Electromagnetism and Relativity 2023/24

Tutorial Sheet 9: More electrostatics

1. A charge Q resides at the centre of a spherical shell of copper, of inner radius a and outer radius b. The copper shell carries no net charge.

   (i) Use Gauss's law to find the field $\underline{E}(\underline{r})$ everywhere, and sketch the field lines.

   (ii) Work out the surface charge density on both the inner and the outer surface of the copper shell.

   (iii) How would the field for $|\underline{r}| > b$ be altered if the charge was moved off-centre within the shell?

2. A point charge q is placed at a position (x_0, y_0, z_0) near a grounded conducting 'corner' which occupies the region $\{(x \text{ or y}) < 0\}$, $-\infty < z < \infty$.

   (i) Use the method of images to find the electrostatic potential $\phi(\underline{r})$ in the region $x > 0$, $y > 0$.
   [Hint: you need three image charges.]

   (ii) Assuming that the point charge is located at $x_0 = a$, $y_0 = a$ and $z_0 = 0$ show that the net force acting on the charge is

$$\underline{F} = -\frac{q^2}{4\pi\epsilon_0}\frac{4-\sqrt{2}}{16a^2}\left(\underline{e}_x + \underline{e}_y\right)$$

3. A conducting sphere of radius a carries a charge Q on its surface.

   (i) Find its potential $\phi(r)$ for $r > a$ (choosing $\phi = 0$ very far away), and thus the potential $V(Q) = \phi(a)$ on the surface of the sphere.

   (ii) Explain why the electrostatic energy of the charged sphere can be written as

$$W_e = \int_0^Q V(Q')\,\mathrm{d}Q',$$

   and hence calculate W_e.

   (iii) Show that this gives the same answer as found by the formula $W_e = \frac{1}{2}\epsilon_0 \int |\underline{E}(\underline{r})|^2\,\mathrm{d}V$ for the total energy in the electric field, where the volume integral is over all space.

4. (i) Assuming that the electric charge Ze of an atomic nucleus is uniformly distributed inside a sphere of radius R, obtain the electric field $\underline{E}$ and electric potential ϕ both inside and outside. [Hint: see lectures for a very similar calculation]

   (ii) Obtain an expression for the electrostatic energy $\frac{1}{2}\int \mathrm{d}V \rho\,\phi$ of the nucleus and verify that it is equal to the field energy $\frac{1}{2}\epsilon_0 \int \mathrm{d}V\,|\underline{E}|^2$.

(PTO)

5. (i) Write down the electric field at a point $\underline{r}$ due to two point charges q_1 and q_2 at positions $\underline{r}_1$ and $\underline{r}_2$. Deduce the energy density of the electric field, and separate it into three pieces: one due to the mutual interaction of the two charges, and two due to the self interaction of the field of each charge upon itself.

   (ii) Show that when integrated over all space the two self energy terms each give rise to an infinite contribution to the electric field energy. Explain why these terms, despite being infinite, are of no physical consequence.

   (iii) Integrate the remaining term over all space, and show that it is equal to the potential energy of the interaction between the two charges.

6. A point charge q is placed at distance b from an earthed conducting sphere of radius a. Find the electrostatic potential outside the conducting sphere and force between the sphere and the charge.

   [Hint: consider an image charge of size q' a distance b' from the centre of the sphere, and tune q' and b' until $\phi = 0$ on the surface of the sphere.]

   How would your results change if instead of being earthed the sphere is isolated and uncharged?

7. An uncharged solid spherical conductor is placed in a constant electric field $\underline{E}_0$.

   (i) Find the resulting potential and electric field.

   (ii) Find the induced surface charge density, and sketch the field lines.

   (iii) Deduce the force on the sphere.

8. Two thin concentric, conducting spherical shells have radii a and b with $b > a$. They carry charges $-Q$ and $+Q$ respectively.

   (i) Use Gauss's law and symmetry to find the field $\underline{E}$ at a radius r from the centre. Sketch the field lines.

   (ii) Find the potential difference between the shells, and hence find the capacitance C.

   (iii) Show that C reduces to that for a parallel plate capacitor of the appropriate area, in the limit where b is only slightly bigger than a.

9. (i) Find the electric field and thus the potential due to an infinite straight cylinder of radius a carrying a uniform charge λ per unit length.

   (ii) A transmission line consists of two parallel infinite cylindrical conductors, each of radius a, the axes of which are at a distance d apart, where d is much greater than a. Show that the capacitance of the line per unit length is approximately inversely proportional to $\ln(d/a)$, and give the constant of proportionality.