Electromagnetism and Relativity 2023/24

Tutorial Sheet 4: Rotation and reflection tensors, the inertia tensor

1. If the position vector $\underline{x}$ is rotated through an angle θ about the axis defined by the unit vector $\underline{n}$ show that it coincides with the vector $\underline{y}$ given by
$$\underline{y} = \underline{x}\cos\theta + \left(\underline{x}\cdot\underline{n}\right)\underline{n}\,(1-\cos\theta) - \underline{x}\times\underline{n}\sin\theta\,.$$
If this is written in the form $y_i = R_{ij}(\theta,\underline{n})\,x_j$ find the elements of R_{ij} and show that
$$\begin{aligned} R_{ii} &= 1+2\cos\theta \\ \epsilon_{ijk}\,R_{jk} &= -2n_i\,\sin\theta \end{aligned}$$

2. A rotation through angle θ about a unit vector $\underline{n}$ is given by $y_i = R_{ij}x_j$, with
$$R = \begin{pmatrix} 4/9 & 1/9 & 8/9 \\ 7/9 & 4/9 & -4/9 \\ -4/9 & 8/9 & 1/9 \end{pmatrix}$$
Using the results from the last two parts of question (1), find θ and $\underline{n}$ for this rotation.
Two identical successive rotations are described by the rotation tensor $S_{ij} = R_{ik}R_{kj}$. Find the angle and axis of the rotation associated with S.

3. For a general rotation tensor with components $R_{ij}(\theta,\underline{n})$, verify explicitly that $RR^T = R^TR = I$ by evaluating $R_{ik}\,R_{jk}$ *etc.*
Verify that $\underline{n}$ is an eigenvector of R with eigenvalue 1. By considering the orthonormal triad $(\underline{u},\underline{v},\underline{n})$, verify by evaluating $R_{ij}\,(u_j \pm iv_j)$ that $\underline{u}\pm i\underline{v}$ are eigenvectors of R with eigenvalues $\exp(\mp i\theta)$.

4. Show that the nine quantities $R_{ij}(\theta,n_k)$ transform as a second rank tensor under a change of basis, *i.e.*
$$R'_{ij}(\theta',n'_k) = \ell_{ip}\,\ell_{jq}\,R_{pq}(\theta,n_k)$$

5. Show that the reflection of a vector $\underline{x}$ in the plane normal to the unit vector $\underline{n}$ is given by
$$y_i = \sigma_{ij}\,x_j \qquad \text{where} \qquad \sigma_{ij} = \delta_{ij} - 2\,n_i\,n_j$$
   (i) Show that $\det\sigma = -1$
   (ii) Show that the eigenvalues of σ are $-1,1,1$. Find the corresponding eigenvectors and interpret your results geometrically.

6. Under a reflection in a plane with unit normal $\underline{n}$, the vector $\underline{x}$ transforms into $\underline{y}$, where
$$\underline{y} = \underline{x} - 2\,(\underline{x}\cdot\underline{n})\,\underline{n}\,.$$
If there is a further reflection, in the plane with unit normal $\underline{m}$, such that $\underline{y}\to\underline{z}$, find the relation between $\underline{z}$ and $\underline{x}$, $\underline{n}$, $\underline{m}$.
Find the expression for $\underline{z}$ if the reflections are performed in the *opposite order*, and show that the difference between these two results is $-4(\underline{n}\cdot\underline{m})\left\{\underline{x}\times(\underline{m}\times\underline{n})\right\}$
List the possible cases for which this difference is zero, assuming that $\underline{x}$ is not the null vector. Interpret these results in terms of your images in two plane mirrors.

(PTO)

7. Show that the inertia tensor about the origin, O, for a system of four particles of mass m, one at each of the points $(a,\, -a,\, a)$, $(a,\, a,\, -a)$, $(-a,\, a,\, a)$ and $(-a,\, -a,\, -a)$ [these are the vertices of a regular tetrahedron], is

$$I_{ij}(O) \;=\; 8ma^2\,\delta_{ij}$$

8. Show that the moment of inertia of a uniform, straight rod of mass M and length a, about an axis through one end and perpendicular to the rod, is $\frac{1}{3}Ma^2$. Hence obtain the inertia tensor $I_{ij}(O)$, with O at one end and the rod in the $\underline{e}_3$ direction.

   Find the inertia tensor $I'_{ij}(O)$ in the $\underline{e}'_i$ basis, in which the rod is inclined equally to the three $\underline{e}'_i$ axes, *i.e.* $\underline{e}_3 = (\underline{e}'_1 + \underline{e}'_2 + \underline{e}'_3)/\sqrt{3}\,$.