Electromagnetism and Relativity 2023/24

Tutorial Sheet 11: More magnetostatics

1. A long cylindrical pipe of radius a and length L is filled with mercury of electrical conductivity σ. A potential difference V acts across the two ends of the pipe, creating an electric current through the mercury (which remains stationary).

   (i) Find the current density, assumed uniform, within the mercury. Using Ampère's law, find the magnetic field $\underline{B}$ at radius ρ from the axis.

   (ii) Show that the magnetic field, when acting on an infinitesimal current element $\underline{J}\,\mathrm{d}V$, produces a radial force $\mathrm{d}\underline{F} = f\underline{e}_\rho \mathrm{d}V$, and determine f.

   (iii) In practice this force is balanced by a radial pressure gradient: $f = \dfrac{\partial p}{\partial \rho}$.
   Find the pressure difference Δp between the centre and the edge of the pipe.

2. Show using Ampère's law that the magnetic field inside a cylinder carrying a uniform current density $\underline{J}$ in the direction of the axis of the cylinder is given by

$$\underline{B}(\underline{r}) = \frac{1}{2}\mu_0\,\underline{J}\times\underline{r}$$

   where $\underline{r}$ is a position vector from an origin on the cylinder axis.

   A long straight cylindrical conductor of radius a and carrying current I is placed with axis along the z axis. A cylindrical hole of radius b, parallel to the z axis but displaced off-centre a distance d along the x-axis, is bored in the conductor ($b + d < a$). Show that the magnetic field in the hole is uniform, parallel to the y-axis:

$$\underline{B} = \frac{\mu_0}{2\pi}\frac{d}{(a^2 - b^2)}\,I\,\underline{e}_y$$

   [Hint: use linear superposition of current densities $\underline{J}$ and $-\underline{J}$]

3. A long straight wire carries a uniform current density $\underline{J}$ inside it.

   (i) What is the magnetic field $\underline{B}$ inside the wire?

   (ii) What is the magnetic vector potential $\underline{A}$ inside the wire?
   [Hint: use the expression for curl in cylindrical polar co-ordinates.]

   (iii) Confirm explicitly that your result for $\underline{A}$ satisfies Poisson's equation, *i.e.* that $\nabla^2\underline{A} = -\mu_0\,\underline{J}$, and also that $\underline{\nabla}\cdot\underline{A} = 0$.

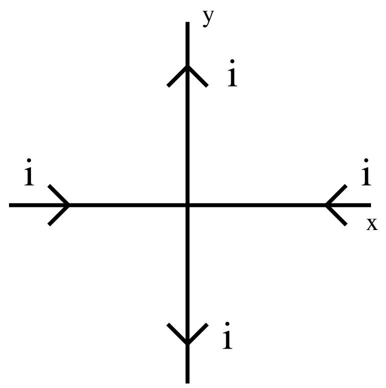
4. The magnetic vector potential may be defined as

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int \mathrm{d}V'\,\frac{\underline{J}(\underline{r}')}{|\underline{r}-\underline{r}'|}$$

   Show directly that $\underline{B} = \underline{\nabla}\times\underline{A}$. Show also that $\underline{\nabla}\cdot\underline{A} = 0$, provided that $\underline{\nabla}\cdot\underline{J} = 0$ and $\underline{J}$ vanishes outside a bounded region of space.

(PTO)

5. Currents I flow along the *entire* x and y axes as shown in the figure



Show that A_x at a point $\underline{r}$ is given by

$$A_x = -\frac{\mu_0 I}{4\pi} \int_{-x}^{x} \frac{\mathrm{d}\xi}{(\xi^2 + y^2 + z^2)^{\frac{1}{2}}}$$

Similarly, find A_y, A_z, and hence $\underline{B}$. Show that if $x = y$ and $z = 0$, then $B_x = B_y = 0$, while $B_z = -\frac{\mu_0 I}{\pi r}$, while if $z^2 \gg x^2 + y^2$,

$$\underline{B} \approx \frac{\mu_0 I}{2\pi} \frac{y\,\underline{e}_x + x\,\underline{e}_y}{r^2}$$

6. A small sphere is uniformly charged throughout its volume and rotating with constant angular velocity $\underline{\omega}$. Show that its magnetic moment is given by

$$\underline{m} = \tfrac{1}{5} Q a^2 \underline{\omega}$$

where Q is the total charge on the sphere. Determine the angular momentum $\underline{L}$ of the sphere in terms of its mass M and hence verify that

$$\underline{m} = \frac{Q}{2M} \underline{L}$$

7. A flat coil having N turns each of radius R carries a current of I. Another flat coil of n turns of radius r, with $r \ll R$, also carries a current of I and is placed co-axially a distance L from the first coil. The currents flow in the same direction. Find the force between the coils.

   [Hint: Find the magnetic field from the first coil on the axis and use $\underline{F} = \underline{\nabla}(\underline{m} \cdot \underline{B})$]

8. A thin spherical shell of radius R carries a uniform surface charge density σ. The shell is rotated at constant angular velocity $\underline{\omega}$ about a diameter. Show that the surface current $\underline{K} = \sigma\,\omega\,r \sin\theta\,\underline{e}_\phi$.

   (i) Determine the boundary conditions which relate the magnetic field just inside the shell to that just outside the shell.

   (ii) Take as an ansatz for the magnetic field a uniform field $\underline{B}_0$ inside the shell, and a dipole field with magnetic moment $\underline{m}$ outside the shell. Show that this ansatz can indeed be tuned to satisfy the boundary conditions, and thus that

$$\underline{B}_0 = \tfrac{2}{3} \mu_0 \sigma R \underline{\omega}$$