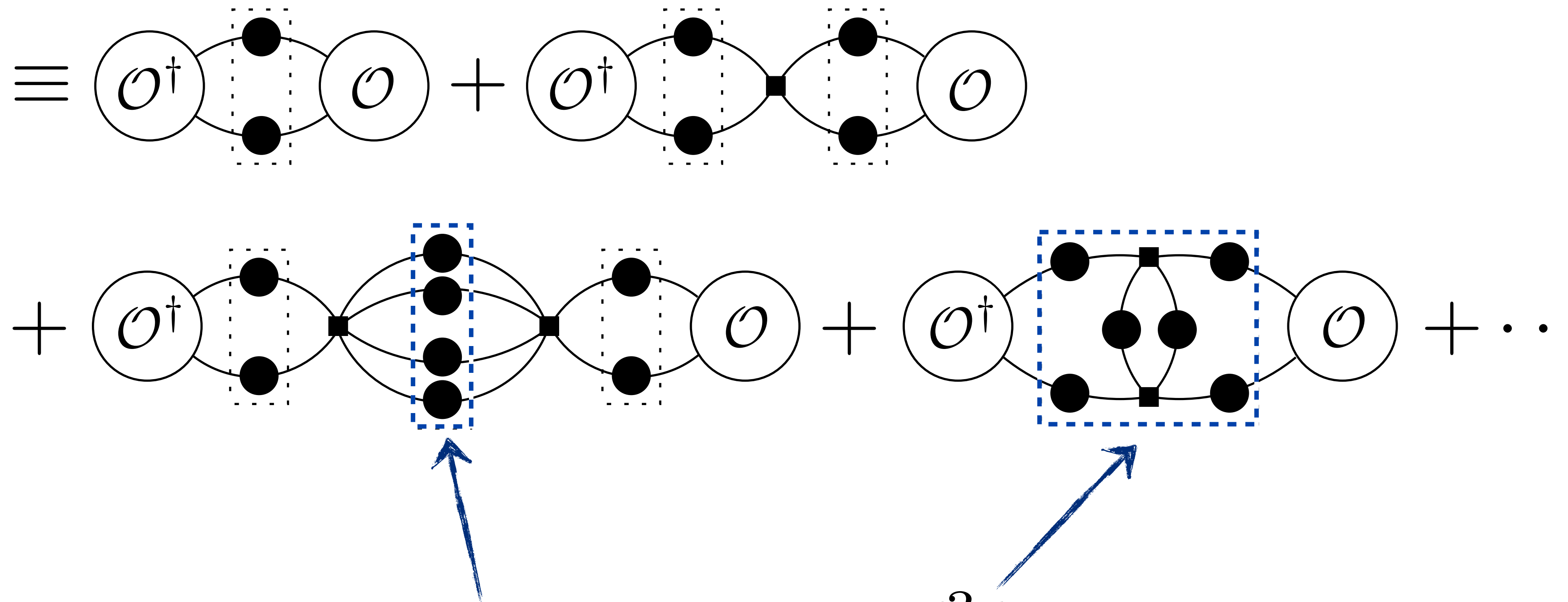
If particles in summed loops cannot all go on shell, then replace

$$\frac{1}{L^3}\sum_{\vec{k}} \longrightarrow \int\frac{d^3k}{(2\pi)^3}$$

difference is order

$$e^{-mL}$$

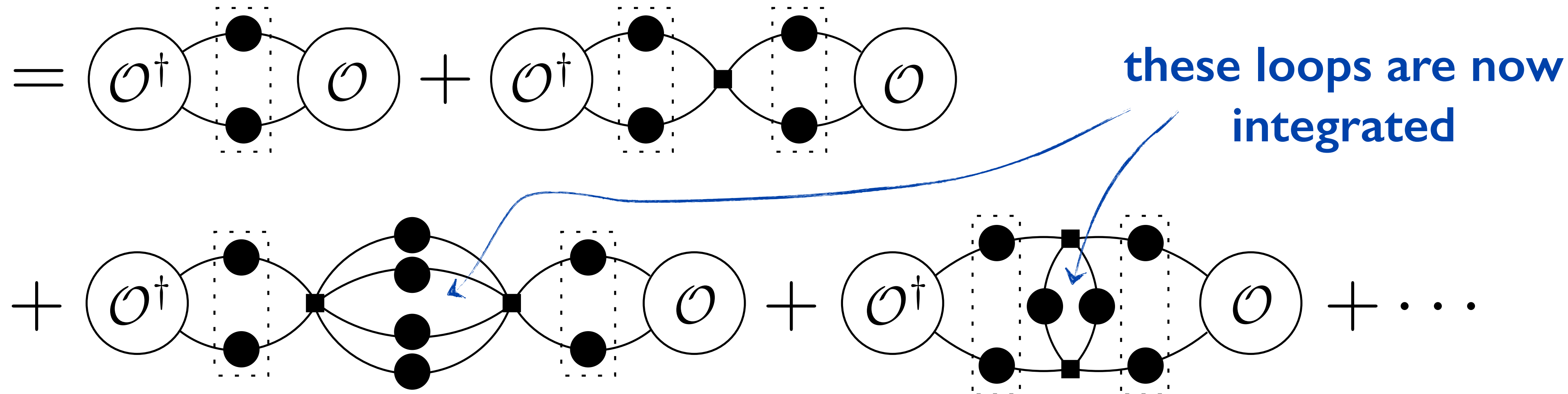
$C_L(P) \equiv$



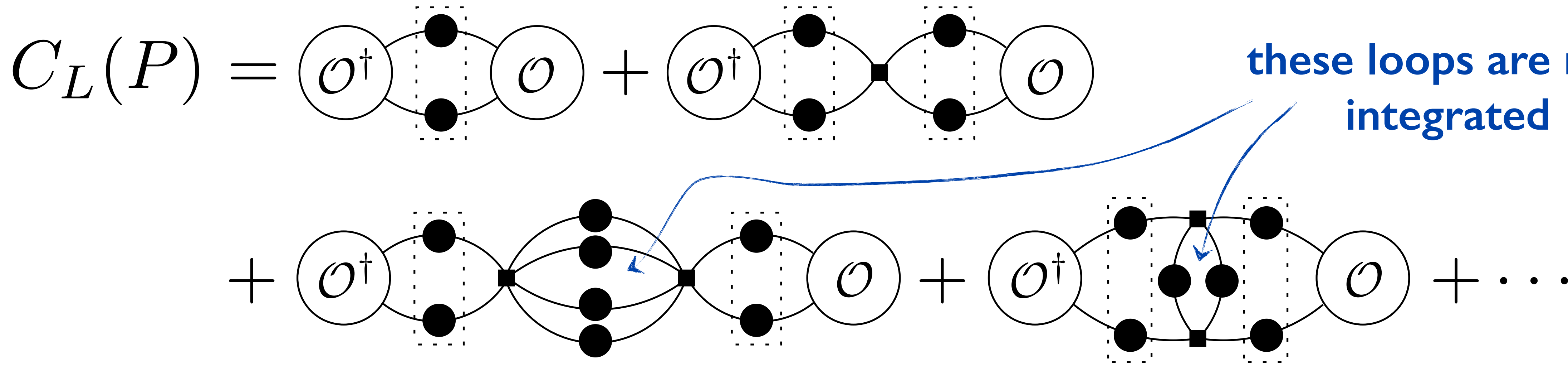
$$\frac{1}{L^3}\sum_{\vec{k}} \longrightarrow \int_{\vec{k}} \equiv \int \frac{d^3k}{(2\pi)^3}$$

Since $E^* < 4m$, **only two** particles with total momentum $(E, \vec{P})$ **can go on-shell**

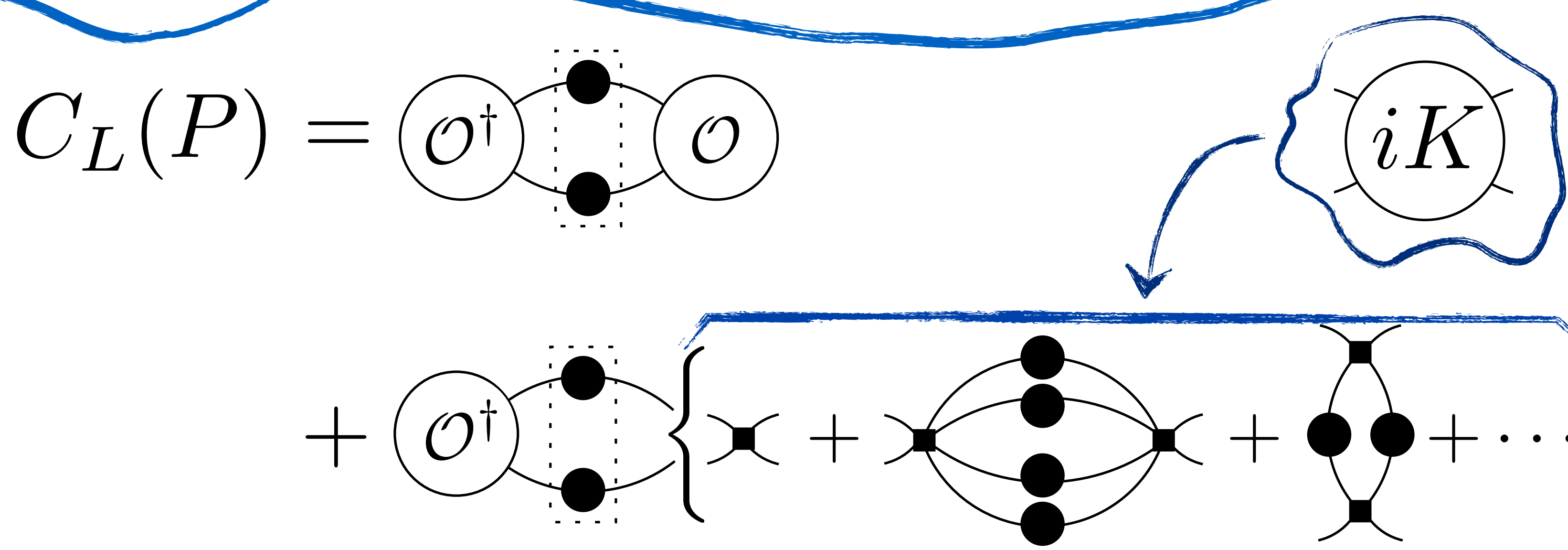
$C_L(P) =$

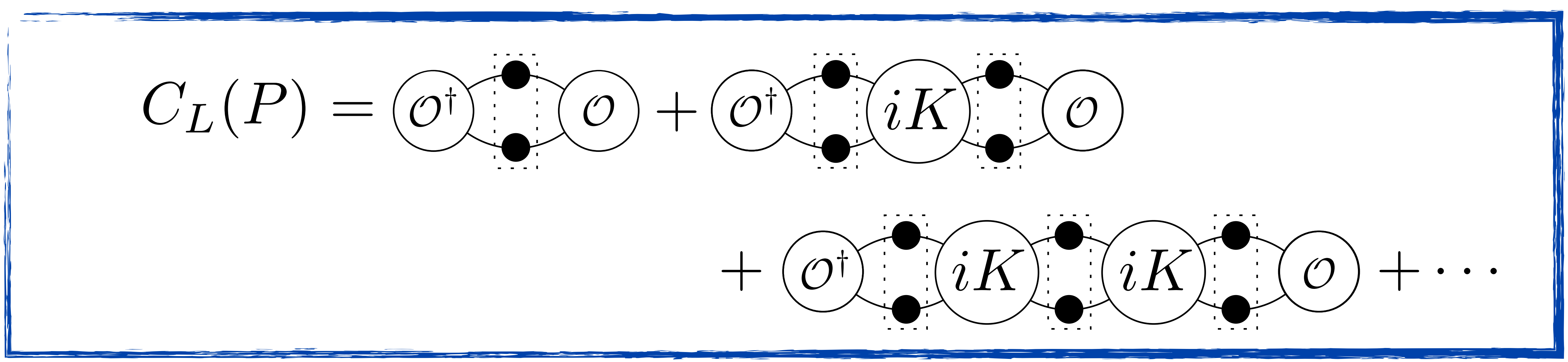


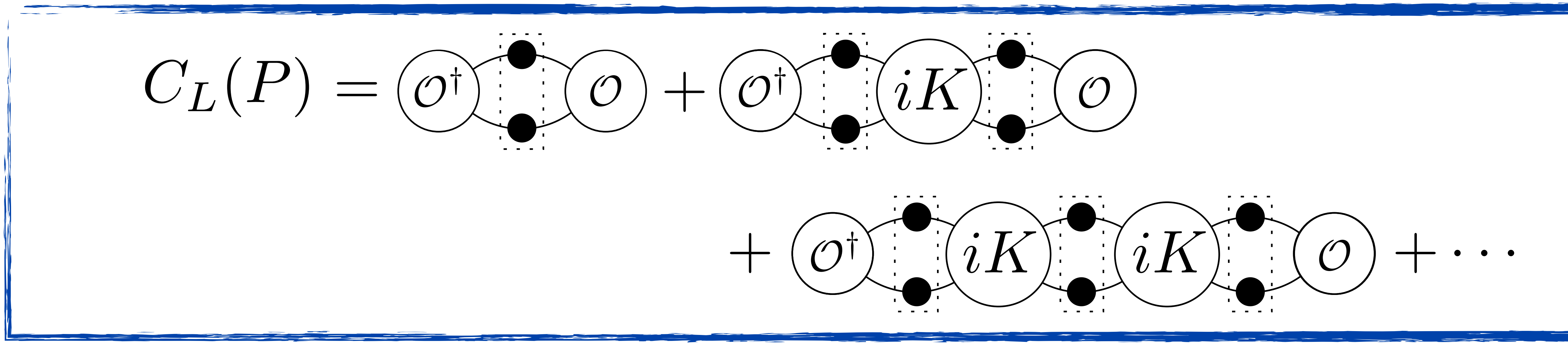
these loops are now integrated



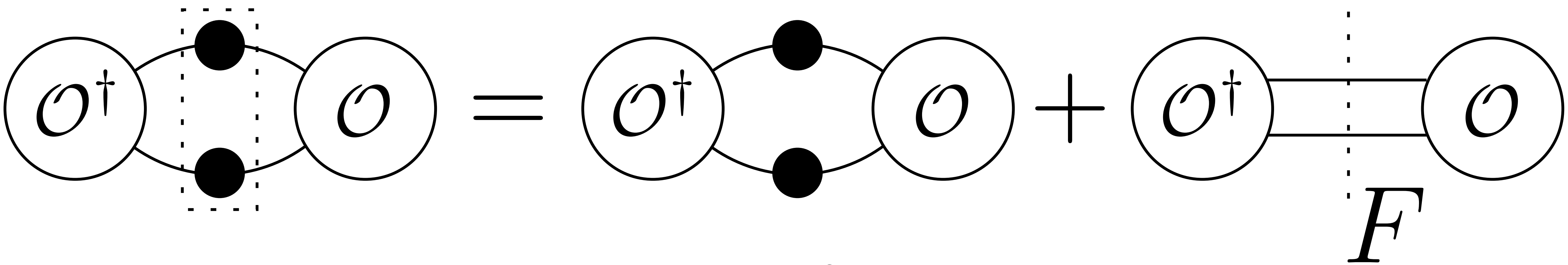
infinite-volume Bethe-Salpeter kernel





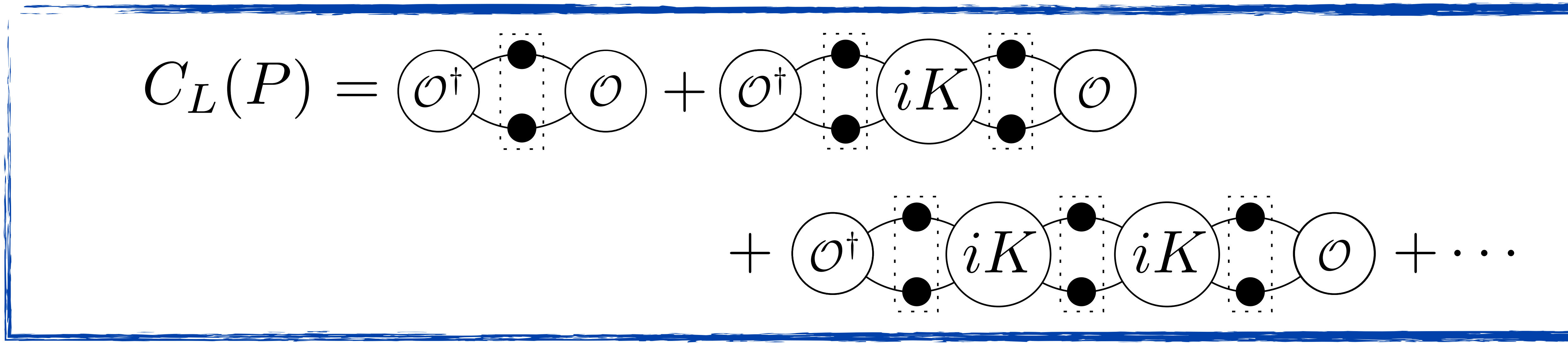


Next we introduce an important identity



$$\frac{1}{L^3}\sum_{\vec{k}} \qquad \int_{\vec{k}}$$

contains all power-law corrections



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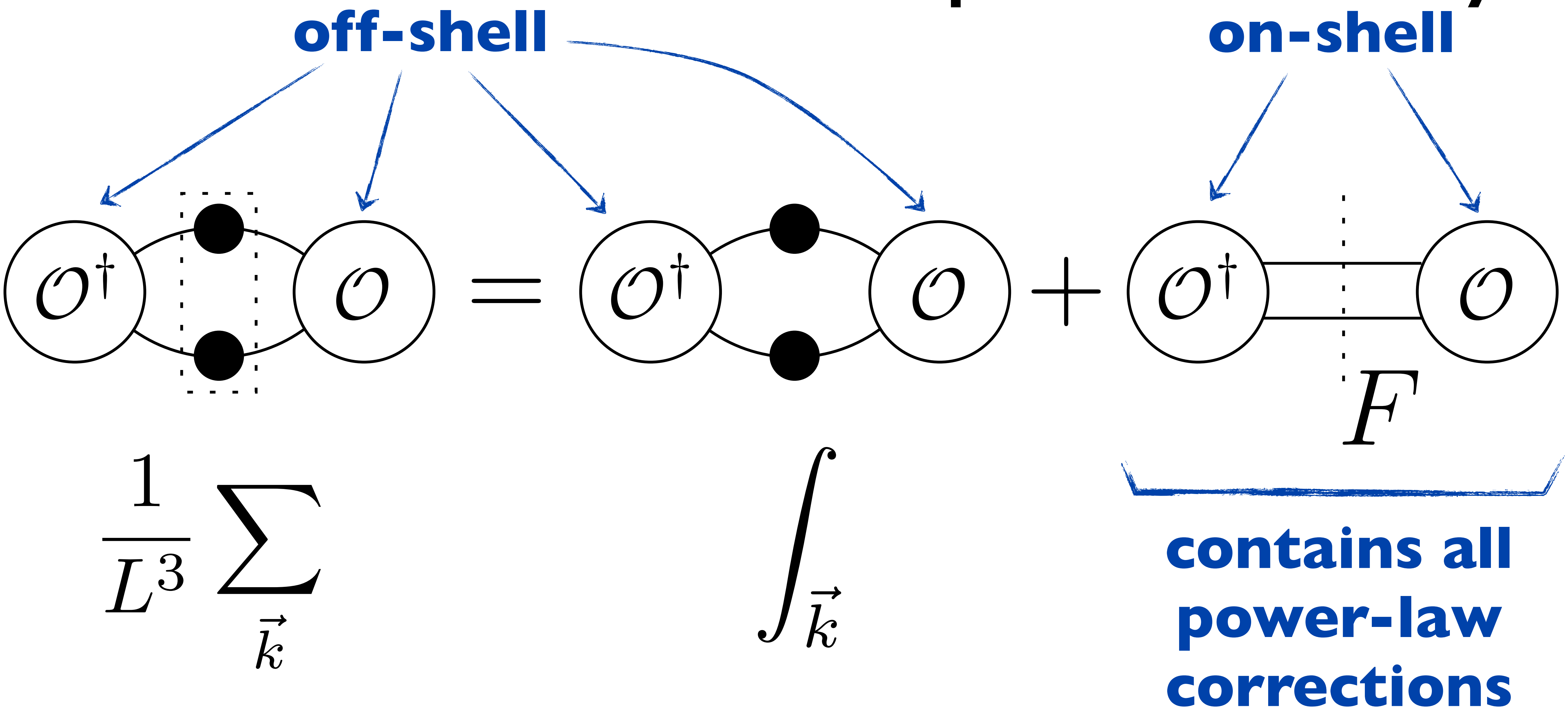
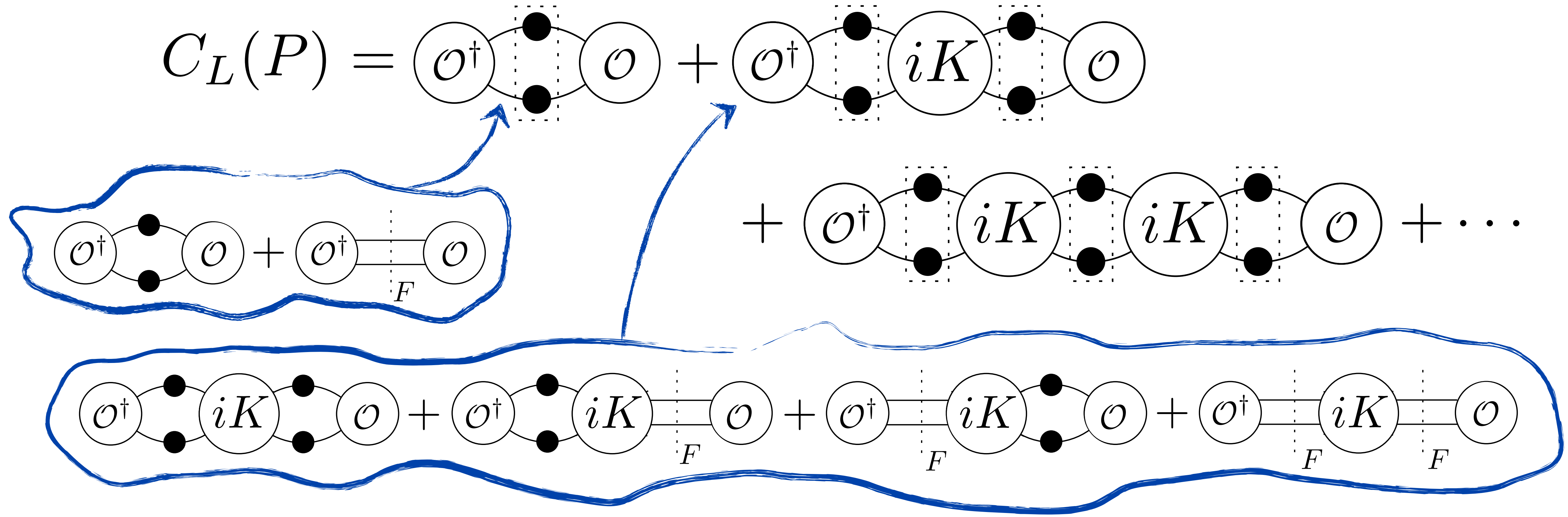
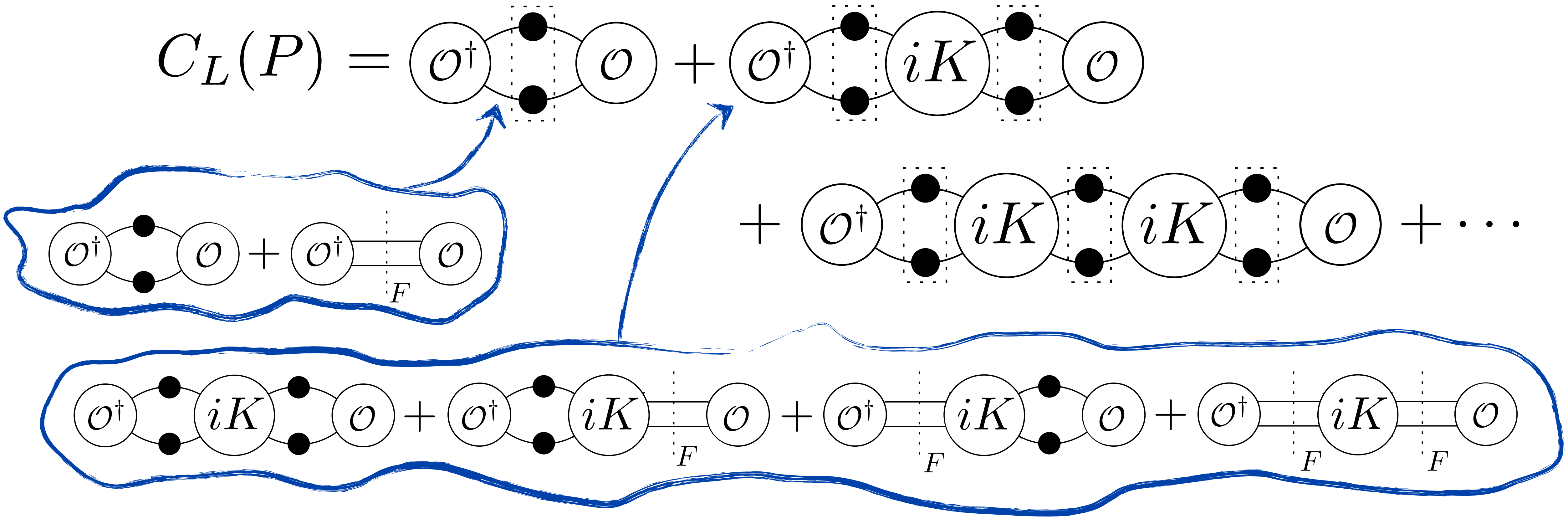


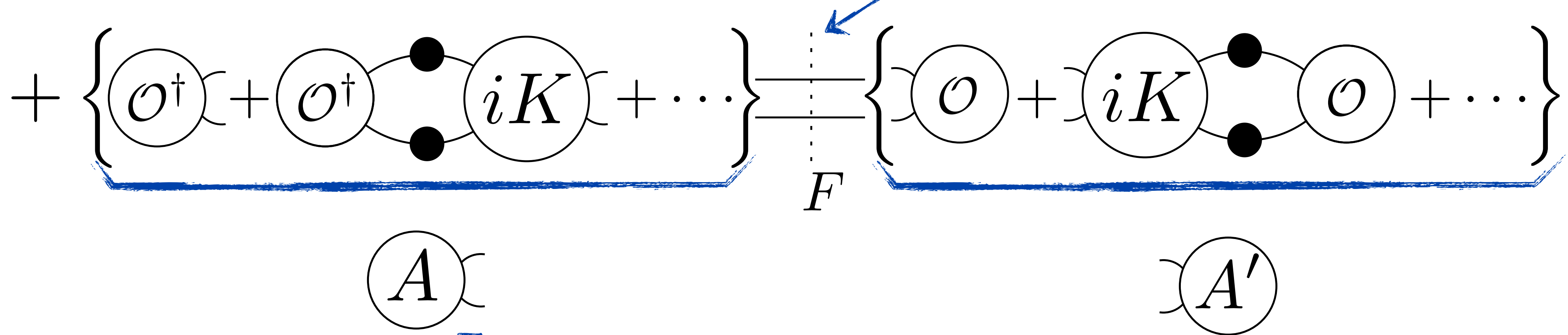

C_L(P) =
𝒪†
𝒪
iK
F
+ ⋯



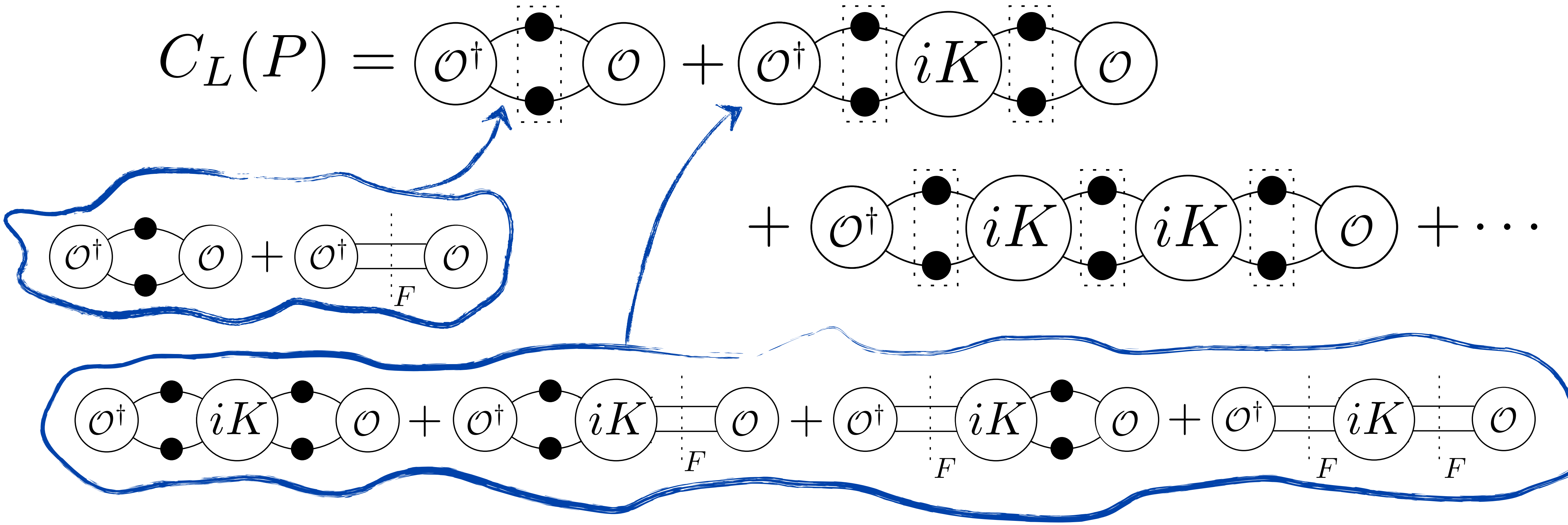
Now regroup by number of F cuts

$$C_L(P) = C_\infty(P)$$

zero F cuts



$\langle \pi\pi, \mathrm{out}|\mathcal{O}^\dagger|0\rangle$ $\langle 0|\mathcal{O}|\pi\pi, \mathrm{in}\rangle$



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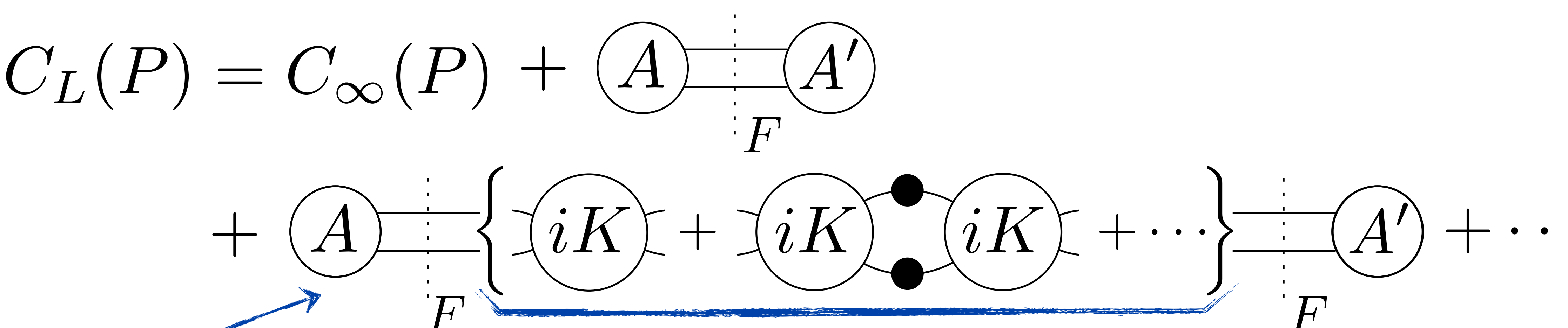
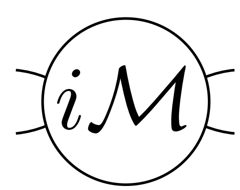


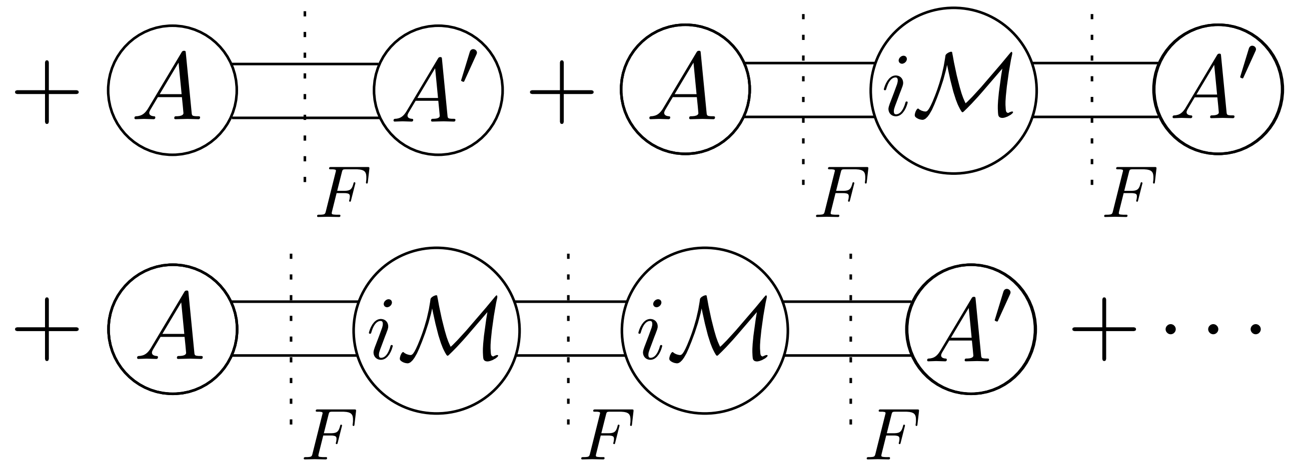


two F cuts

As Promised!

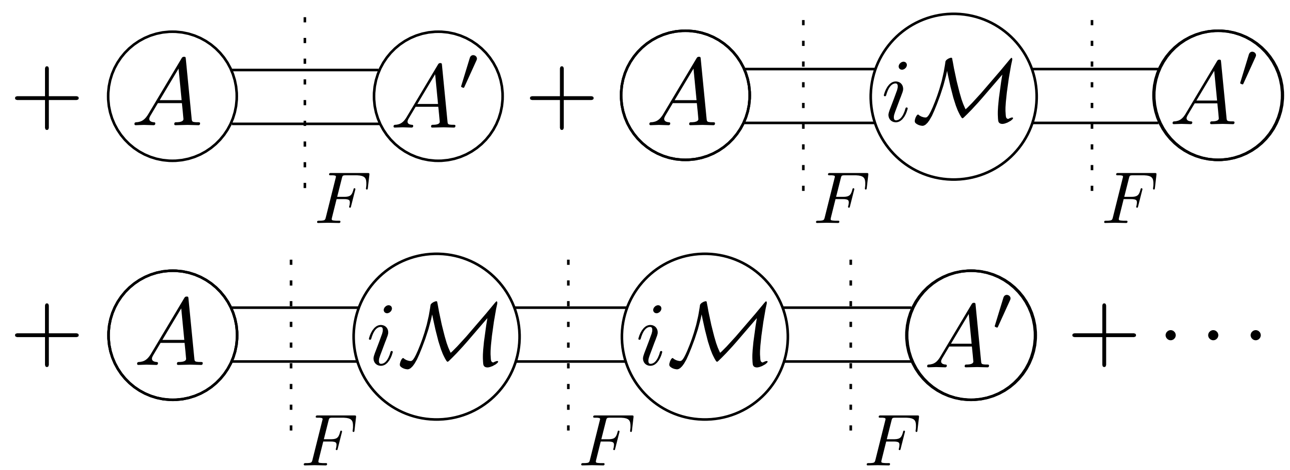
infinite-volume on-shell two-to-two scattering amplitude

$$C_L(P) = C_\infty(P)$$



$$C_L(P) = C_\infty(P) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2\to 2} iF]^n A$$

$$C_L(P) = C_\infty(P)$$



$$C_L(P) = C_\infty(P) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2\to2} iF]^n A$$

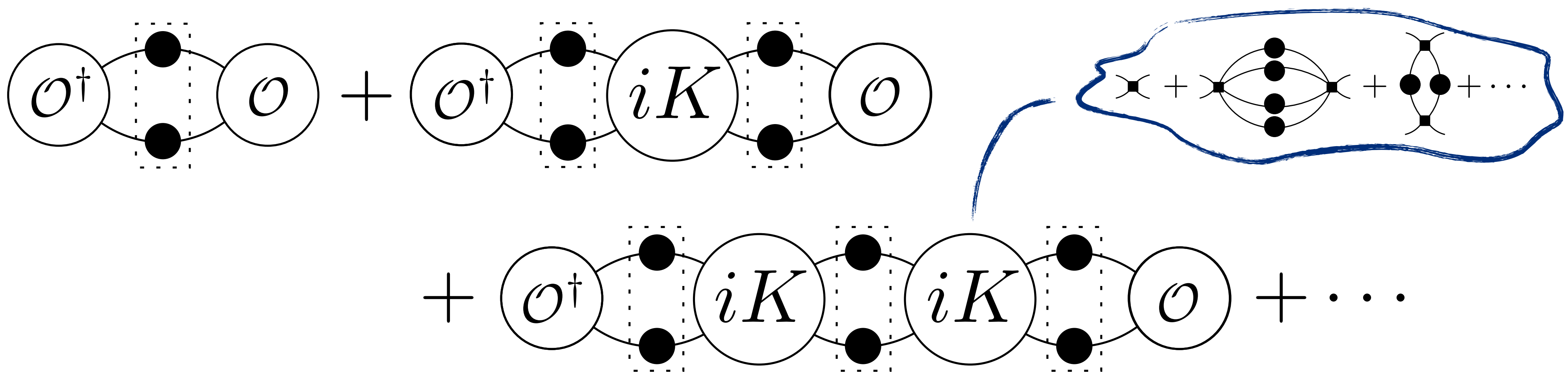
$$C_L(P) = C_\infty(P) + A' iF \frac{1}{1 - i\mathcal{M}_{2\to2} iF} A$$

no poles **no poles** **no poles**

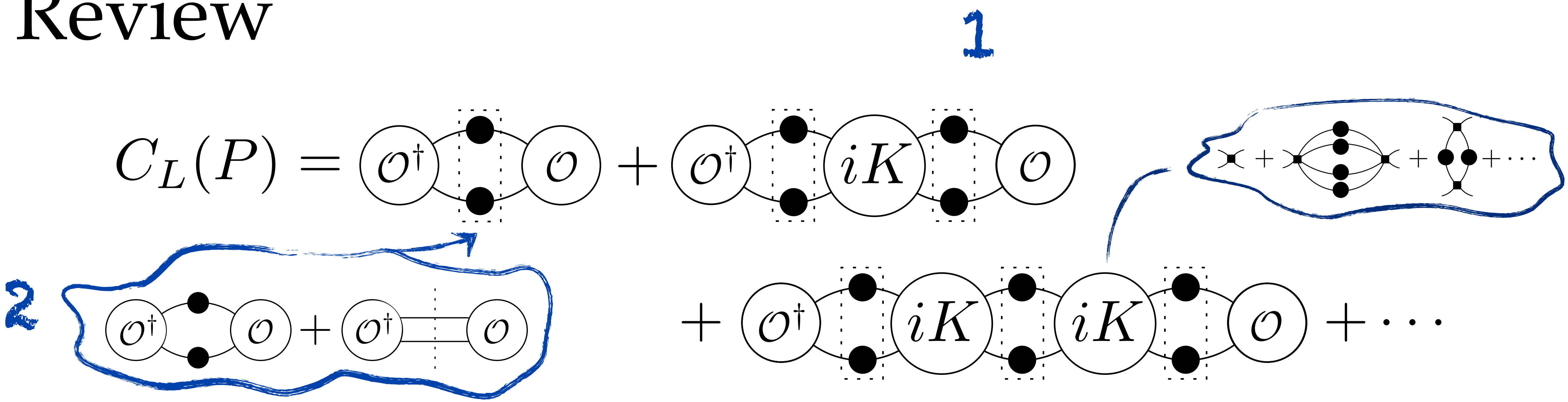
$C_L(P)$ **diverges whenever** $iF \dfrac{1}{1 - i\mathcal{M}_{2\to2} iF}$ **diverges**

Review

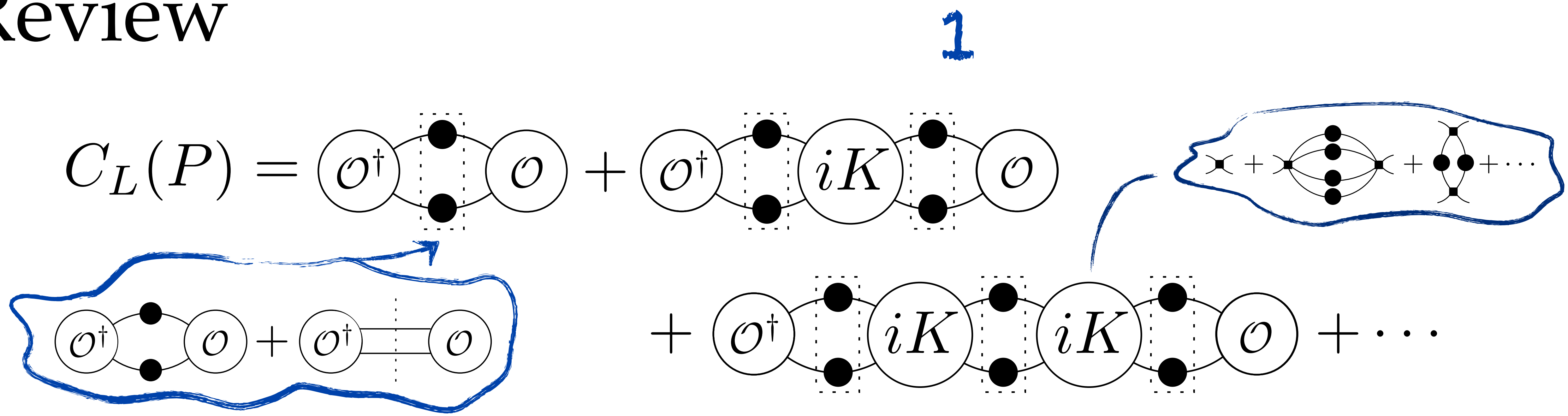
1

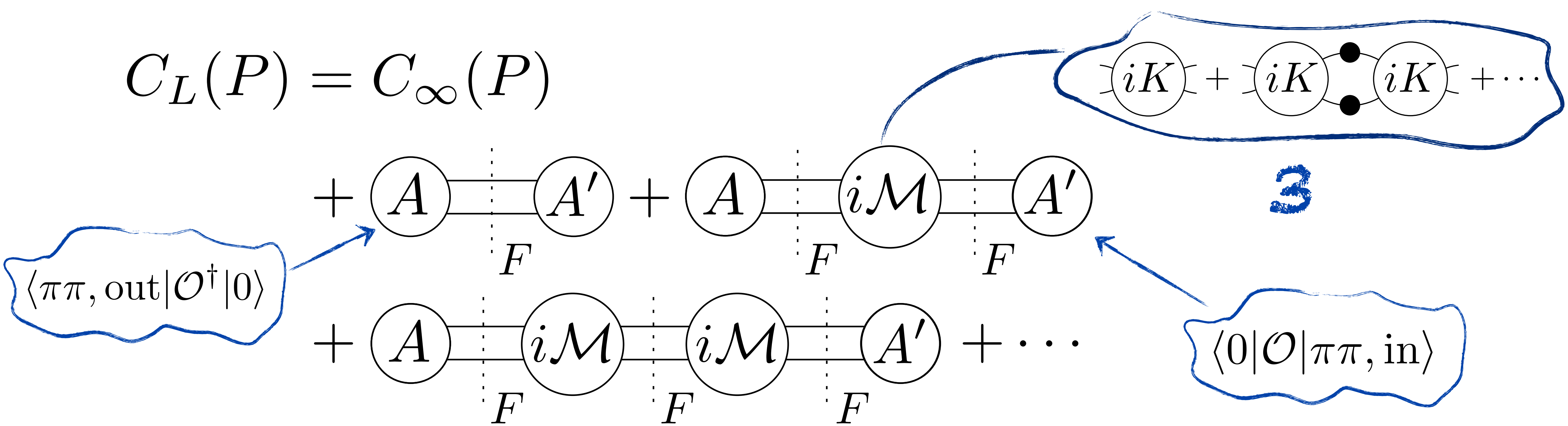
$C_L(P) =$ 

Review



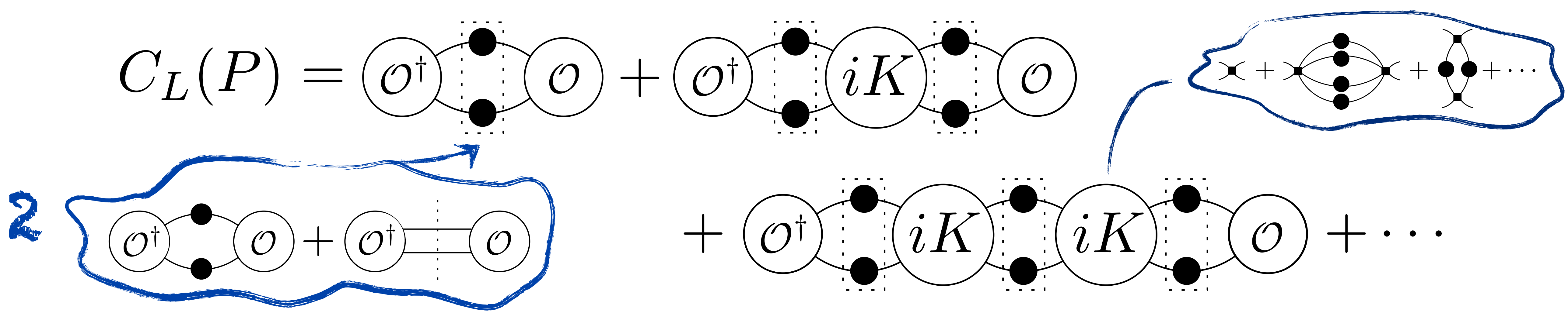
Review

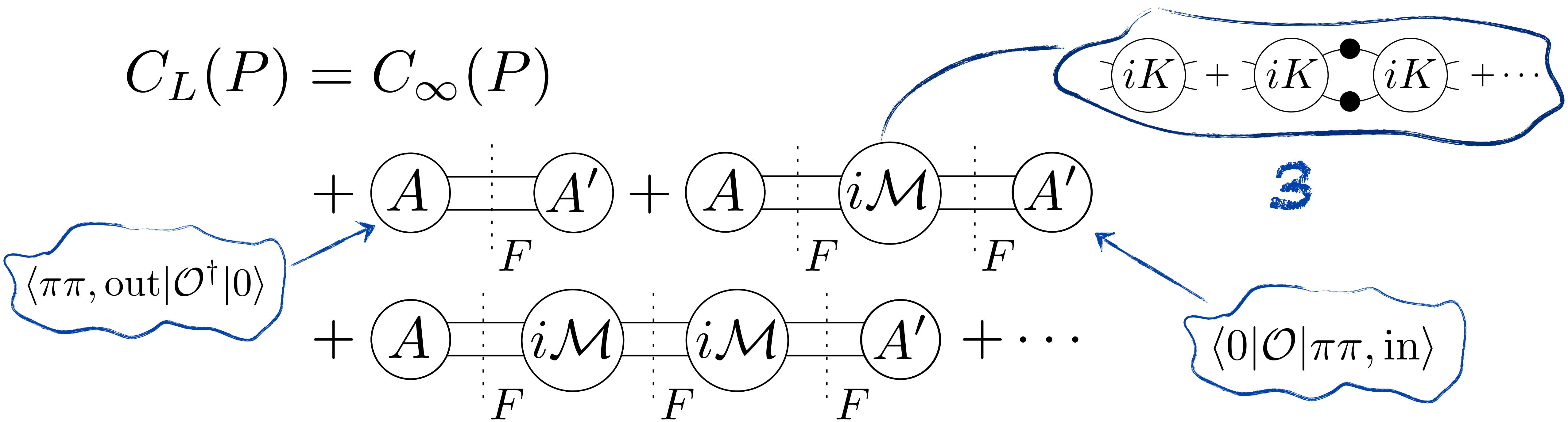

1
2
$C_L(P) =$


$C_L(P) = C_\infty(P)$
3
F
$\langle \pi\pi, \mathrm{out}|\mathcal{O}^\dagger|0\rangle$
$\langle 0|\mathcal{O}|\pi\pi, \mathrm{in}\rangle$

Review

1





4

$$C_L(P) = C_\infty(P) + A' iF \frac{1}{1 - i\mathcal{M}_{2\to 2} iF} A$$

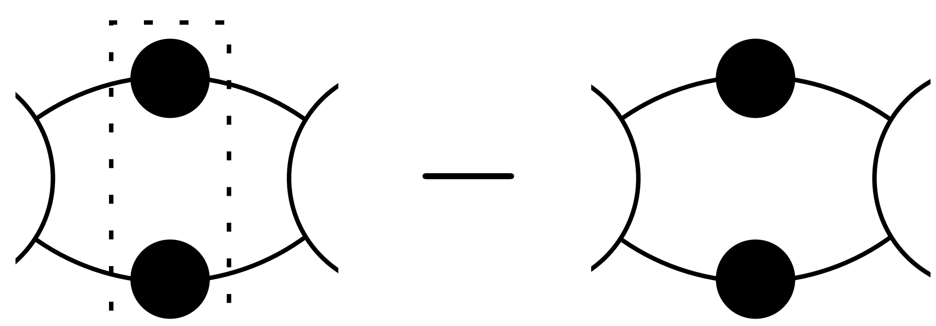
Quantization Condition

At fixed $(L, \vec{P})$, finite-volume spectrum
is all solutions to

$$\Delta_{L,P}(E) = \det[1 - i\mathcal{M}_{2\to2}\, iF] = 0$$

diagonal matrix in
angular momentum space

kinematic, not diagonal
(related to Lüscher
Zeta function)

$F \equiv$ 

Quantization Condition

$$\Delta_{L,P}(E) = \det[1 - i\mathcal{M}_{2\to 2} iF] = 0$$

...is it useful?

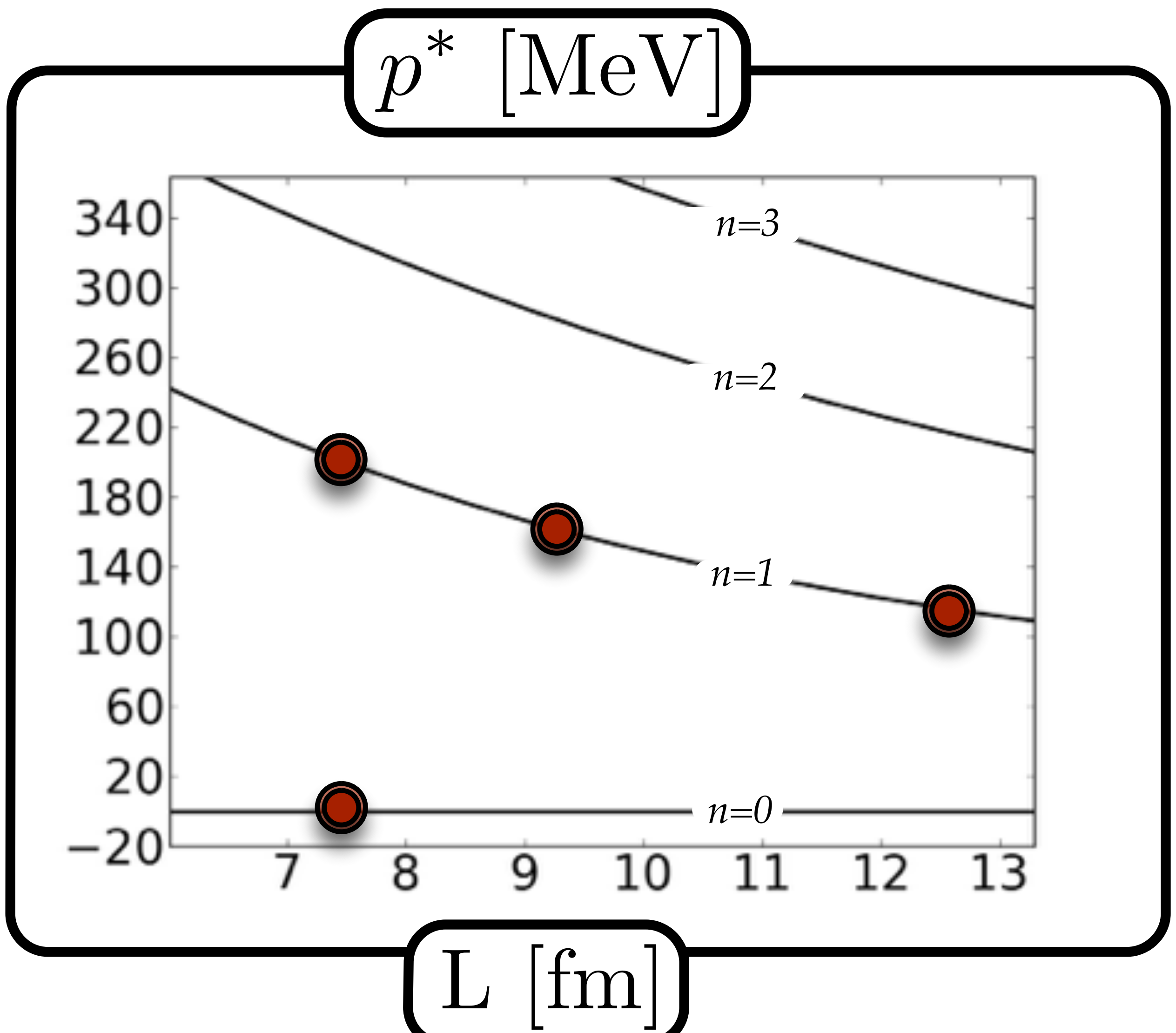
At low energies, s-wave dominates

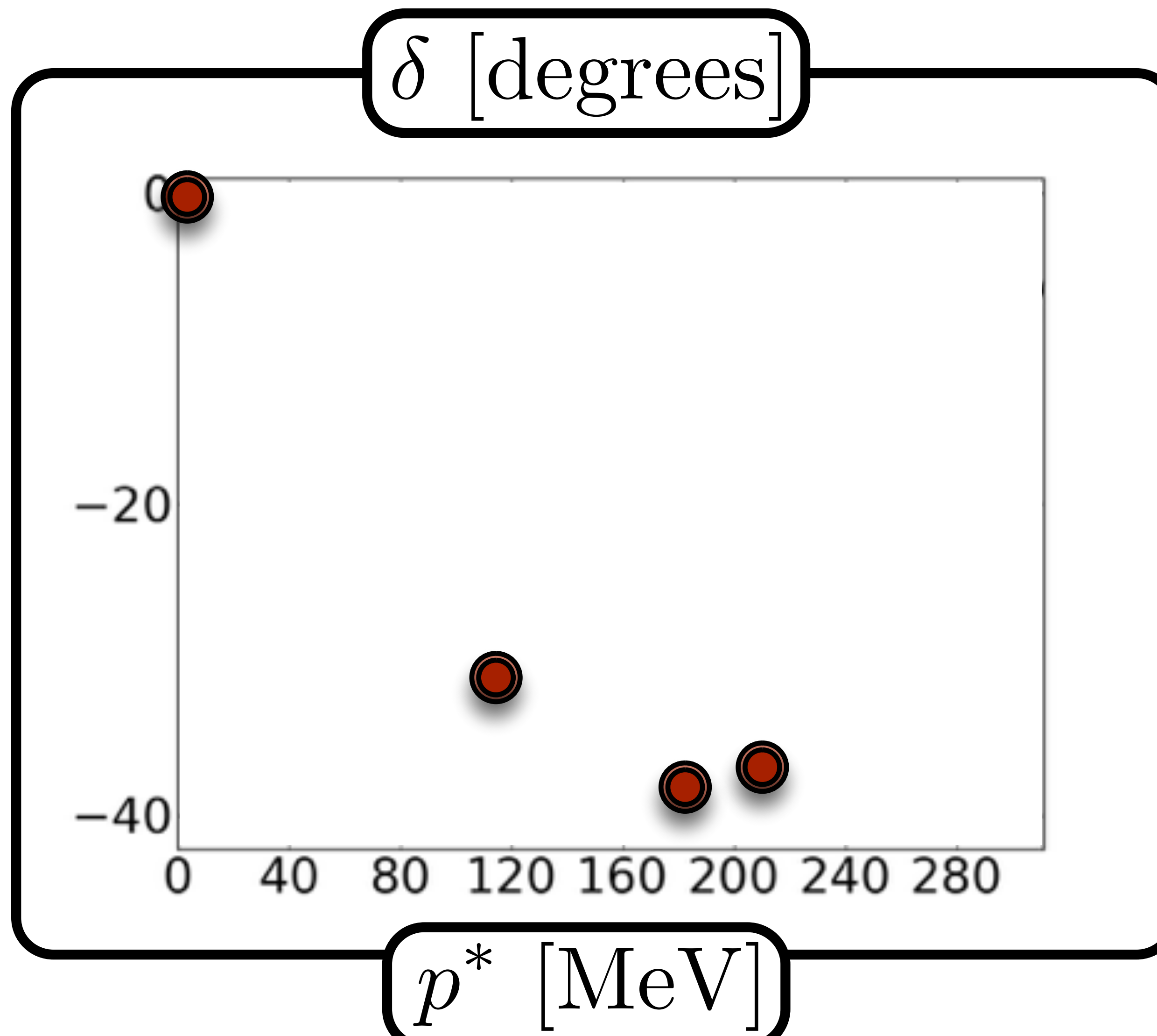
$$\boxed{[\mathcal{M}^s_{2\to 2}(E^*_n)]^{-1} = -F^s(E_n, \vec{P}, L)}$$

$$\left(F^s(E, \vec{P}, L) \equiv \frac{1}{2}\left[\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right] \frac{1}{2\omega_k 2\omega_{P-k}(E - \omega_k - \omega_{P-k} + i\epsilon)} \right)$$

$$\boxed{\left[\mathcal{M}^s_{2\to 2}(E^*_n)\right]^{-1} = -F^s(E_n, \vec{P}, L)}$$

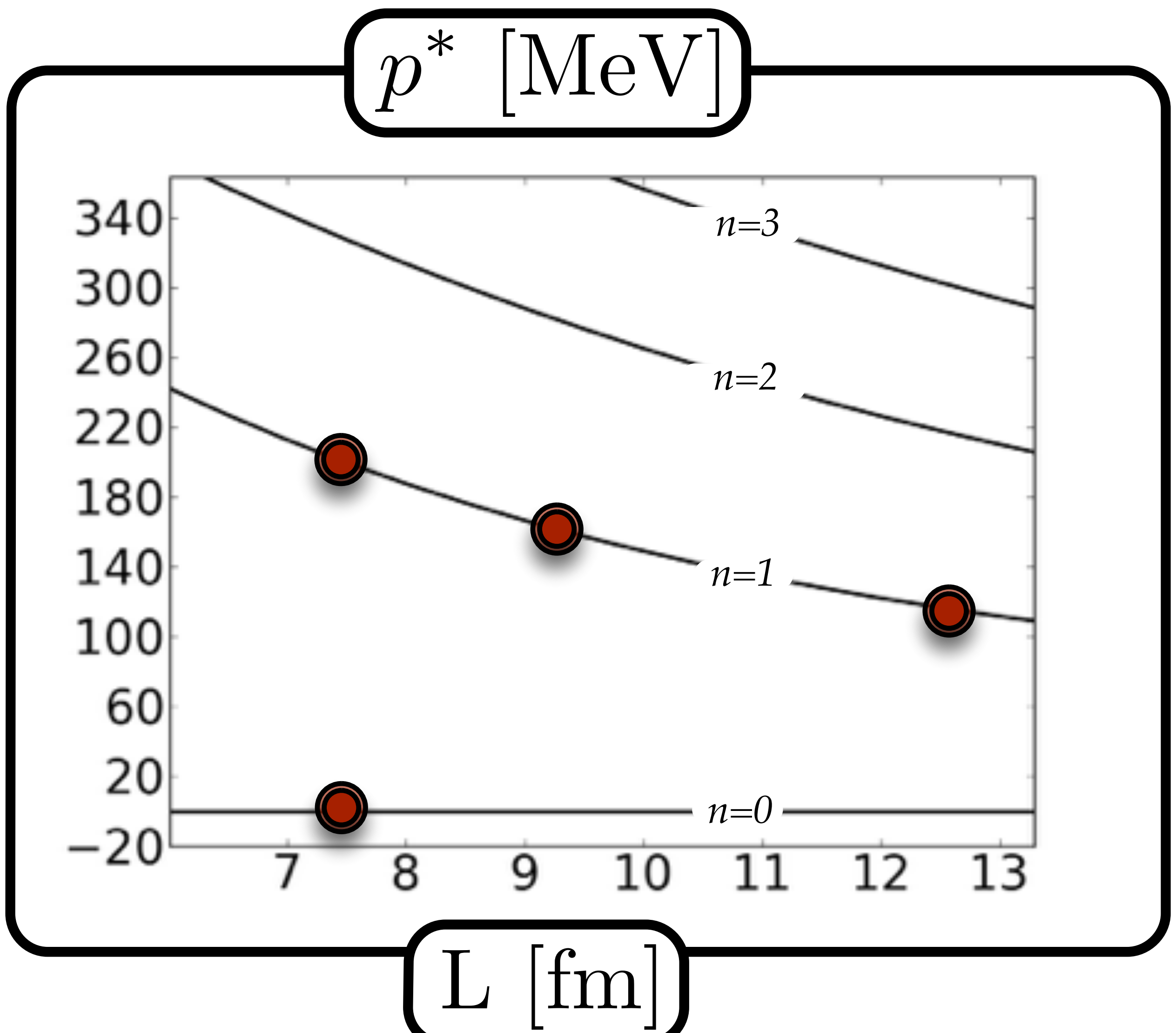
$$\mathcal{M}^s_{2\to 2}(E) = \frac{16\pi E}{p\cot\delta(p) - ip}$$

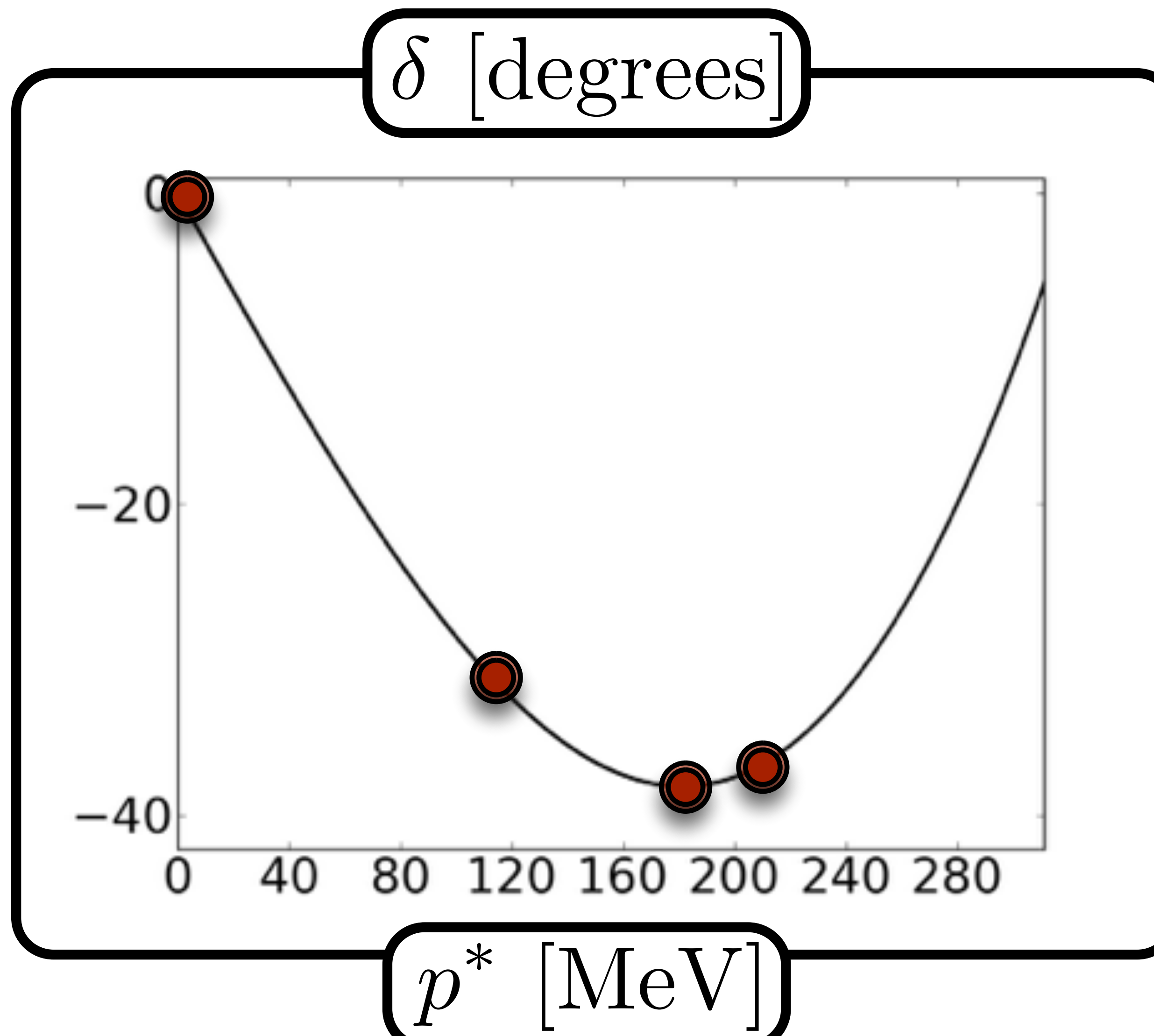




$$\boxed{\left[\mathcal{M}^s_{2\to 2}(E^*_n)\right]^{-1} = -F^s(E_n, \vec{P}, L)}$$

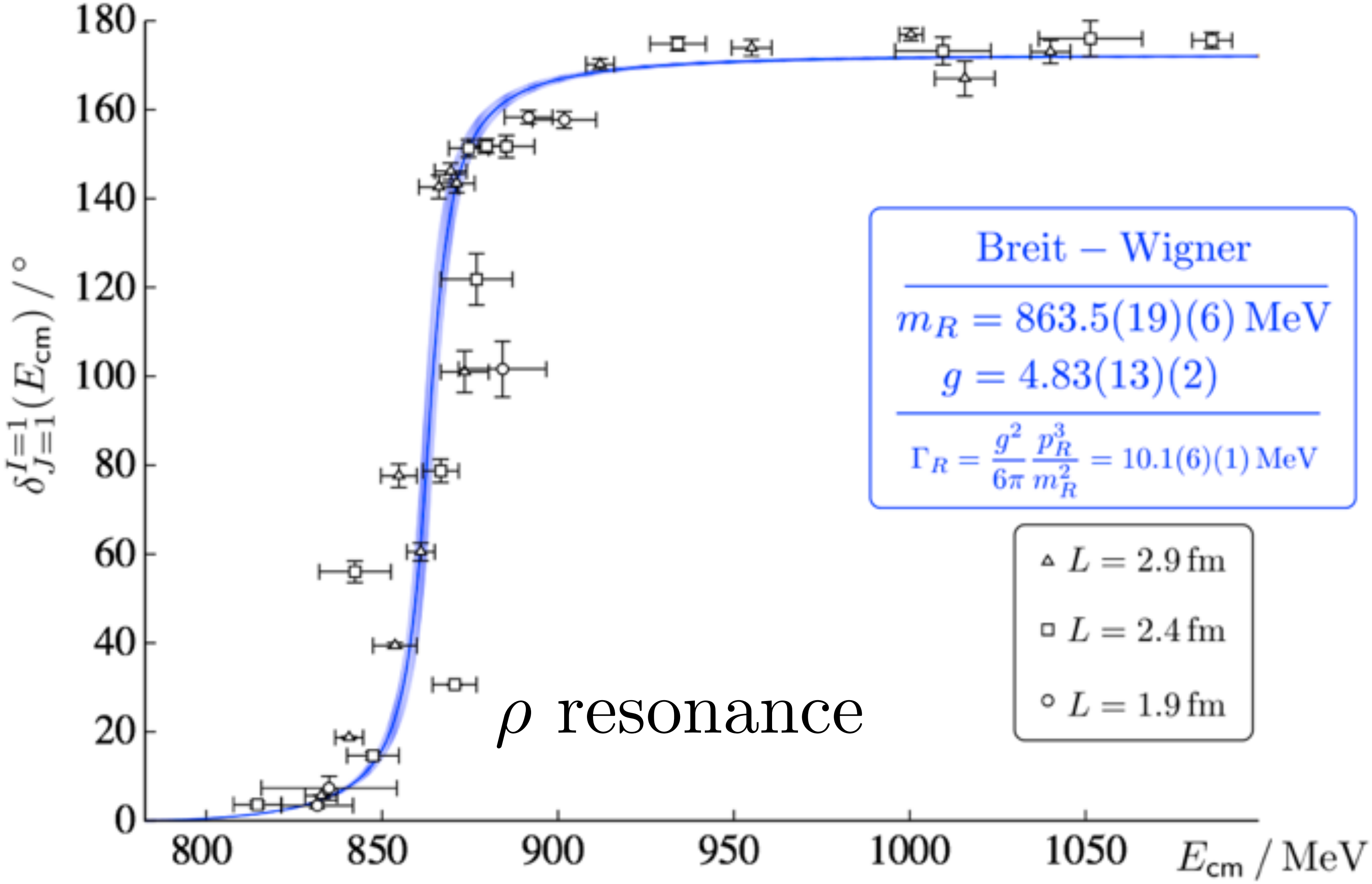
$$\mathcal{M}^s_{2\to 2}(E) = \frac{16\pi E}{p\cot\delta(p) - ip}$$





$$p_n \cot \delta_{J=1}(p_n) = -16\pi E_n^* \, \mathrm{Re}\, F_{10;10}(E_n, \vec{P}, L)$$

$\pi\pi \to \pi\pi$



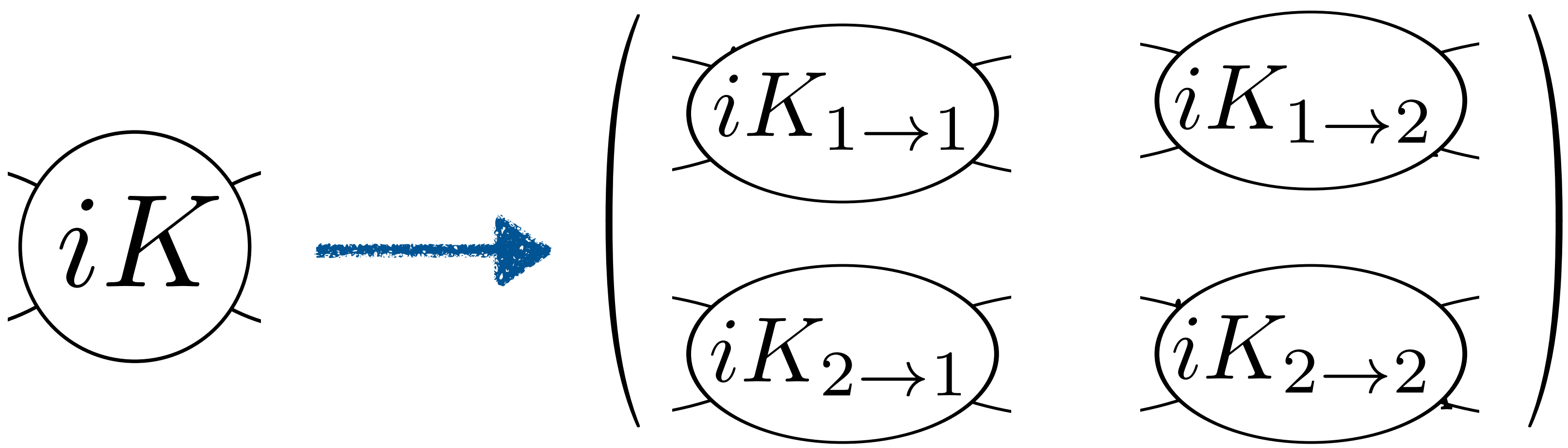
$m_\pi = 391$ MeV

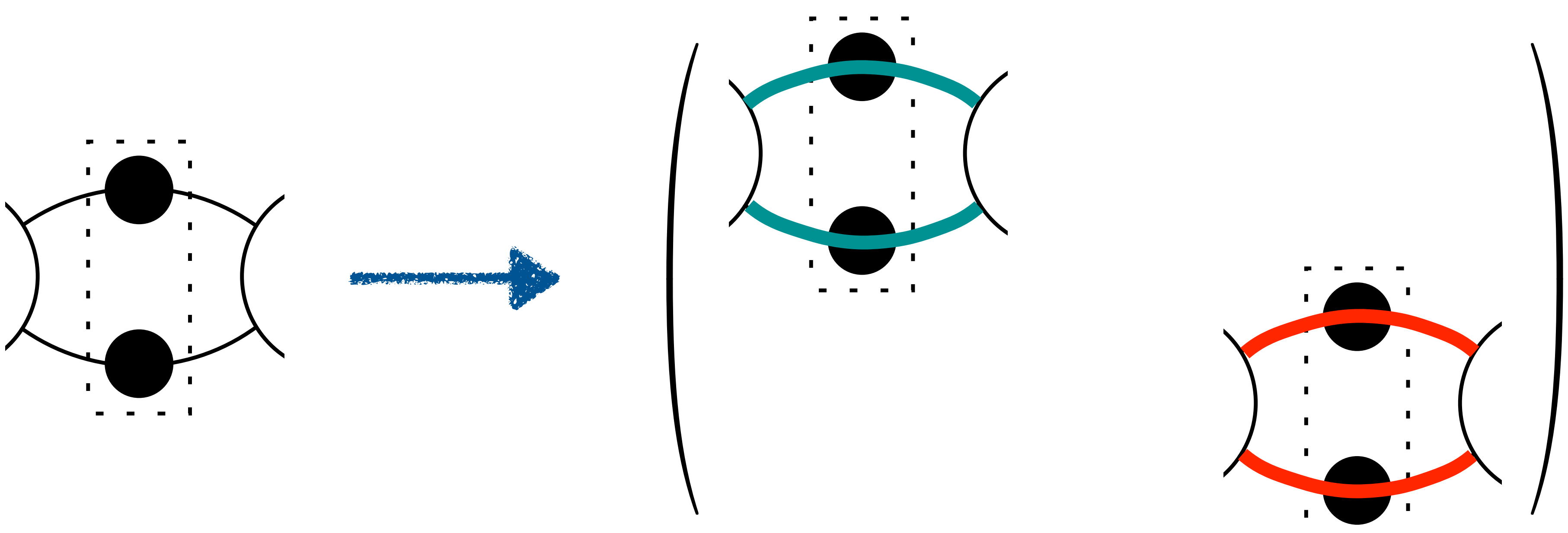
from Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505

Scattering of multiple two-particle channels

$$\pi\pi \to \overline{K}K \qquad \pi K \to \eta K$$

Make following replacements





Scattering of multiple two-particle channels

$$\pi\pi \to \overline{K}K \qquad \pi K \to \eta K$$

One finds

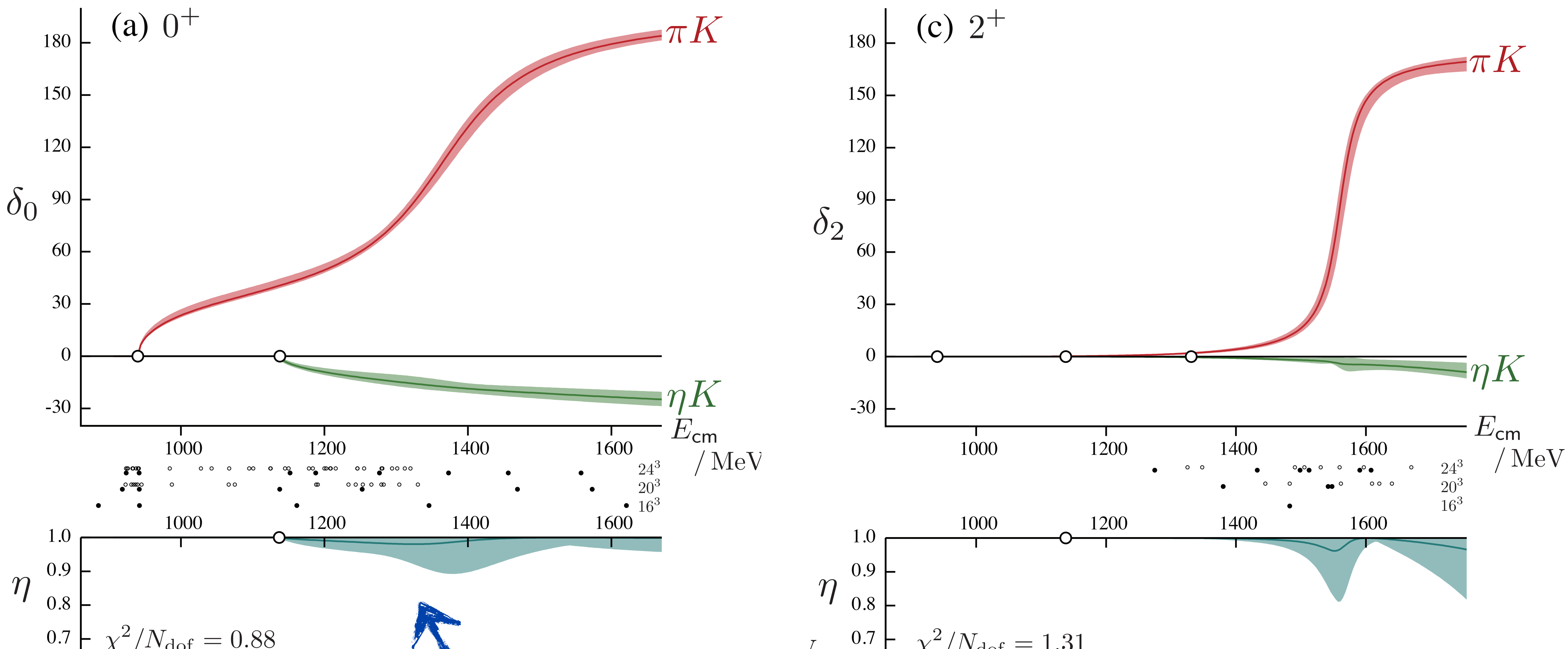
$$\det\left[1 - \begin{pmatrix} i\mathcal{M}_{1\to 1} & i\mathcal{M}_{1\to 2} \\ i\mathcal{M}_{2\to 1} & i\mathcal{M}_{2\to 2} \end{pmatrix} \begin{pmatrix} iF_1 & 0 \\ 0 & iF_2 \end{pmatrix}\right] = 0$$

MTH, S. R. Sharpe, *Phys.Rev. D86* (2012) 016007

R. A. Briceño, Z. Davoudi, *Phys.Rev. D88* (2013) 094507

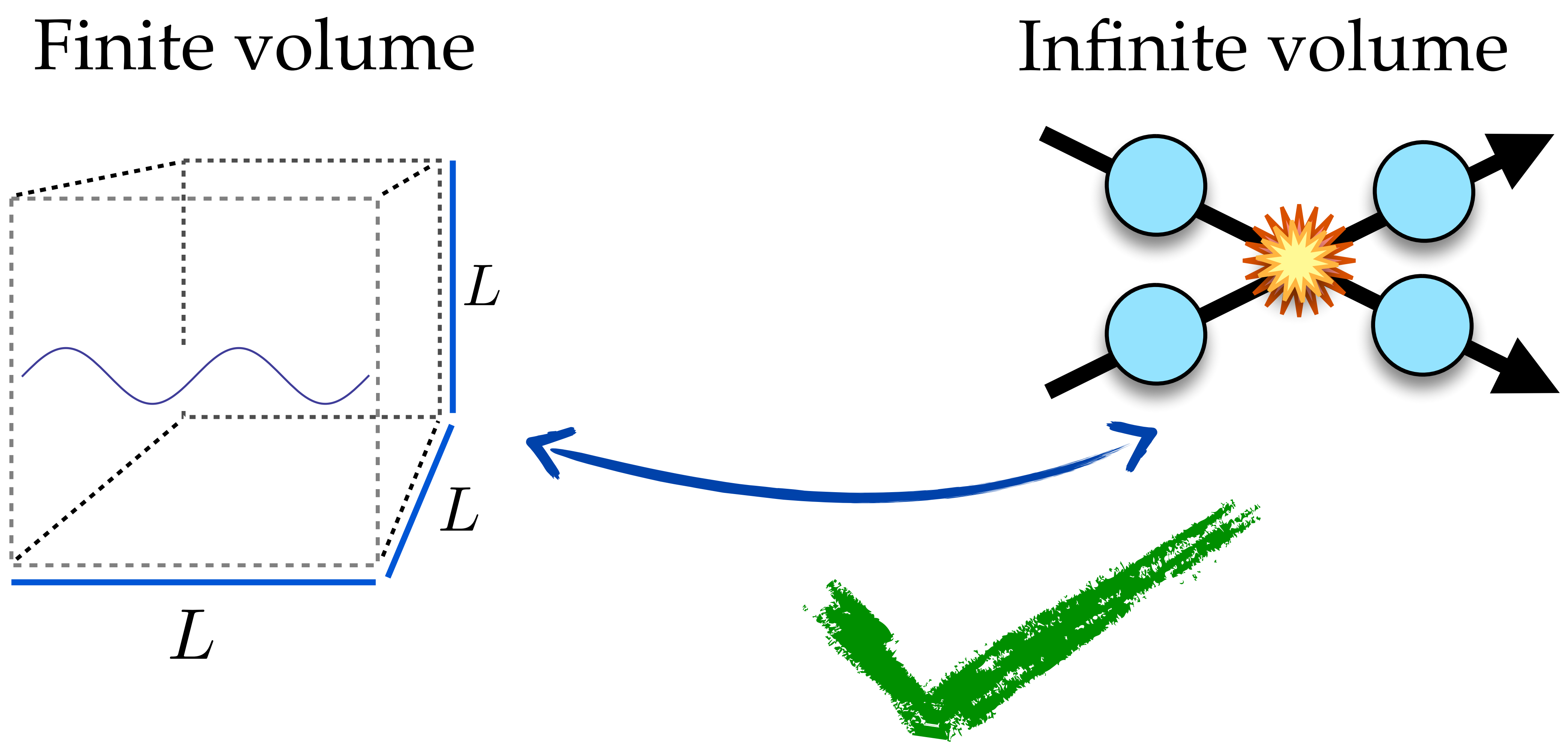
Already implemented in LQCD calculation

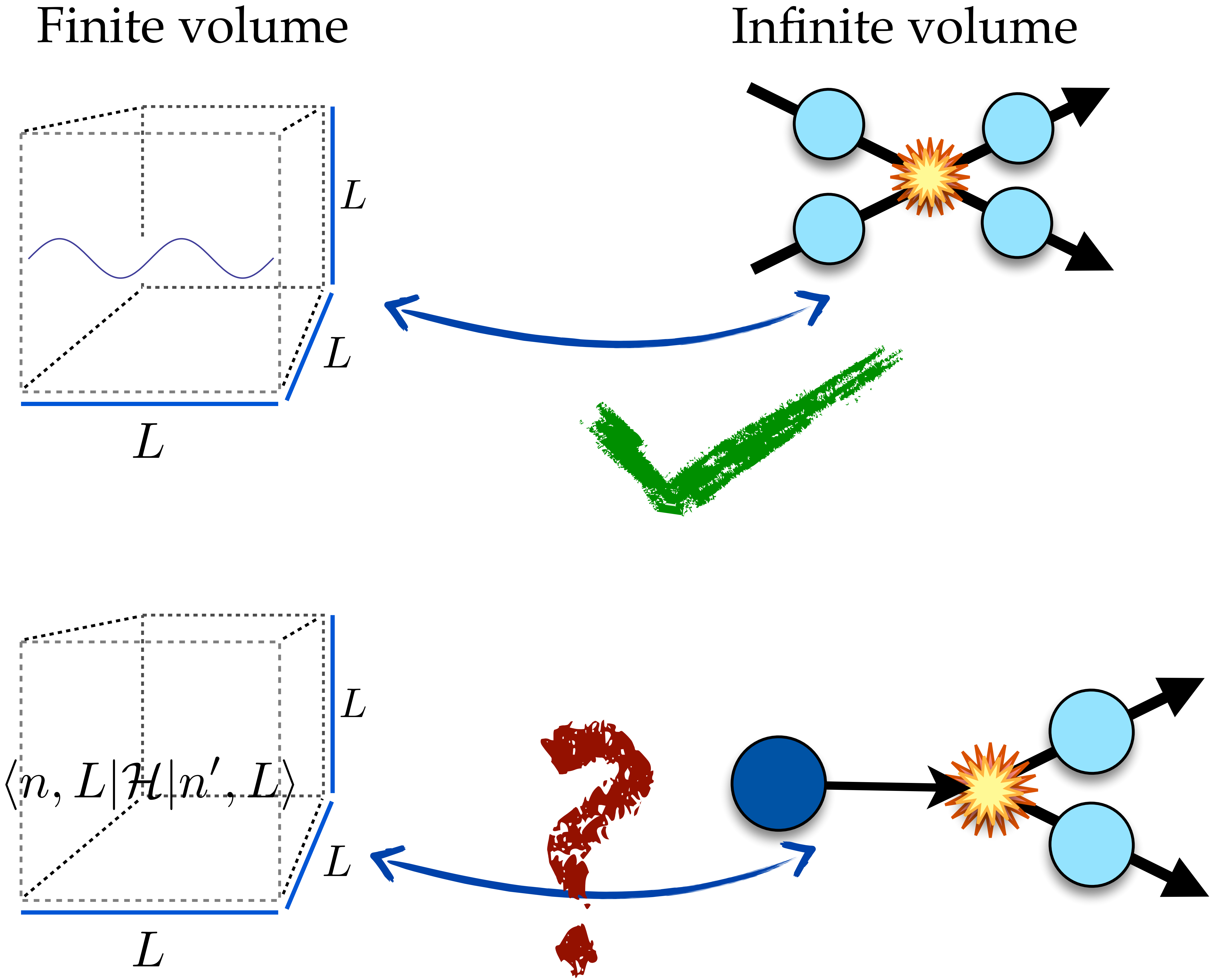
$$\pi K \to \eta K$$



$$\mathcal{M}(\pi K \to \eta K) \sim \sqrt{1-\eta^2}$$

from Dudek, Edwards, Thomas, Wilson in arXiv:1406:4158


Finite volume
Infinite volume
L
L
L



Finite volume
Infinite volume
L
L
L
L
L
L
$\langle n, L|\mathcal{H}|n', L\rangle$

Relate finite- and infinite-volume states

$$C_L(x_0 - y_0, \mathbf{P}) \equiv \int_L d^3x \int_L d^3y \, e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(y) | 0 \rangle$$

Relate finite- and infinite-volume states

$$C_L(x_0 - y_0, \mathbf{P}) \equiv \int_L d^3x \int_L d^3y\, e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(y)|0\rangle$$

before $P_0, \mathbf{P}$

now $x_0 - y_0, \mathbf{P}$

Relate finite- and infinite-volume states

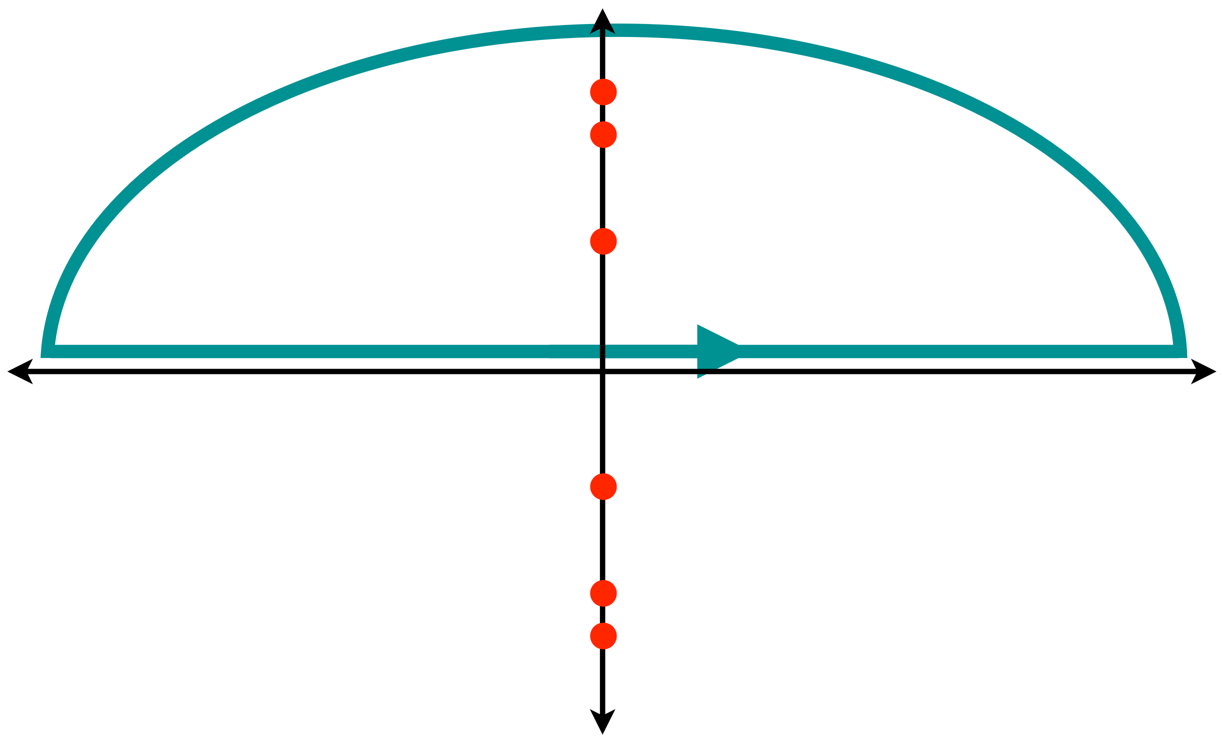
$$C_L(x_0 - y_0, \mathbf{P}) \equiv \int_L d^3x \int_L d^3y \, e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(y)|0\rangle$$

$$= L^3 \int \frac{dP_0}{2\pi} \, e^{iP_0(x_0-y_0)} \left[C_\infty(P) + A' \frac{1}{(iF)^{-1} - i\mathcal{M}_{2\to 2}} A \right]$$

Relate finite- and infinite-volume states

$$C_L(x_0 - y_0, \mathbf{P}) \equiv \int_L d^3x \int_L d^3y \, e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(y)|0\rangle$$

$$= L^3 \int \frac{dP_0}{2\pi} \, e^{iP_0(x_0-y_0)} \left[C_\infty(P) + A' \frac{1}{(iF)^{-1} - i\mathcal{M}_{2\to 2}} A\right]$$



C_L **analytic structure**

Relate finite- and infinite-volume states

$$C_L(x_0 - y_0, \mathbf{P}) \equiv \int_L d^3x \int_L d^3y \, e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(y) | 0 \rangle$$

$$= L^3 \int \frac{dP_0}{2\pi} \, e^{iP_0(x_0-y_0)} \left[C_\infty(P) + A' \frac{1}{(iF)^{-1} - i\mathcal{M}_{2\to 2}} A \right]$$

$$= \sum_n e^{-E_{n,L}(x_0-y_0)} \overbrace{\langle 0 | \mathcal{O}(0, \mathbf{P}) | \pi\pi, \mathrm{in} \rangle_n} \mathcal{R}_n \overbrace{\langle \pi\pi, \mathrm{out} | \mathcal{O}^\dagger(0, -\mathbf{P}) | 0 \rangle_n}$$

$$\mathcal{R}_n \equiv \text{Residue of } \frac{1}{(iF)^{-1} - i\mathcal{M}_{2\to 2}} \text{ at } E_{n,L}$$

Relate finite- and infinite-volume states

$$C_L(x_0 - y_0, \mathbf{P}) \equiv \int_L d^3x \int_L d^3y \, e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(y) | 0 \rangle$$

$$= L^3 \int \frac{dP_0}{2\pi} e^{iP_0(x_0-y_0)} \left[C_\infty(P) + A' \frac{1}{(iF)^{-1} - i\mathcal{M}_{2\to2}} A \right]$$

$$= \sum_n e^{-E_{n,L}(x_0-y_0)} \langle 0 | \mathcal{O}(0, \mathbf{P}) \underbrace{| \pi\pi, \mathrm{in} \rangle_n \mathcal{R}_n \langle \pi\pi, \mathrm{out} |}_{|n, L, \text{“}\pi\pi\text{”}\rangle\langle n, L, \text{“}\pi\pi\text{”}|} \mathcal{O}^\dagger(0, -\mathbf{P}) | 0 \rangle_n$$

$$\mathcal{R}_n \equiv \text{ Residue of } \frac{1}{(iF)^{-1} - i\mathcal{M}_{2\to2}} \text{ at } E_{n,L}$$

Relation on states: Single channel

$$|n, L, \text{“}\pi\pi\text{”}\rangle\langle n, L, \text{“}\pi\pi\text{”}| = \begin{pmatrix} |\pi\pi, \text{in}, J=0\rangle & |\pi\pi, \text{in}, J=1\rangle & \cdots \end{pmatrix} \begin{pmatrix} \\ \mathcal{R}_n \\ \\ \end{pmatrix} \begin{pmatrix} \langle\pi\pi, \text{out}, J=0| \\ \langle\pi\pi, \text{out}, J=1| \\ \vdots \end{pmatrix}$$

$$\mathcal{R}_n \equiv \text{Residue of } \frac{1}{(iF)^{-1} - i\mathcal{M}_{2\to2}} \text{ at } E_{n,L}$$

- Matrix in angular momentum space
- Depends on $\mathcal{M}_{2\to2}$, $d\mathcal{M}_{2\to2}/dE$, L
- Generalization of the Lellouch-Lüscher factor
- Just a normalization factor in case of a single channel

Relation on states: Coupled channels

$$|n,L\rangle\langle n,L| = \big(|\phi_1\phi_2, J=0\rangle \quad |\phi_3\phi_4, J=0\rangle \quad \cdots\big)\begin{pmatrix} \\ & \mathcal{R}_n & \\ \\ \end{pmatrix}\begin{pmatrix} \langle\phi_1\phi_2, J=0| \\ \langle\phi_3\phi_4, J=0| \\ \langle\phi_1\phi_2, J=1| \\ \langle\phi_3\phi_4, J=1| \\ \vdots \end{pmatrix}$$

$$\mathcal{R}_n \equiv \text{Residue of } \frac{1}{(iF)^{-1} - i\mathcal{M}_{2\to2}} \text{ at } E_{n,L}$$

- Matrix in combined <u>angular momentum and channel space</u>
- Quantifies mixing of angular momentum and channels due to finite volume

Back to…

How can one use numerical Lattice QCD to determine

$$\langle \phi_1(p), \phi_2(P_f - p), \mathrm{out} | \widetilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q}) | \Phi, P_i \rangle \,\textbf{?}$$

Back to…

How can one use numerical Lattice QCD to determine

$$\langle \phi_1(p), \phi_2(P_f - p), \mathrm{out}|\widetilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q})|\Phi, P_i\rangle\,\textbf{?}$$

Contract the relation for two-particle states with $\widetilde{\mathcal{J}}(\mathbf{Q})|\Phi\rangle$

for single particle state, difference between finite and infinite-volume is exponentially suppressed e^{-mL}

Back to...

How can one use numerical Lattice QCD to determine

$$\langle \phi_1(p), \phi_2(P_f - p), \mathrm{out} | \widetilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q}) | \Phi, P_i \rangle \,\textbf{?}$$

Contract the relation for two-particle states with $\widetilde{\mathcal{J}}(\mathbf{Q})|\Phi\rangle$

$$\langle \Phi | \widetilde{\mathcal{J}}^\dagger(\mathbf{Q}) \boxed{ |n, L\rangle\langle n, L| = \big(|\phi_1\phi_2, J=0\rangle \quad |\phi_3\phi_4, J=0\rangle \quad \cdots \big) \begin{pmatrix} & & \\ & \mathcal{R}_n & \\ & & \end{pmatrix} \begin{pmatrix} \langle \phi_1\phi_2, J=0| \\ \langle \phi_3\phi_4, J=0| \\ \langle \phi_1\phi_2, J=1| \\ \langle \phi_3\phi_4, J=1| \\ \vdots \end{pmatrix} } \widetilde{\mathcal{J}}(\mathbf{Q}) | \Phi \rangle$$

Back to...

How can one use numerical Lattice QCD to determine

$$\langle \phi_1(p), \phi_2(P_f - p), \mathrm{out}|\widetilde{\mathcal{J}}_A(x_0 = 0, \mathbf{Q})|\Phi, P_i\rangle \,\textbf{?}$$

Contract the relation for two-particle states with $\widetilde{\mathcal{J}}(\mathbf{Q})|\Phi\rangle$

$$\langle \Phi|\widetilde{\mathcal{J}}^\dagger(\mathbf{Q}) \boxed{|n, L\rangle\langle n, L| = \begin{pmatrix} |\phi_1\phi_2, J=0\rangle & |\phi_3\phi_4, J=0\rangle & \cdots \end{pmatrix} \begin{pmatrix} & & \\ & \mathcal{R}_n & \\ & & \end{pmatrix} \begin{pmatrix} \langle \phi_1\phi_2, J=0| \\ \langle \phi_3\phi_4, J=0| \\ \langle \phi_1\phi_2, J=1| \\ \langle \phi_3\phi_4, J=1| \\ \vdots \end{pmatrix}} \widetilde{\mathcal{J}}(\mathbf{Q})|\Phi\rangle$$

In the paper we present an alternative derivation in which we explicitly calculate three-point correlators in finite-volume

Master equation

$$|\langle E_{n_f}, \mathbf{P}_f, L | \widetilde{\mathcal{J}}_A(0,\mathbf{Q}) | E_i, \mathbf{P}_i, L\rangle| = \frac{1}{\sqrt{2E_i}} \sqrt{\mathcal{A}^\dagger_{n_f} \mathcal{R}_{n_f} \mathcal{A}_{n_f}}$$

$$\mathcal{A}_{n_f}(2\pi)^3 \delta^3(\mathbf{P}_2 + \mathbf{Q} - \mathbf{P}_f) \equiv \begin{pmatrix} \langle \phi_1\phi_2, J=0 | \widetilde{\mathcal{J}}_A(\mathbf{Q}) | \Phi\rangle \\ \langle \phi_3\phi_4, J=0 | \widetilde{\mathcal{J}}_A(\mathbf{Q}) | \Phi\rangle \\ \langle \phi_1\phi_2, J=1 | \widetilde{\mathcal{J}}_A(\mathbf{Q}) | \Phi\rangle \\ \langle \phi_3\phi_4, J=1 | \widetilde{\mathcal{J}}_A(\mathbf{Q}) | \Phi\rangle \\ \vdots \end{pmatrix}$$

column vector containing all transition amplitudes with given quantum numbers

R. A. Briceño, MTH, A. Walker-Loud, arXiv:1406.5965

Master equation

$$\left|\langle E_{n_f}, \mathbf{P}_f, L | \widetilde{\mathcal{J}}_A(0, \mathbf{Q}) | E_i, \mathbf{P}_i, L \rangle\right| = \frac{1}{\sqrt{2E_i}} \sqrt{\mathcal{A}^\dagger_{n_f} \mathcal{R}_{n_f} \mathcal{A}_{n_f}}$$

Model independent & non-perturbative

Universal: lattice QCD, lattice EFT, etc

Not quite arbitrary quantum numbers:
degenerate or non-degenerate masses, arbitrary momenta and angular momenta.... but no intrinsic spin

Asymmetric volumes and boundary conditions:
periodic, anti-periodic, any linear combination and any rectangular prism

R. A. Briceño, MTH, A. Walker-Loud, arXiv:1406.5965

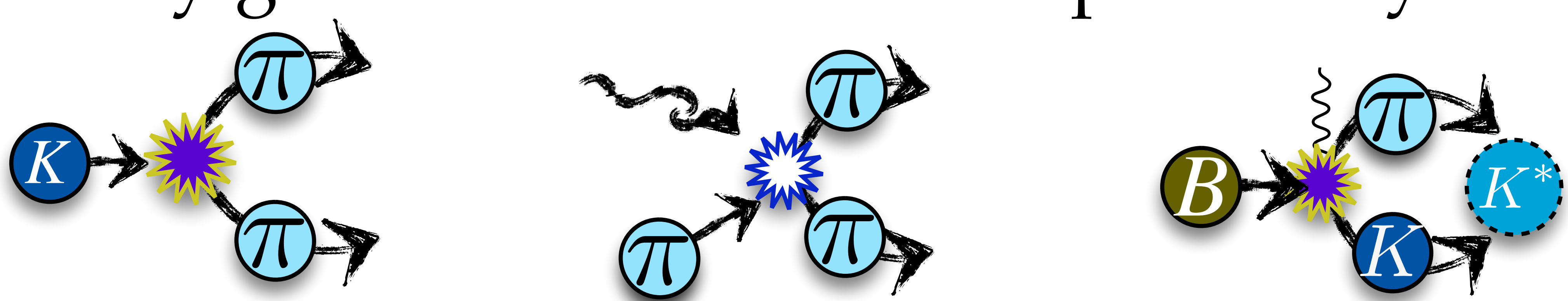
Conclusion

$$\det[1 - i\mathcal{M}_{2\to 2} iF] = 0$$

$$|\langle E_{n_f}, \mathbf{P}_f, L|\widetilde{\mathcal{J}}_A(0,\mathbf{Q})|E_i, \mathbf{P}_i, L\rangle| = \frac{1}{\sqrt{2E_i}}\sqrt{\mathcal{A}^\dagger_{n_f}\mathcal{R}_{n_f}\mathcal{A}_{n_f}}$$

Presented formalism for extracting
two-to-two scattering and
one-to-two matrix elements using numerical LQCD

Completely general result for scalar particle systems



Next steps… include spin, **three particle states**

see MTH, S. R. Sharpe, arXiv:1408.5933 to appear in *Phys.Rev. D*

Three-to-three scattering $\pi\pi\pi \to \pi\pi\pi$

$$C_L(E, \vec{P}) \equiv \int_L d^4x\, e^{i(Ex^0 - \vec{P}\cdot\vec{x})} \langle 0 | \mathrm{T}\sigma(x)\sigma^\dagger(0)|0\rangle$$

Require $m < E^* < 5m$ **odd-particle quantum numbers**

$5m$

$E_2^*(L, \vec{P})$

$E_1^*(L, \vec{P})$

$E_0^*(L, \vec{P})$

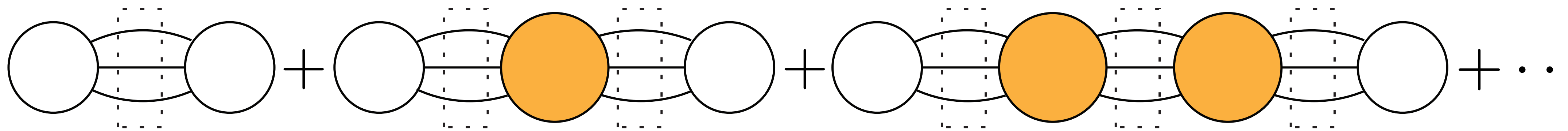
m

$i\mathcal{M}_{2\to 2}$

$i\mathcal{M}_{3\to 3}$

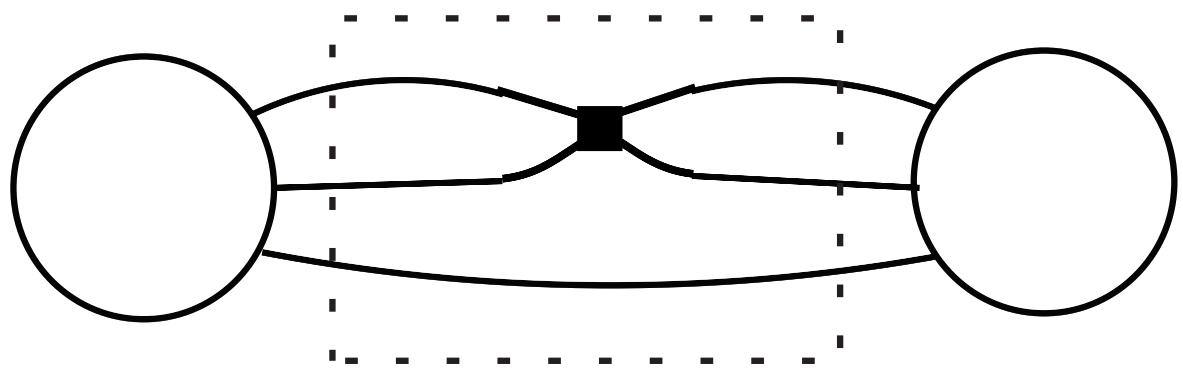
Assume no two-particle bound state

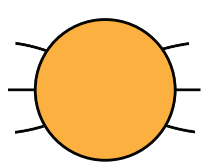
New skeleton expansion

$C_L(E,\vec{P}) \overset{?}{=}$ 

(propagators still fully dressed)

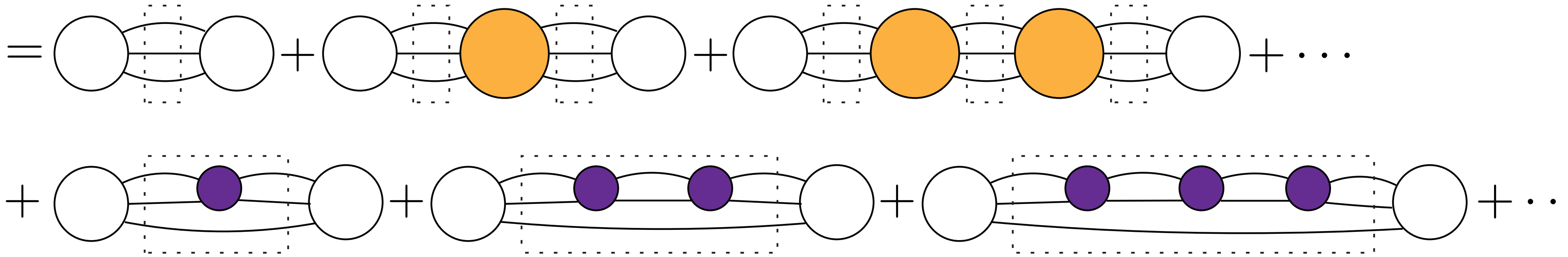
No! We also need diagrams like



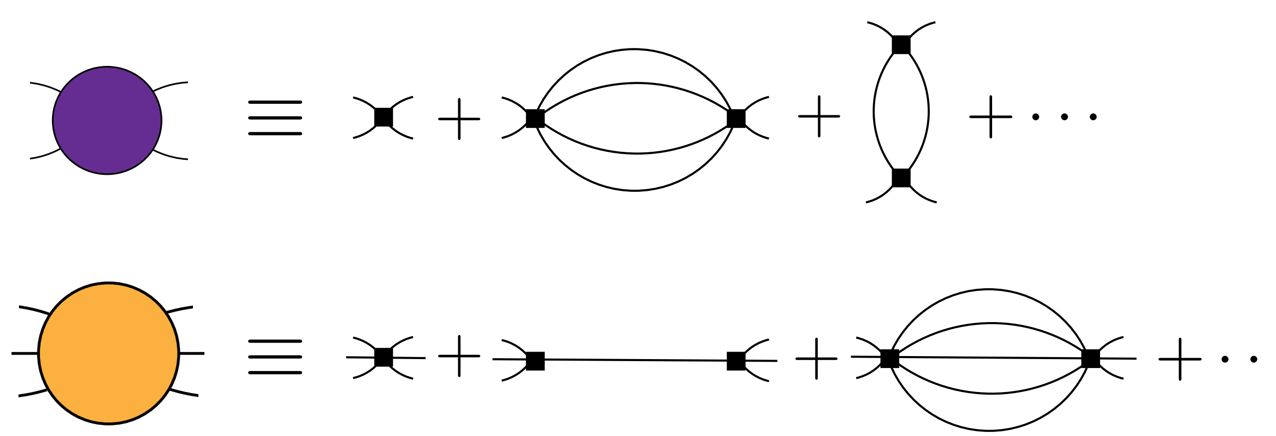
( **should only contain connected diagrams)**

New skeleton expansion

$C_L(E,\vec{P}) =$

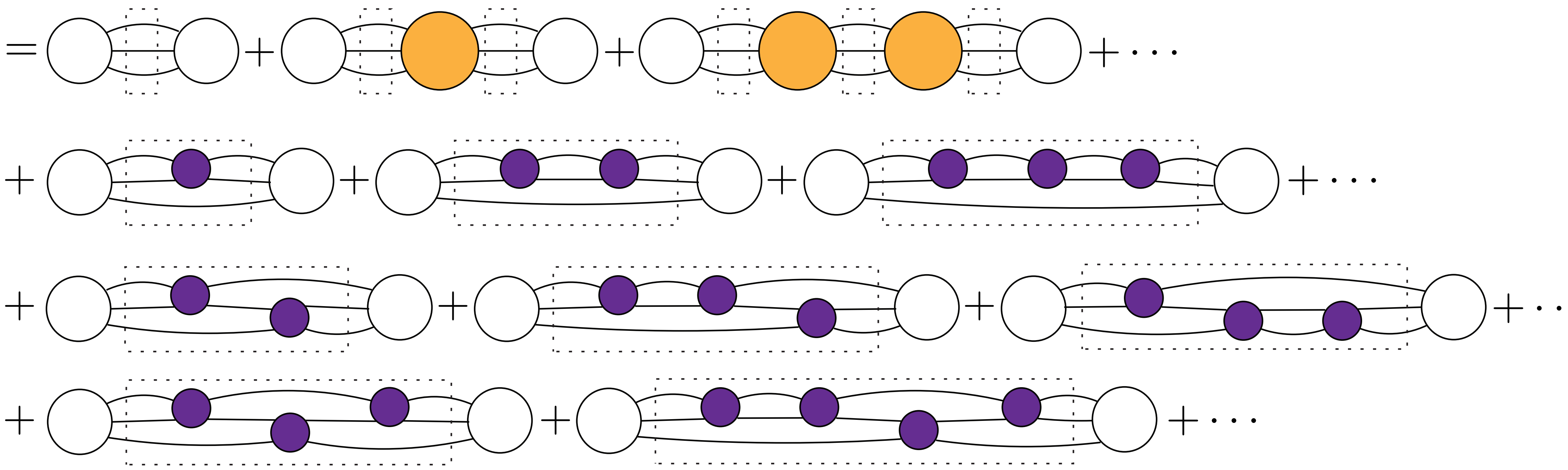


Kernel definitions:

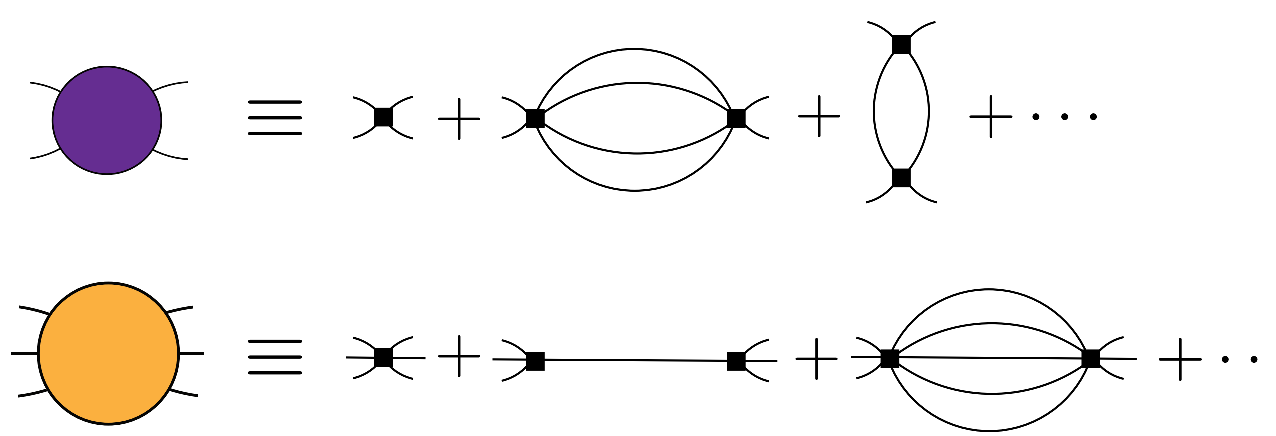


New skeleton expansion

$C_L(E, \vec{P}) =$

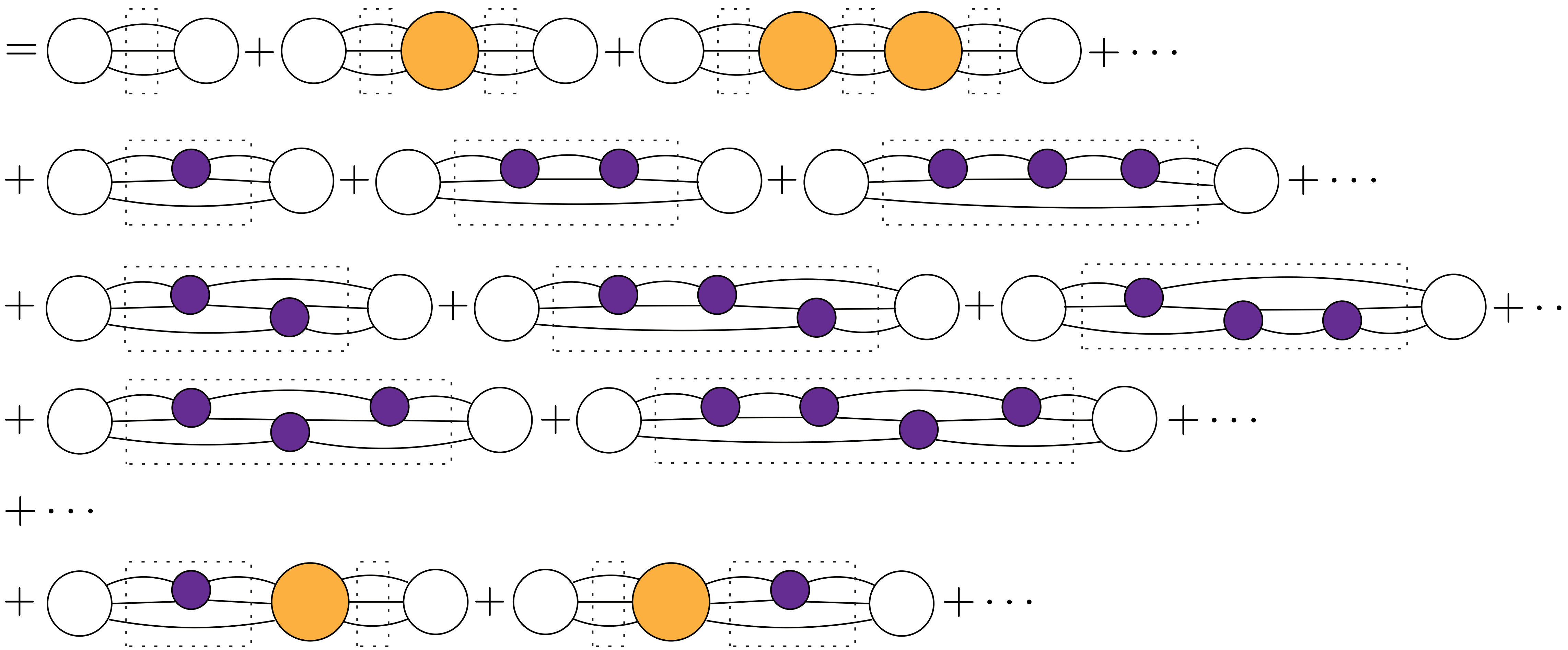


Kernel definitions:

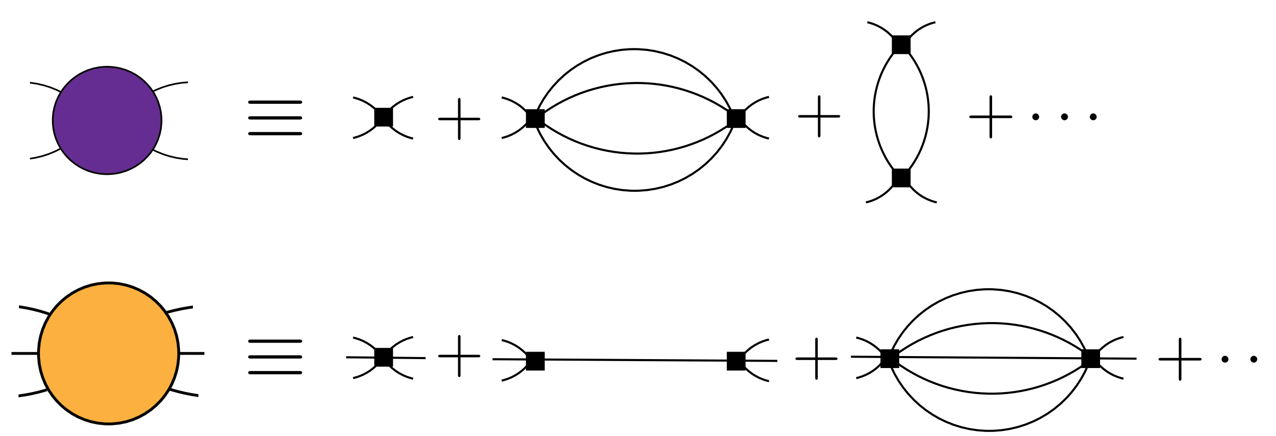


New skeleton expansion

$C_L(E,\vec{P}) =$

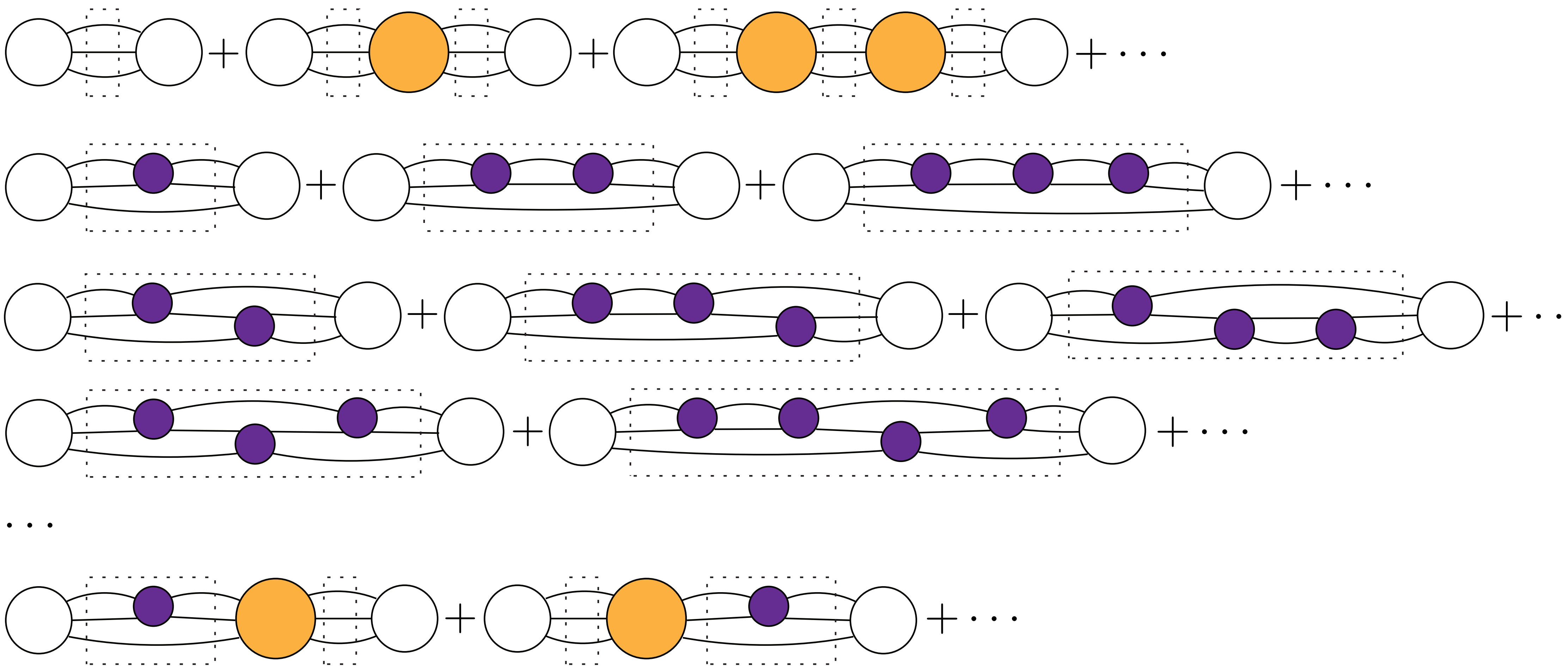


Kernel definitions:



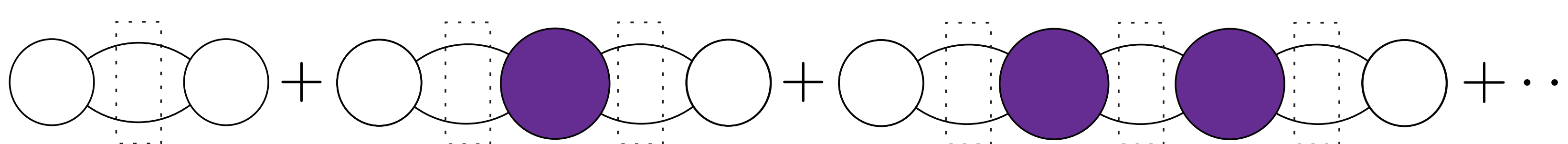
New skeleton expansion

$C_L(E, \vec{P}) =$



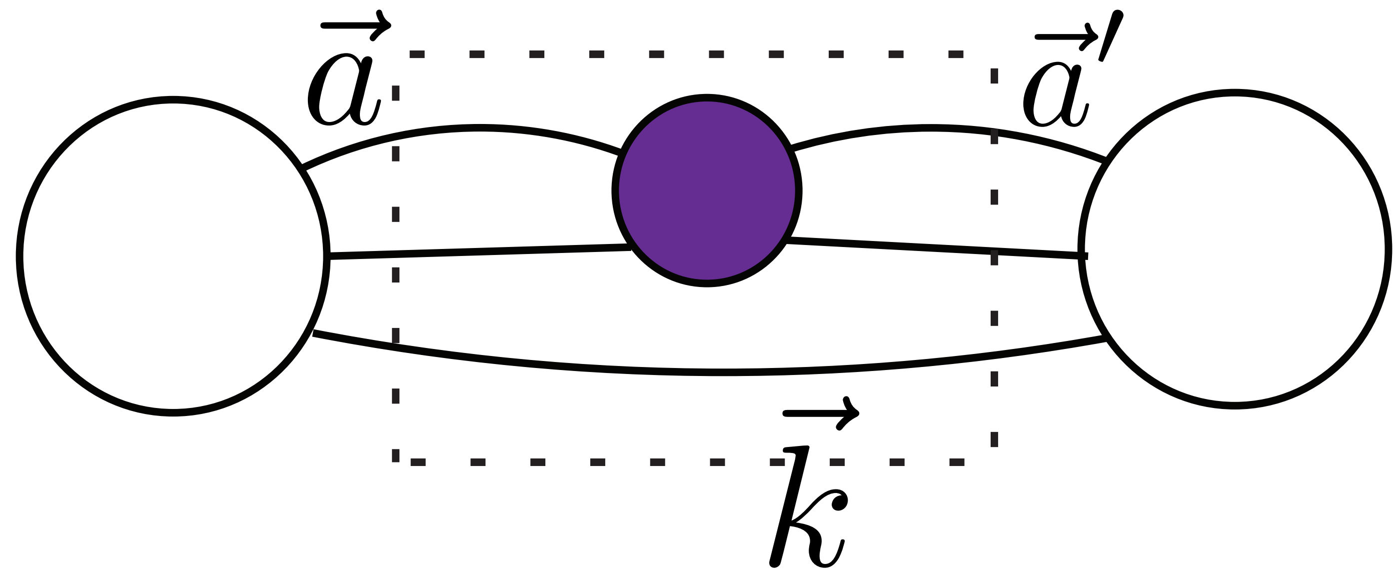
Compare to two-particle skeleton expansion

$C_L(E, \vec{P}) =$

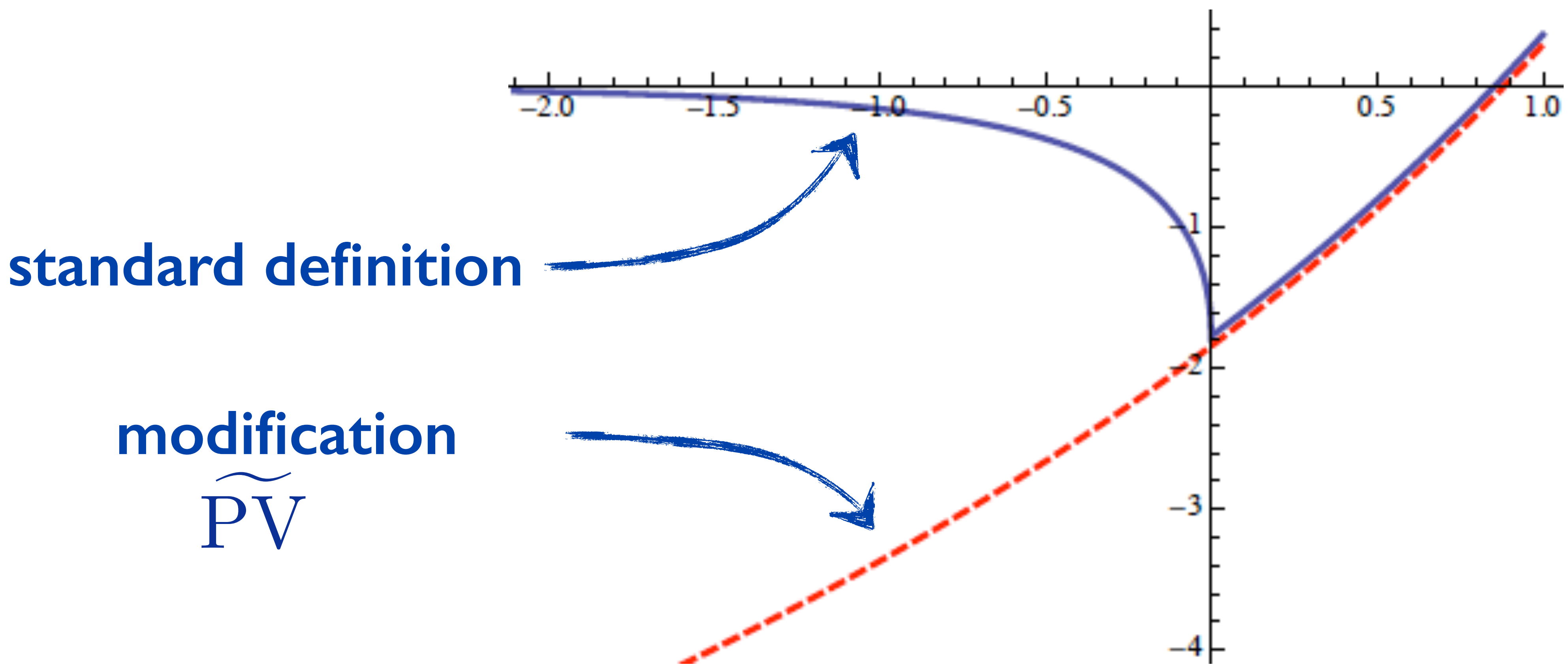


Cusps

$$\sum_{\vec{a},\vec{a}',\vec{k}}$$



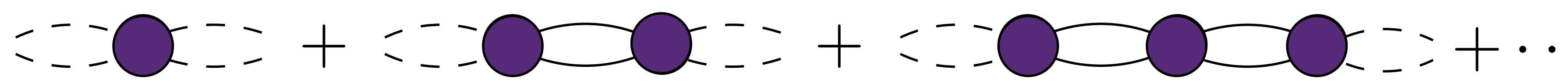
Relate $\vec{a}$ sum to an integral, gives cusp in $\vec{k}$



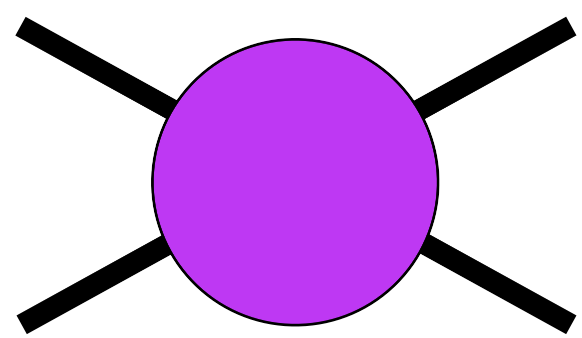
Leads to new infinite-volume quantities

has a cusp

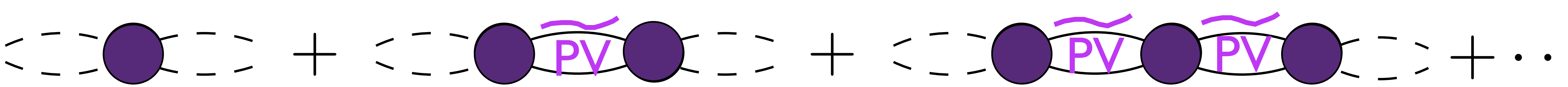
$i\mathcal{M}_{2\to 2} =$



$$i\widetilde{\mathcal{K}}_{2\to 2} =$$



$=$



has no cusp

Three-particle result

At fixed $(L, \vec{P})$, finite-volume spectrum
is all solutions to

$$\Delta_{L,P}(E) = \det[1 - \underbrace{i\widetilde{\mathcal{K}}_{df,3\to3}} \underbrace{iF_3}] = 0$$

matrix in $\vec{k}, \ell, m$
space

depends on kinematics
and two-particle
scattering

MTH, S. R. Sharpe, arXiv:1408.5933 to appear in *Phys.Rev. D*

K. Polejaeva, A. Rusetsky *Eur. Phys. J.* A48 (2012) 67
R.A. Briceño, Z. Davoudi, *Phys. Rev.* D87 (2013) 094507

Conclusion

$$\det[1 - i\mathcal{M}_{2\to 2}\, iF] = 0$$

$$|\langle E_{n_f}, \mathbf{P}_f, L|\widetilde{\mathcal{J}}_A(0,\mathbf{Q})|E_i, \mathbf{P}_i, L\rangle| = \frac{1}{\sqrt{2E_i}}\sqrt{\mathcal{A}^\dagger_{n_f}\mathcal{R}_{n_f}\mathcal{A}_{n_f}}$$

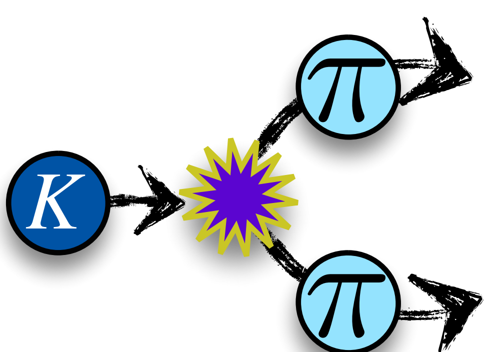
Presented formalism for extracting

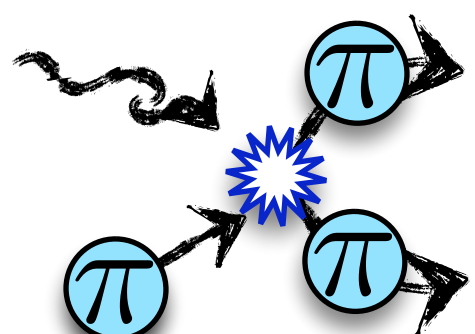
two-to-two scattering

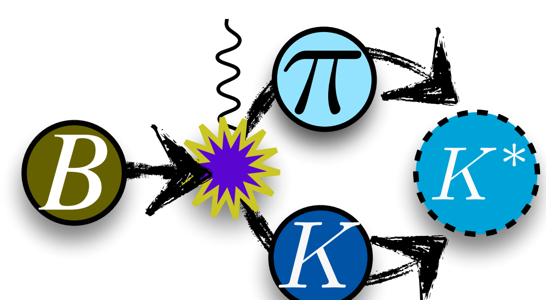
one-to-two matrix elements

three-to-three scattering using numerical LQCD

Completely general result for scalar particle systems







Next steps… include spin, one to three transitions

Finite vs infinite volume matrix elements

$$\langle \pi(p)\pi(-p), \mathrm{out}|\mathcal{H}|K\rangle \overset{?}{\approx} \langle n, L, \text{``}\pi\pi\text{''}|\mathcal{H}|0, L, \text{``}K\text{''}\rangle$$

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complex number **real number**

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units do not match

$$\langle \pi(p')|\pi(p)\rangle = 2\omega_p\,(2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{p}') \qquad \omega_p = \sqrt{\mathbf{p}^2 + m_\pi^2}$$

$$\langle n', L|n, L\rangle = \delta_{n,n'}$$

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$$\xcancel{\langle \pi(p)\pi(-p), \mathrm{out}|\mathcal{H}|K\rangle \overset{?}{\sim} \langle n, L, \text{“}\pi\pi\text{”}|\mathcal{H}|0, L, \text{“}K\text{”}\rangle}$$

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At the very least, we must account for different normalizations

Warm up… non-interacting pions

$$|0, L, \text{``}K\text{''}\rangle = \sqrt{\frac{1}{2M_K L^3}}\,|\mathbf{0}, L, K\rangle_{\rm rel}$$

unit normalization

relativistic normalization

Warm up… non-interacting pions

$$|0, L, \text{“}K\text{”}\rangle = \sqrt{\frac{1}{2M_K L^3}}\,|\mathbf{0}, L, K\rangle_{\rm rel}$$

unit normalization

relativistic normalization

$$\langle \mathbf{p}, \infty, K|\mathbf{k}, \infty, K\rangle_{\rm rel} = 2\omega_k (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{k})$$

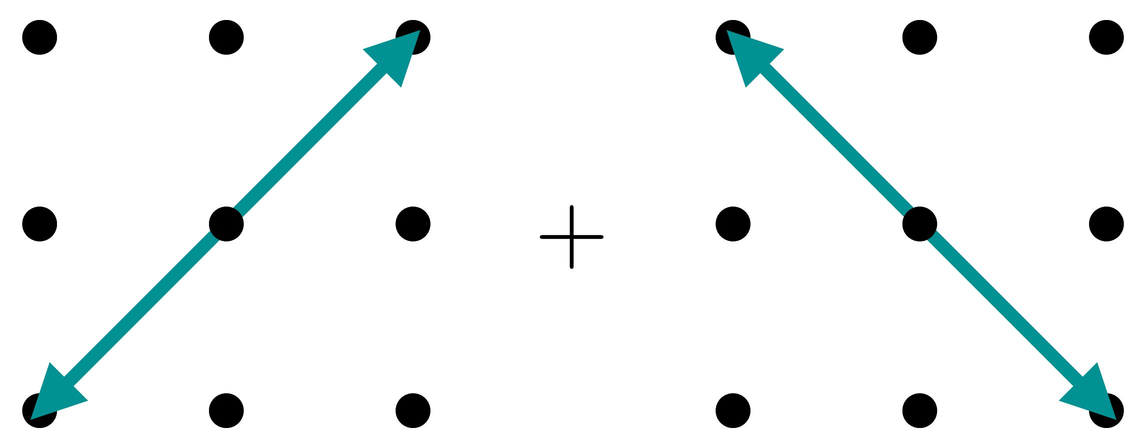
becomes

$$\langle \mathbf{p}, L, K|\mathbf{k}, L, K\rangle_{\rm rel} = 2\omega_k L^3 \delta_{kp}$$

Warm up… non-interacting pions

$$|0, L, \text{``}K\text{''}\rangle = \sqrt{\frac{1}{2M_K L^3}}|\mathbf{0}, L, K\rangle_{\text{rel}}$$

$$\langle n, L, \text{``}\pi\pi\text{''}| = \sqrt{\frac{2}{\nu_n}\frac{1}{M_K^2 L^6}}\Big[\langle n00, L, \pi|\langle -n00, L, \pi|_{\text{rel}} + \cdots\Big]$$



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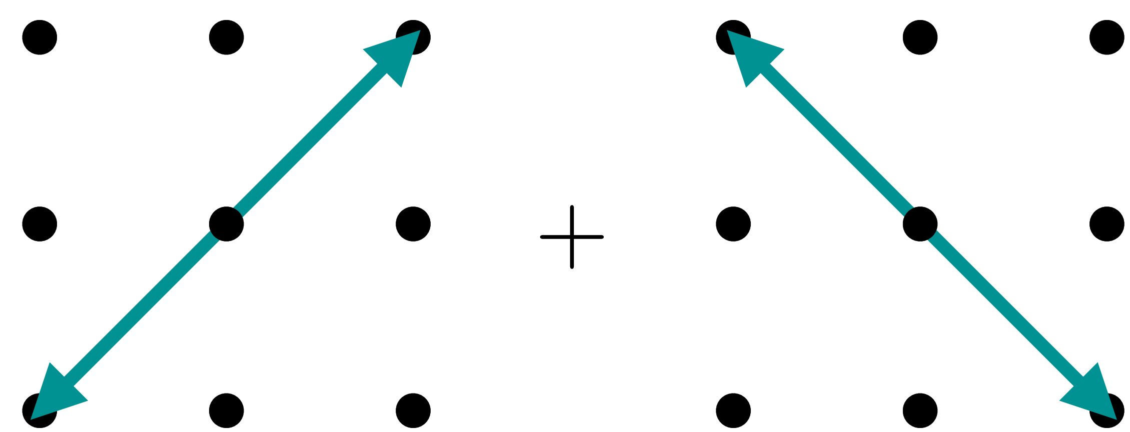
number of integer vectors $\mathbf{z}$ such that $\mathbf{z}^2 = n$.



Warm up… non-interacting pions

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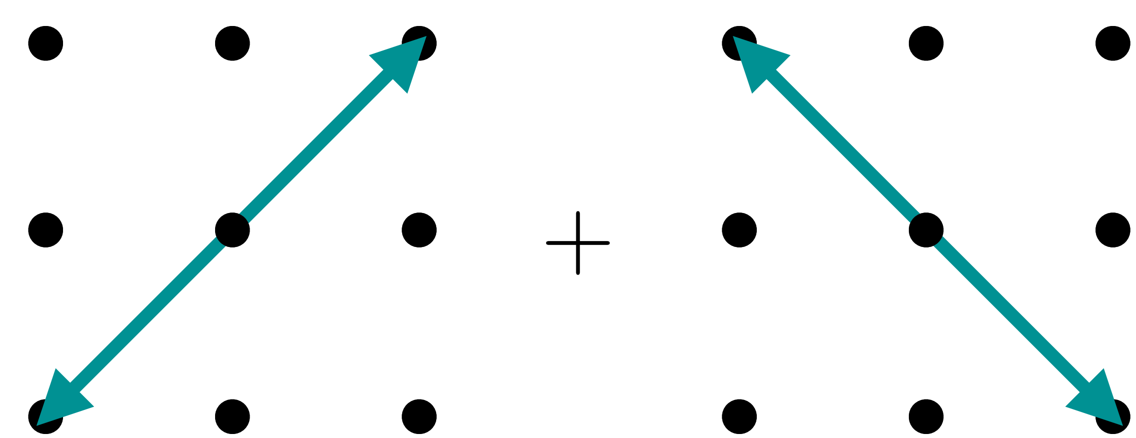

$$\boxed{|\langle n, L, \text{“}\pi\pi\text{”}|\mathcal{H}|0, L, \text{“}K\text{”}\rangle|^2 = \frac{\nu_n}{4M_K^3 L^9}|\langle \pi(p)\pi(-p), \mathrm{out}|\mathcal{H}|K\rangle|^2}$$

Notation

$$\boxed{|\langle n, L, \text{“}\pi\pi\text{”}|\mathcal{H}|0, L, \text{“}K\text{”}\rangle|^2 = \frac{\nu_n}{4M_K^3 L^9} |\langle \pi(p)\pi(-p), \mathrm{out}|\mathcal{H}|K\rangle|^2}$$

Notation

$$|\langle n, L, \text{“}\pi\pi\text{”}|\mathcal{H}|0, L, \text{“}K\text{”}\rangle|^2 = \frac{\nu_n}{4M_K^3 L^9}|\langle \pi(p)\pi(-p), \mathrm{out}|\mathcal{H}|K\rangle|^2$$

can be re-expressed as…

$$|\langle n, L, \text{“}\pi\pi\text{”}|\widetilde{\mathcal{H}}(\mathbf{Q}=\mathbf{0})|0, L, \text{“}K\text{”}\rangle| = \frac{1}{\sqrt{2M_K}}\sqrt{[\mathcal{A}^\dagger_{K\to\pi\pi}\mathcal{R}\mathcal{A}_{K\to\pi\pi}]}$$

$$\widetilde{\mathcal{H}}(\mathbf{Q}) = \int d\mathbf{x}\, e^{-i\mathbf{Q}\cdot\mathbf{x}}\,\mathcal{H}(x_0=0, \mathbf{x})$$

$$\mathcal{A}_{K\to\pi\pi}(2\pi)^3\delta^3(\mathbf{p}+\mathbf{k}) \equiv \langle \pi(p)\pi(k)|\widetilde{\mathcal{H}}(\mathbf{Q}=\mathbf{0})|K\rangle$$

$$\mathcal{R} = \frac{\nu_n}{2M_K^2 L^3}$$