Electromagnetism and Relativity 2023/24

Tutorial Sheet 10: Magnetostatics

1. The plane $z = 0$ forms an interface between two conductors of conductivity σ_1 and σ_2. In the first conductor a steady current $\underline{J}_1$ flows making an angle θ_1 with the normal to the interface.

   (i) Find the current in the second conductor.
   [Hints: Apply the divergence theorem to $\underline{\nabla} \cdot \underline{J} = 0$, and Stokes' theorem to $\underline{\nabla} \times \underline{E} = 0$ with $\underline{J} = \sigma \underline{E}$, to determine the changes in the normal and tangential components of $\underline{J}$ across the boundary.]

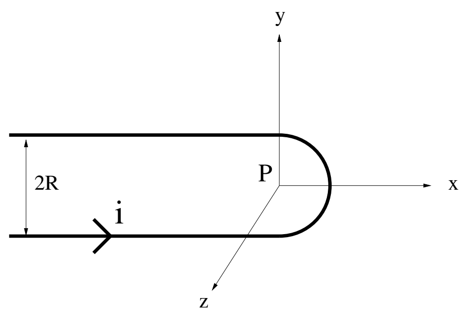
   (ii) Using the boundary condition on the normal component of $\underline{E}$ at the boundary, show that the charge density on the interface is
   $$\epsilon_0 \frac{\sigma_2 - \sigma_1}{\sigma_1 \sigma_2} J_1 \cos\theta_1$$

2. (i) Show that the magnetic field $\underline{B}$ on the axis of a circular current loop of radius a is
   $$\underline{B} = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} \underline{e}_z$$
   where I is the current and z is the distance along the axis from the centre of the loop.

   (ii) An insulating disc of radius a has uniform surface charge density σ. It rotates at angular velocity ω about a perpendicular axis through its centre. What is the surface current density $\underline{K}(\underline{r})$ at $\underline{r}$ from its centre? Find the contribution $\mathrm{d}\underline{B}(z)$ to the magnetic field on the axis of the disc from a ring with radii between ρ and $\rho + \mathrm{d}\rho$, and thus find the magnetic field on the axis of the spinning disc. Show that as $z \to \infty$,
   $$\underline{B}(z) \sim \frac{1}{8} \mu_0 \sigma \omega \frac{a^4}{z^3} \underline{e}_z$$

   (iii) What would the corresponding results be for a spinning ring of inner radius a and outer radius b? Recover the result in part (i) by taking the limit $b \to a$.

3. A long wire is bent into the shape of a hair-pin, and lies in the $x - y$ plane as shown in the figure.



A current I flows in the hair-pin. Find the magnetic field at the centre (P) of the semi-circle.

[The answer is: $\underline{B} = \frac{\mu_0 I}{4\pi R} (2 + \pi)\, \underline{e}_z$]

(PTO)

4. A coil of length L and radius a has N turns per unit length. Find the field on the axis of the coil a distance z from one end, and show that in the limit of an infinitely long thin coil you recover the standard result $B = N\mu_0 I$ uniformly along the axis.

   [Hint: start from the result of question 2, part (i). For large N, you can consider turns as continuous in z', so that the number of turns in $\mathrm{d}z'$ is $N\mathrm{d}z'$.]

5. (i) A wire with uniform charge density λ per unit length is bent into a ring of radius a and rotates with angular velocity ω about an axis through its centre and perpendicular to the plane of the ring. Find the magnetic field on the axis at a distance z from the ring. Show that the radial (from the z axis) magnetic field at a distance $\rho \ll z$ from the axis is given by

   $$B_\rho = \frac{3\mu_0 \lambda \omega a^3 \rho z}{4(a^2+z^2)^{5/2}}$$

   [Hint: for the last part use $\underline{\nabla} \cdot \underline{B} = 0$ in cylindrical co-ordinates.]

   (ii) Hence calculate the electric and magnetic forces exerted on a second co-axial ring of the same radius, carrying charge density λ', and also rotating with angular velocity ω, placed at a distance L from the first, with $L \gg a$. Show that the total force is zero if $\omega = \sqrt{\frac{2}{3}} cL/a^2$ where $c = 1/\sqrt{\mu_0 \epsilon_0}$. Why is this impossible to achieve in practice?

6. Use Ampère's Law and Gauss' Law to show the following results for a solenoid of infinite length:

   (i) $B_\phi = 0$ everywhere;

   (ii) $B_\rho = 0$ everywhere;

   (iii) $B_z = 0$ outside the solenoid;

   (iv) $B_z = \mu_0 N I$ inside the solenoid.

   where we used cylindrical coordinates (ρ, ϕ, z) throughout.

7. Two very long thin wires carrying equal and opposite currents of I are placed parallel to the x-axis at $y = 0$ and $z = \pm a$. Calculate the magnetic field $\underline{B}$ in the $y - z$ plane and show that its gradient $\frac{\partial B}{\partial y}$ on the y-axis is greatest when $y = \pm a/\sqrt{3}$.

8. Return to question (5), but this time instead of placing the two rings far apart, put them very close together, so that $L \ll a$. Again compute both electrical and magnetic forces, and show that now they balance when $\omega = c/a$. Is this now possible?