MP2A: Vectors, Tensors and Fields

[U03869 PHY-2-MP2A]

Brian Pendleton

(Course Lecturer)

- email: bjp@ph.ed.ac.uk
- room: JCMB 4413
- telephone: 0131-650-5241
- web: http://www.ph.ed.ac.uk/∼bjp/

March 26, 2010

Abstract

(Mathematical) Physics: Construct a (mathematical) model to explain known facts and make new predictions. The tools of the trade are mathematical methods.

Timetable

- Tuesday 11:00-12:00 Lecture (LTC)
- Thursday 14:00-17:00 Tutorial Workshop (Teaching Studio 1206C)
- Friday 11:00-12:00 Lecture (LTC)

Contents

Syllabus

- Vectors, the geometric approach, scalar and cross products, triple products, the equation of a line and plane
- Vector spaces, Cartesian bases, handedness of basis
- Indices and the summation convention, the Kronecker delta and Levi-Cevita epsilon symbols, product of two epsilons
- Rotations of bases, orthogonal transformations, proper and improper transformations, transformation of vectors and scalars
- Cartesian tensors, definition, general properties, invariants, examples of the conductivity and inertia tensors
- Eigenvalues and eigenvectors of real symmetric matrices, diagonalisation of inertia tensor
- Fields, potentials, grad, div and curl and their physical interpretation, del-squared, vector identities involving grad
- Polar coordinates
- Line integrals, vector integration, conservative forces
- Surface and volume integrals, the divergence and Stokes' theorem together with some applications

Books

- Any Mathematical Methods book that you are comfortable with.
- K. F. Riley, M. P. Hobson and S. J. Bence,
  Mathematical Methods for Physics and Engineering, (CUP 1998).
- P. C. Matthews,
  Vector Calculus, (Springer 1998).
- M. R. Spiegel,
  Vector Analysis, (Schaum, McGraw-Hill 1974).
- M. L. Boas,
  Mathematical Methods in the Physical Sciences, (Wiley 2006).
- G. B. Arfken and H. J. Weber,
  Mathematical Methods for Physicists, (Academic Press 2001).
- D. E. Bourne and P. C. Kendall,
  Vector Analysis and Cartesian Tensors, (Chapman and Hall 1993).

1 Review of Vectors

1.1 Physics Terminology

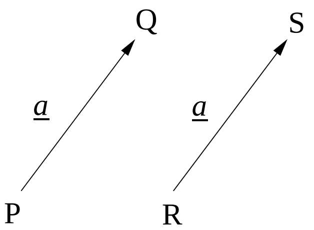
Scalar : quantity specified by a single number;

Vector : quantity specified by a number (magnitude) and a **direction**;

e.g. speed is a scalar, velocity is a vector

1.2 Geometrical Approach

A vector is *represented* by a 'directed line segment' with a length and direction proportional to the magnitude and direction of the vector (in appropriate units). A vector can be considered as a class of equivalent directed line segments *e.g.*

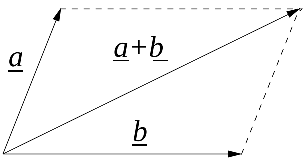


Both displacements from P to Q and from R to S are represented by the same vector. Also, different quantities can be represented by the same vector *e.g.* a displacement of a cm, or a velocity of a ms^{-1} or $\ldots$, where a is the magnitude or **length** of vector $\underline{a}$

Notation: Textbooks often denote vectors by boldface: $\boldsymbol{a}$ but here we use underline: $\underline{a}$ (or sometimes $\vec{a}$). (Alternatively we can write $\boldsymbol{PQ} \equiv \overrightarrow{PQ} \equiv \boldsymbol{RS} \equiv \overrightarrow{RS}$.) Denote a vector by $\underline{a}$ and its magnitude by $|\underline{a}|$ or a. *Always* underline a vector to distinguish it from its magnitude. A unit vector is often, but not always, denoted by a hat $\hat{\underline{a}} = \underline{a} \,/\, a$ and represents a direction. $\underline{n}$ is usually taken to be a unit vector (without a hat).

Addition of vectors – parallelogram law

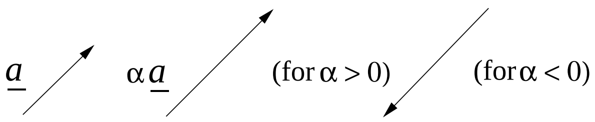
i.e.



$$\begin{aligned} \underline{a} + \underline{b} &= \underline{b} + \underline{a} && \text{(commutative) ;} \\ (\underline{a} + \underline{b}) + \underline{c} &= \underline{a} + (\underline{b} + \underline{c}) && \text{(associative) .} \end{aligned}$$

Multiplication by scalars

A vector $\underline{a}$ may be multiplied by a scalar α to give a new vector $\alpha\,\underline{a}$, *e.g.*



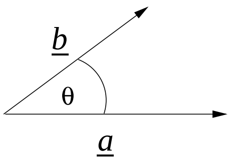
Also, for scalars α, β and vectors $\underline{a}$ and $\underline{b}$

$$\begin{aligned} |\alpha\underline{a}| &= |\alpha||\underline{a}| \\ \alpha(\underline{a} + \underline{b}) &= \alpha\underline{a} + \alpha\underline{b} && \text{(distributive)} \\ \alpha(\beta\underline{a}) &= (\alpha\beta)\underline{a} && \text{(associative)} \\ (\alpha + \beta)\underline{a} &= \alpha\underline{a} + \beta\underline{a} \,. \end{aligned}$$

1.3 Scalar or dot product

The scalar product (also known as the dot product) between two vectors is *defined* to be

$$\boxed{\underline{a}\cdot\underline{b} \equiv ab\cos\theta, \text{ where } \theta \text{ is the angle between } \underline{a} \text{ and } \underline{b}}$$



$$\boxed{\underline{a}\cdot\underline{b} \text{ is a scalar – } i.e. \text{ a single number.}}$$

Notes on scalar product

(*i*) $\underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a}$ (commutative) ; $\underline{a}\cdot(\underline{b}+\underline{c}) = \underline{a}\cdot\underline{b} + \underline{a}\cdot\underline{c}$ (distributive)

(*ii*) $\underline{n}\cdot\underline{a} = a\cos\theta$ = the scalar projection of $\underline{a}$ onto $\underline{n}$, where $\underline{n}$ is a unit vector

(*iii*) $(\underline{n}\cdot\underline{a})\,\underline{n} = a\cos\theta\,\underline{n}$ = the vector projection of $\underline{a}$ onto $\underline{n}$

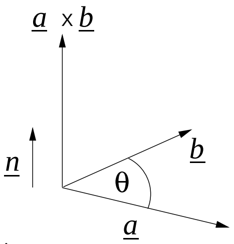
(*iv*) A vector may be resolved with respect to some direction $\underline{n}$ into a parallel component $\underline{a}_{\parallel} = (\underline{n}\cdot\underline{a})\underline{n}$ and a perpendicular component $\underline{a}_{\perp} = \underline{a} - \underline{a}_{\parallel}$. You should check that $\underline{a}_{\perp}\cdot\underline{n} = 0$

(*v*) $\boxed{\underline{a}\cdot\underline{a} \equiv |\underline{a}|^2 \equiv a^2}$ which defines the magnitude $|\underline{a}|$ of a vector. For a unit vector $\hat{\underline{a}}\cdot\hat{\underline{a}} = 1$

1.4 The vector or 'cross' product

$$\boxed{\underline{a}\times\underline{b} \;\equiv\; ab\sin\theta\,\underline{n}\text{ , where } \underline{n} \text{ is in the 'right-hand screw direction'}}$$

i.e. $\underline{n}$ is a unit vector normal to the plane of $\underline{a}$ and $\underline{b}$, in the direction of a right-handed screw for rotation of $\underline{a}$ to $\underline{b}$ (through $< \pi$ radians).



$$\boxed{\underline{a}\times\underline{b} \text{ is a vector – } i.e. \text{ it has a direction and a length.}}$$

[It is also called the **wedge** product – and in this case denoted by $\underline{a}\wedge\underline{b}$.]

Notes on vector product

(*i*) $\underline{a}\times\underline{b} = -\underline{b}\times\underline{a}$ (*not* commutative)

(*ii*) $\underline{a}\times\underline{b} = 0$ if $\underline{a},\underline{b}$ are parallel

(iii) $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$

(iv) $\underline{a} \times (\alpha \underline{b}) = \alpha \underline{a} \times \underline{b}$

1.5 The Scalar Triple Product

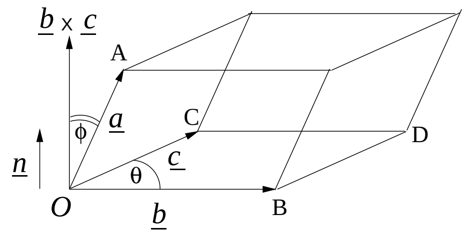
The scalar triple product is defined as follows

$$(\underline{a}, \underline{b}, \underline{c}) \quad \equiv \quad \underline{a} \cdot (\underline{b} \times \underline{c})$$

Notes

(i) If $\underline{a}$, $\underline{b}$ and $\underline{c}$ are three concurrent edges of a parallelepiped, the volume is $(\underline{a}, \underline{b}, \underline{c})$.
To see this, note that:

$$\begin{aligned} \text{area of the base} &= \text{area of parallelogram } OBDC \\ &= b\, c \sin\theta = |\underline{b} \times \underline{c}| \\ \text{height} &= a \cos\phi = \underline{n} \cdot \underline{a} \\ \text{volume} &= \text{area of base} \times \text{height} \\ &= b\, c \sin\theta\, \underline{n} \cdot \underline{a} \\ &= \underline{a} \cdot (\underline{b} \times \underline{c}) \end{aligned}$$



(ii) If we choose $\underline{c}, \underline{a}$ to define the base then a similar calculation gives volume $= \underline{b} \cdot (\underline{c} \times \underline{a})$
We deduce the following symmetry/antisymmetry properties:

$$(\underline{a}, \underline{b}, \underline{c}) = (\underline{b}, \underline{c}, \underline{a}) = (\underline{c}, \underline{a}, \underline{b}) = -(\underline{a}, \underline{c}, \underline{b}) = -(\underline{b}, \underline{a}, \underline{c}) = -(\underline{c}, \underline{b}, \underline{a})$$

(iii) If $\underline{a}, \underline{b}$ and $\underline{c}$ are **coplanar** (*i.e.* all three vectors lie in the same plane) then $V = (\underline{a}, \underline{b}, \underline{c}) = 0$, and vice-versa.

1.6 The Vector Triple Product

There are *several* ways of combining 3 vectors to form a new vector.
e.g. $\underline{a} \times (\underline{b} \times \underline{c})$; $(\underline{a} \times \underline{b}) \times \underline{c}$, etc. Note carefully that *brackets are important*, since the cross product is *not* associative

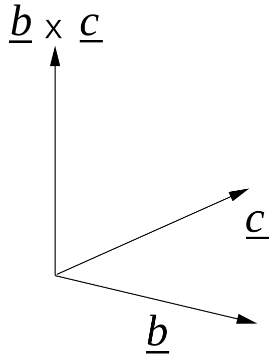
$$\underline{a} \times (\underline{b} \times \underline{c}) \;\neq\; (\underline{a} \times \underline{b}) \times \underline{c}\,.$$

Expressions involving two (or more) vector products can be simplified by using the identity

$$\boxed{\underline{a} \times (\underline{b} \times \underline{c}) \;=\; (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c}}\,.$$

This is a result you **must** know – memorise it! This is sometimes known as the 'bac-cab rule', but you must write the vectors in front of the scalar products to see this: $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}\,(\underline{a} \cdot \underline{c}) - \underline{c}\,(\underline{a} \cdot \underline{b})$

To show this, first note that $\underline{b}\times\underline{c}$ is $\perp$ to the $(\underline{b},\, \underline{c})$ plane. Now $\underline{a}\times(\underline{b}\times\underline{c})$ is $\perp$ to $\underline{b}\times\underline{c}$, so it must lie in this plane. Hence we can write



$$\underline{a}\times(\underline{b}\times\underline{c}) = \beta\,\underline{b} + \gamma\,\underline{c}$$

with β, γ scalars which must be linear in $\underline{a}$ & $\underline{c}$, $\underline{a}$ & $\underline{b}$ respectively. Taking the scalar product with $\underline{a}$ gives $\underline{a}\cdot(\underline{a}\times(\underline{b}\times\underline{c})) = 0 = \beta(\underline{a}\cdot\underline{b}) + \gamma(\underline{a}\cdot\underline{c})$, and from this we can write $\beta = \alpha(\underline{a}\cdot\underline{c})$, $\gamma = -\alpha(\underline{a}\cdot\underline{b})$ for some constant α, to give

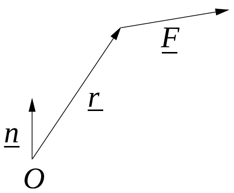
$$\underline{a}\times(\underline{b}\times\underline{c}) = \alpha\left[(\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}\right]$$

The constant may be determined by considering the particular case when $\underline{c} \parallel \underline{a}$ and $\underline{b} \perp \underline{c}$ (*e.g.* $\underline{b} = b\underline{e}_x$, $\underline{c} = c\underline{e}_y$, $\underline{a} = a\underline{e}_y$), to give $\underline{a}\times(\underline{b}\times\underline{c}) = ac\underline{b} = \alpha(ac\underline{b} - 0)$ or $\alpha = 1$. (Exercise: work through this.)

1.7 Some examples in Physics

(i) Torque

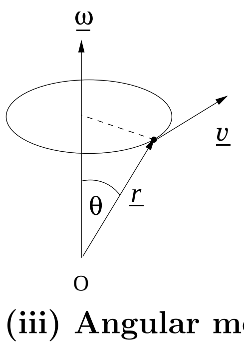
The **torque** or **couple** or **moment** of a force about the origin is defined as $\underline{T} = \underline{r}\times\underline{F}$ where $\underline{r}$ is the position vector of the point where the force is acting and $\underline{F}$ is the force vector at that point. Thus torque about the origin is a vector quantity.



The magnitude of the torque about an axis through the origin in direction $\underline{n}$ is given by $\underline{n}\cdot(\underline{r}\times\underline{F})$. Note that this is a scalar quantity formed by a scalar triple product

(ii) Angular velocity

Consider a point in a rigid body rotating with **angular velocity** $\underline{\omega}$: $|\underline{\omega}|$ is the angular speed of rotation measured in radians per second and $\underline{\hat{\omega}}$ lies along the axis of rotation. Let the position vector of the point with respect to an origin O on the axis of rotation be $\underline{r}$.



You should convince yourself that the point's velocity is $\underline{v} = \underline{\omega}\times\underline{r}$ by checking that this gives the right direction for $\underline{v}$; that it is perpendicular to the plane of $\underline{\omega}$ and $\underline{r}$; that the magnitude $|\underline{v}| = \omega r\sin\theta = \omega\rho$, where ρ is the radius of the circle in which the point is travelling.

(iii) Angular momentum

Now consider the **angular momentum** of a particle, this is defined by $\underline{L} = \underline{r}\times(m\underline{v})$ where m is the mass of the particle.

Using the above expression for $\underline{v}$ we obtain

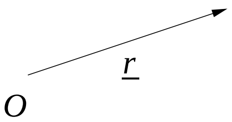
$$\underline{L} = m\underline{r}\times(\underline{\omega}\times\underline{r}) = m\left[r^2\,\underline{\omega} - (\underline{r}\cdot\underline{\omega})\,\underline{r}\right]$$

where we have used the identity for the vector triple product. Note that only if $\underline{r}$ is perpendicular to $\underline{\omega}$ do we obtain $\underline{L} = mr^2\underline{\omega}$, which means that only then are $\underline{L}$ and $\underline{\omega}$ in the same direction. Also note that $\underline{L} = 0$ if $\underline{\omega}$ and $\underline{r}$ are parallel.

2 Equations of Points, Lines and Planes

2.1 Position vector

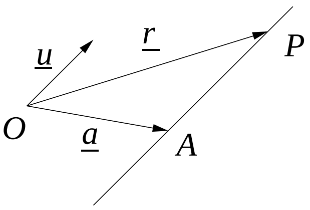
A **position vector** is a vector bound to some origin and gives the position of a point relative to that origin. It is often denoted by $\underline{r}$ (or $\overrightarrow{OP}$ or $\underline{x}$).



The equation for a point is simply $\underline{r} = \underline{a}$ where $\underline{a}$ is some vector.

2.2 The Equation of a Line

Suppose that P lies on a line which passes through a point A which has a position vector $\underline{a}$ with respect to an origin O. Let P have position vector $\underline{r}$ relative to O and let $\underline{u}$ be a vector through the origin in a direction parallel to the line.



We may write

$$\underline{r} = \underline{a} + \lambda \underline{u}$$

which is the **parametric equation of the line** *i.e.* as we vary the parameter λ from $-\infty$ to ∞, $\underline{r}$ describes all points on the line.

Rearranging and using $\underline{u} \times \underline{u} = 0$, we can also write this as

$$(\underline{r} - \underline{a}) \times \underline{u} = 0$$

or

$$\boxed{\underline{r} \times \underline{u} = \underline{c}}$$

where $\underline{c} = \underline{a} \times \underline{u}$ is normal to the plane containing the line and origin.

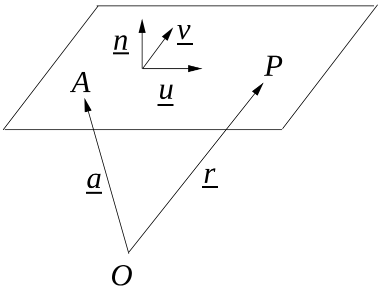
Physical example: If angular momentum $\underline{L}$ of a particle and its velocity $\underline{v}$ are known, we still don't know the position exactly because the solution of $\underline{L} = m\underline{r} \times \underline{v}$ is a line $\underline{r} = \underline{r}_0 + \lambda \underline{v}$.

Notes

(i) $\underline{r} \times \underline{u} = \underline{c}$ is an **implicit equation** for a line

(ii) $\underline{r} \times \underline{u} = 0$ is the equation of a line through the origin.

2.3 The Equation of a Plane



$\underline{r}$ is the position vector of an arbitrary point P on the plane
$\underline{a}$ is the position vector of a fixed point A in the plane
$\underline{u}$ and $\underline{v}$ are parallel to the plane but non-collinear: $\underline{u} \times \underline{v} \neq 0$.

We can express the vector $\overrightarrow{AP}$ in terms of $\underline{u}$ and $\underline{v}$, so that:

$$\underline{r} = \underline{a} + \overrightarrow{AP} = \underline{a} + \lambda \underline{u} + \mu \underline{v}$$

for some λ and μ. This is the **parametric equation of the plane**.

We define the unit normal to the plane

$$\underline{n} = \frac{\underline{u} \times \underline{v}}{|\underline{u} \times \underline{v}|} \ .$$

Since $\underline{u} \cdot \underline{n} = \underline{v} \cdot \underline{n} = 0$, we have the implicit equation

$$(\underline{r} - \underline{a}) \cdot \underline{n} = 0 \ .$$

Alternatively, we can write this as

$$\boxed{\underline{r} \cdot \underline{n} \ = \ p}$$

where $p = \underline{a} \cdot \underline{n}$ is the perpendicular distance of the plane from the origin.

This is a very important equation which you must be able to recognise.

Note: $\underline{r} \cdot \underline{n} = 0$ is the equation for a plane through the origin (with unit normal $\underline{n}$).

2.4 Examples of Dealing with Vector Equations

Before going through some worked examples let us state two simple rules which will help you to avoid many common mistakes

(i) **Always** check that the quantities on both sides of an equation are of the same type. For example, any equation of the form *vector* = *scalar* is clearly wrong. (The only exception to this is when we write *vector* = 0 instead of $\underline{0}$.)

(ii) **Never** try to divide by a vector – there is no such operation!

Example 1: Is the following set of equations consistent?

$$\underline{r} \times \underline{b} = \underline{c} \tag{1}$$
$$\underline{r} = \underline{a} \times \underline{c} \tag{2}$$

Geometrical interpretation: the first equation is the (implicit) equation for a line whereas the second equation is the (explicit) equation for a point. Thus the question is whether the point is on the line. If we insert equation (2) for $\underline{r}$ into the LHS of equation (1) we find

$$\underline{r} \times \underline{b} = (\underline{a} \times \underline{c}) \times \underline{b} = -\underline{b} \times (\underline{a} \times \underline{c}) = -\underline{a}\,(\underline{b} \cdot \underline{c}) + \underline{c}\,(\underline{a} \cdot \underline{b}) \tag{3}$$

Now from (1) we have that $\underline{b} \cdot \underline{c} = \underline{b} \cdot (\underline{r} \times \underline{b}) = 0$ thus (3) becomes

$$\underline{r} \times \underline{b} = \underline{c}\,(\underline{a} \cdot \underline{b}) \tag{4}$$

so that, on comparing (1) and (4), we require

$$\underline{a} \cdot \underline{b} = 1$$

for the equations to be consistent.

Example 2: Solve the following set of equations for $\underline{r}$.

$$\underline{r} \times \underline{a} \;=\; \underline{b} \tag{5}$$
$$\underline{r} \times \underline{c} \;=\; \underline{d} \tag{6}$$

Geometrical interpretation: both equations are equations for lines, *e.g.* (5) is for a line parallel to $\underline{a}$ where $\underline{b}$ is normal to the plane containing the line and the origin. The problem is to find the intersection of two lines – assuming the equations are consistent and the lines do indeed have an intersection.

Are these equations consistent? Take the scalar product of (5) with $\underline{c}$, and of (6) with $\underline{a}$:

$$(\underline{r} \times \underline{a}) \cdot \underline{c} \;=\; \underline{b} \cdot \underline{c} \tag{7}$$
$$(\underline{r} \times \underline{c}) \cdot \underline{a} \;=\; \underline{d} \cdot \underline{a} \tag{8}$$

Using the cyclic properties of the scalar triple product, we must have $\underline{b} \cdot \underline{c} = -\underline{d} \cdot \underline{a}$ for consistency.

To solve (5) and (6), we take the vector product of equation (5) with $\underline{d}$, which gives

$$\underline{b} \times \underline{d} = (\underline{r} \times \underline{a}) \times \underline{d} = -\underline{d} \times (\underline{r} \times \underline{a}) = -\underline{r}\,(\underline{a} \cdot \underline{d}) + \underline{a}\,(\underline{d} \cdot \underline{r})$$

From (6) we see that $\underline{d} \cdot \underline{r} = \underline{r} \cdot (\underline{r} \times \underline{c}) = 0$, so the solution is

$$\underline{r} = -\frac{\underline{b} \times \underline{d}}{\underline{a} \cdot \underline{d}} \qquad (\text{for } \ \underline{a} \cdot \underline{d} \neq 0)$$

Alternatively, we could have taken the vector product of $\underline{b}$ with equation (6) to obtain

$$\underline{b} \times \underline{d} = \underline{b} \times (\underline{r} \times \underline{c}) = \underline{r}\,(\underline{b} \cdot \underline{c}) - \underline{c}\,(\underline{b} \cdot \underline{r}) \,.$$

From equation (5), we find $\underline{b} \cdot \underline{r} = 0$, hence

$$\underline{r} = \frac{\underline{b} \times \underline{d}}{\underline{b} \cdot \underline{c}} \qquad (\text{for } \ \underline{b} \cdot \underline{c} \neq 0)$$

in agreement with our first solution (when $\underline{b} \cdot \underline{c} = -\underline{d} \cdot \underline{a}$)

What happens when $\underline{a} \cdot \underline{d} = \underline{b} \cdot \underline{c} = 0$? In this case the above approach does not give an expression for $\underline{r}$. However from (8) we see $\underline{a} \cdot \underline{d} = 0$ implies that $\underline{a} \cdot (\underline{r} \times \underline{c}) = 0$ so that $\underline{a}$, $\underline{c}$, $\underline{r}$ are coplanar. We can therefore write $\underline{r}$ as a linear combination of $\underline{a}, \underline{c}$:

$$\underline{r} = \alpha \, \underline{a} + \gamma \, \underline{c} \, . \tag{9}$$

To determine the scalar α we can take the vector product with $\underline{c}$ to find

$$\underline{d} = \alpha \, \underline{a} \times \underline{c} \tag{10}$$

(because $\underline{r} \times \underline{c} = \underline{d}$ from (6) and $\underline{c} \times \underline{c} = 0$). In order to extract α we need to convert the vectors in (10) into scalars. We do this by taking, for example, a scalar product with $\underline{b}$

$$\underline{b} \cdot \underline{d} = \alpha \, \underline{b} \cdot (\underline{a} \times \underline{c})$$

so that

$$\alpha = \frac{-\underline{b} \cdot \underline{d}}{(\underline{a}\,,\ \underline{b}\,,\ \underline{c})} \, .$$

Similarly, one can determine γ by taking the vector product of (9) with $\underline{a}$:

$$\underline{b} = \gamma \, \underline{c} \times \underline{a}$$

then taking a scalar product with $\underline{b}$ to obtain finally

$$\gamma = \frac{\underline{b} \cdot \underline{b}}{(\underline{a}\,,\ \underline{b}\,,\ \underline{c})} \quad .$$

Example 3: Solve for $\underline{r}$ the vector equation

$$\underline{r} + (\underline{n} \cdot \underline{r}) \, \underline{n} + 2\underline{n} \times \underline{r} + 2\underline{b} = 0 \tag{11}$$

where $\underline{n} \cdot \underline{n} = 1$.

In order to unravel this equation we can try taking scalar and vector products of the equation with the vectors involved. However straight away we see that taking various products with $\underline{r}$ will not help, since it will produce terms that are quadratic in $\underline{r}$. Instead, we want to eliminate $(\underline{n} \cdot \underline{r})$ and $(\underline{n} \times \underline{r})$ so we try taking scalar and vector products with $\underline{n}$.

Taking the scalar product of $\underline{n}$ with both sides of equation (11) one finds

$$\underline{n} \cdot \underline{r} + (\underline{n} \cdot \underline{r})(\underline{n} \cdot \underline{n}) + 0 + 2\underline{n} \cdot \underline{b} = 0$$

so that, since $(\underline{n} \cdot \underline{n}) = 1$, we have

$$\underline{n} \cdot \underline{r} = -\underline{n} \cdot \underline{b} \tag{12}$$

Taking the vector product of $\underline{n}$ with equation (11) gives

$$\underline{n} \times \underline{r} + 0 + 2 \left[\underline{n}(\underline{n} \cdot \underline{r}) - \underline{r} \right] + 2\underline{n} \times \underline{b} = 0$$

so that

$$\underline{n} \times \underline{r} = 2 \left[\underline{n}(\underline{b} \cdot \underline{n}) + \underline{r} \right] - 2\underline{n} \times \underline{b} \tag{13}$$

where we have used (12). Substituting (12) and (13) into (11) one (eventually) obtains

$$\underline{r} = \frac{1}{5} \left[-3(\underline{b} \cdot \underline{n}) \, \underline{n} + 4(\underline{n} \times \underline{b}) - 2\underline{b} \right] \tag{14}$$

3 Vector Spaces and Orthonormal Bases

3.1 Review of linear vector spaces

Let V denote a linear vector space. Then vectors in V obey the following rules for addition and multiplication by scalars

$$\begin{aligned} \underline{a} + \underline{b} &\in V \qquad \text{if} \quad \underline{a}, \underline{b} \in V \\ \alpha \underline{a} &\in V \qquad \text{if} \quad \underline{a} \in V \\ \alpha(\underline{a} + \underline{b}) &= \alpha \underline{a} + \alpha \underline{b} \\ (\alpha + \beta)\underline{a} &= \alpha \underline{a} + \beta \underline{a} \end{aligned}$$

The space contains a zero vector or null vector, $\underline{0}$, so that, for example $\underline{a} + (-\underline{a}) = \underline{0}$. We usually omit the underline from the zero vector.

Of course as we have seen, vectors in $I\!R^3$ (usual 3-dimensional real space) obey these axioms. Other simple examples are a plane through the origin which forms a two-dimensional space and a line through the origin which forms a one-dimensional space.

3.2 Linear Independence

Let $\underline{a}$ and $\underline{b}$ be two vectors in a plane through the origin, and consider the equation

$$\boxed{\alpha \underline{a} + \beta \underline{b} = 0}$$

If this is satisfied for *non-zero* α and β then $\underline{a}$ and $\underline{b}$ are said to be *linearly dependent*,

$$i.e. \quad \underline{b} = -\frac{\alpha}{\beta}\,\underline{a}\ .$$

Clearly $\underline{a}$ and $\underline{b}$ are *collinear* (either parallel or anti-parallel).

If this equation can be satisfied *only* for $\alpha = \beta = 0$, then $\underline{a}$ and $\underline{b}$ are *linearly independent*; they are obviously *not* collinear, and no λ can be found such that $\underline{b} = \lambda \underline{a}$.

Notes

(i) If $\underline{a}$, $\underline{b}$ are linearly independent then any vector $\underline{r}$ in the plane may be written uniquely as a linear combination

$$\underline{r} = \alpha \underline{a} + \beta \underline{b}$$

(ii) We say $\underline{a}$, $\underline{b}$ *span* the plane, or $\underline{a}$, $\underline{b}$ form a *basis* for the plane.

(iii) We call (α, β) a *representation* of $\underline{r}$ in the basis formed by $\underline{a}$, $\underline{b}$, and we say that α, β are the *components* of $\underline{r}$ in this basis.

In three dimensions three vectors are linearly dependent if we can find non-trivial α, β, γ (*i.e.* not all zero) such that

$$\boxed{\alpha \underline{a} + \beta \underline{b} + \gamma \underline{c} = 0}$$

otherwise $\underline{a}$, $\underline{b}$, $\underline{c}$ are linearly independent (no one is a linear combination of the other two).

Notes

(i) If $\underline{a}$, $\underline{b}$ and $\underline{c}$ are linearly independent they span $I\!R^3$ and form a basis, *i.e.* for any vector $\underline{r}$ we can find scalars α, β, γ such that

$$\underline{r} = \alpha \underline{a} + \beta \underline{b} + \gamma \underline{c} \; .$$

(ii) The *triple* of numbers (α, β, γ) is the *representation* of $\underline{r}$ in this basis, and α, β, γ are the *components* of $\underline{r}$ in this basis.

(iii) The geometrical interpretation of linear dependence in three dimensions is that

$$\boxed{\text{three linearly dependent vectors} \quad \Leftrightarrow \quad \text{three coplanar vectors}}$$

To see this, note that if $\alpha \underline{a} + \beta \underline{b} + \gamma \underline{c} = 0$ then

$$\begin{array}{llll} \text{for } \alpha \neq 0: & \underline{a} \cdot (\underline{b} \times \underline{c}) = 0 & \Rightarrow & \underline{a}, \underline{b}, \underline{c} \;\; \text{are coplanar} \\ \text{for } \alpha = 0: & \underline{b} \text{ is collinear with } \underline{c} & \Rightarrow & \underline{a}, \underline{b}, \underline{c} \;\; \text{are coplanar} \end{array}$$

These ideas can be generalised to vector spaces of arbitrary dimension. For a space of dimension n one can find at most n linearly independent vectors.

3.3 Standard orthonormal basis: Cartesian basis

A basis in which the basis vectors are *orthogonal* and *normalised* (of unit length) is called an **orthonormal** basis.

You have already have encountered the idea of *Cartesian coordinates* in which points in space are labelled by coordinates (x, y, z). As usual, we introduce orthonormal basis vectors denoted by $\underline{i}$, $\underline{j}$ and $\underline{k}$ or $\underline{e}_x$, $\underline{e}_y$ and $\underline{e}_z$ which point along the x, y and z-axes, respectively. It is usually understood that the basis vectors are related by the right-hand screw rule, with $\underline{i} \times \underline{j} = \underline{k}$ and so on, cyclically.

In the 'xyz' notation the components of a vector $\underline{a}$ are a_x, a_y, a_z, and a vector is written in terms of the basis vectors as

$$\underline{a} \;\; = \;\; a_x \, \underline{i} + a_y \, \underline{j} + a_z \, \underline{k} \qquad \text{or} \qquad \underline{a} = a_x \, \underline{e}_x + a_y \, \underline{e}_y + a_z \, \underline{e}_z \; .$$

Also note that *in this basis* the basis vectors themselves are represented by

$$\underline{i} = \underline{e}_x = (1, 0, 0) \quad \underline{j} = \underline{e}_y = (0, 1, 0) \quad \underline{k} = \underline{e}_z = (0, 0, 1)$$

In the following we shall sometimes use the 'xyz' notation but very rarely the 'ijk' notation.

3.4 Introduction to Suffix or Index notation

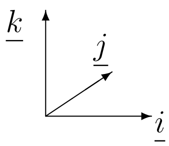
A more systematic labelling of orthonormal basis vectors for $I\!R^3$ is to use $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$. Instead of $\underline{i}$ we write $\underline{e}_1$, instead of $\underline{j}$ we write $\underline{e}_2$, and instead of $\underline{k}$ we write $\underline{e}_3$. Then, from the definition of the scalar product in Section (1.3), we get

$$\boxed{\underline{e}_1 \cdot \underline{e}_1 = \underline{e}_2 \cdot \underline{e}_2 = \underline{e}_3 \cdot \underline{e}_3 = 1 \qquad \text{and} \qquad \underline{e}_1 \cdot \underline{e}_2 = \underline{e}_2 \cdot \underline{e}_3 = \underline{e}_3 \cdot \underline{e}_1 = 0} \tag{15}$$

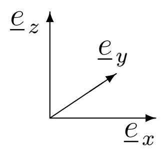
Similarly the components of any vector $\underline{a}$ in 3-d space are denoted by a_1, a_2 and a_3.

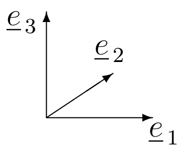
This scheme is known as the *suffix* or *index* notation. Its great advantages over 'xyz' notation are that it clearly generalises easily to any number of dimensions, and it greatly simplifies manipulations and the verification of various identities (see later in the course).

Old Notation **New Notation**



or





$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ $\quad$ $\underline{r} = x\,\underline{e}_x + y\,\underline{e}_y + z\,\underline{e}_z$ $\quad$ $\underline{r} = x_1\,\underline{e}_1 + x_2\,\underline{e}_2 + x_3\,\underline{e}_3 \equiv r_1\,\underline{e}_1 + r_2\,\underline{e}_2 + r_3\,\underline{e}_3$

Thus any vector $\underline{a}$ is written in this new notation as

$$\underline{a} = a_1\,\underline{e}_1 + a_2\,\underline{e}_2 + a_3\,\underline{e}_3 = \sum_{i=1}^{3} a_i\,\underline{e}_i\,.$$

$$\boxed{\text{The last summation will often be abbreviated to } \underline{a} = \sum_i a_i\,\underline{e}_i}$$

Notes

(i) The three numbers a_i, $i = 1, 2, 3$, are called the (Cartesian) *components* of $\underline{a}$ with respect to the basis set $\{\underline{e}_i\}$.

(ii) We may write $\underline{a} = \sum_{i=1}^{3} a_i\,\underline{e}_i = \sum_{j=1}^{3} a_j\,\underline{e}_j = \sum_{\alpha=1}^{3} a_\alpha\,\underline{e}_\alpha$ where the summed indices i, j, α are called 'dummy', 'repeated' or 'summation' indices. We can choose any letter for them.

(iii) The components a_i of a vector $\underline{a}$ may be obtained using the orthonormality properties of equation (15):

$$\underline{a} \cdot \underline{e}_1 = (a_1\,\underline{e}_1 + a_2\,\underline{e}_2 + a_3\,\underline{e}_3) \cdot \underline{e}_1 = a_1$$

$$\boxed{a_1 \text{ is the projection of } \underline{a} \text{ in the direction of } \underline{e}_1.}$$

Similarly for the components a_2 and a_3. So in general we may write

$$\boxed{\underline{a} \cdot \underline{e}_i = a_i \quad \text{or sometimes} \quad (\underline{a})_i}$$

where in this equation i is a 'free' index and may take values $i = 1, 2, 3$. In this way we are in fact condensing three equations into one.

(iv) In terms of these components, the scalar product is

$$\underline{a} \cdot \underline{b} = (a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3) \cdot (b_1 \underline{e}_1 + b_2 \underline{e}_2 + b_3 \underline{e}_3)$$

Using the orthonormality of the basis vectors (equation (15)), this becomes

$$\boxed{\underline{a} \cdot \underline{b} = \sum_{i=1}^{3} a_i \, b_i}$$

In particular the magnitude of a vector is now

$$a = |\underline{a}| = \sqrt{\underline{a} \cdot \underline{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2} \, .$$

(v) From Notes 3 and 4 above we can define **direction cosines** l_1, l_2, l_3 of the vector $\underline{a}$ as the cosines of the angles between the vector and the basis axes, namely

$$l_i \equiv \cos\theta_i = \frac{\underline{a} \cdot \underline{e}_i}{a} = \frac{a_i}{a} \, , \qquad i = 1, 2, 3 \, .$$

It follows that

$$\sum_{i=1}^{3} l_i^2 \equiv l_1^2 + l_2^2 + l_3^2 = 1 \, .$$

If $\underline{a}$ has direction cosines l_i, $\underline{b}$ has direction cosines m_i, and θ is the angle between $\underline{a}$ and $\underline{b}$, then

$$\underline{a} \cdot \underline{b} = \sum_{i=1}^{3} a_i b_i = ab \sum_{i=1}^{3} l_i m_i = ab \cos\theta \, ,$$

or

$$\cos\theta = \sum_{i=1}^{3} l_i m_i \, .$$

4 Using Suffix Notation

4.1 Free Indices and Summation Indices

Consider, for example, the vector equation

$$\underline{a} - (\underline{b} \cdot \underline{c}) \, \underline{d} + 3\underline{n} = 0 \qquad (16)$$

The basis vectors are linearly independent, so this equation must hold for each component separately

$$a_i - (\underline{b} \cdot \underline{c})\, d_i + 3n_i = 0 \quad \text{for} \quad i = 1, 2, 3 \tag{17}$$

The free index i occurs **once** and ***only*** **once** in each term of the equation. In general every term in the equation must be of the same kind, *i.e.* have the same free indices.

Now suppose that we want to write the scalar product that appears in the second term of equation (17) in suffix notation. As we have seen, summation indices are 'dummy' indices and can be relabelled. For example

$$\underline{b} \cdot \underline{c} = \sum_{i=1}^{3} b_i c_i = \sum_{k=1}^{3} b_k c_k$$

This freedom should *always* be used to avoid confusion with other indices in the equation. In this case, we avoid using i as a summation index, as we have already used it as a free index, and rewrite equation (17) as

$$a_i - \left(\sum_{k=1}^{3} b_k c_k \right) d_i + 3n_i = 0 \quad \text{for} \quad i = 1, 2, 3$$

rather than

$$a_i - \left(\sum_{i=1}^{3} b_i c_i \right) d_i + 3n_i = 0 \quad \text{for} \quad i = 1, 2, 3$$

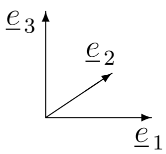
which would lead to great confusion and inevitably lead to mistakes when the brackets are removed – as they will be very soon.

4.2 Handedness of Basis

In the usual Cartesian basis that we've considered up to now, the basis vectors $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$ form a *right-handed* basis: $\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$, $\underline{e}_2 \times \underline{e}_3 = \underline{e}_1$, $\underline{e}_3 \times \underline{e}_1 = \underline{e}_2$.

However, we *could* choose $\underline{e}_1 \times \underline{e}_2 = -\underline{e}_3$, and so on, in which case the basis is said to be *left-handed*.

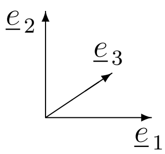
Right handed



$$\begin{aligned} \underline{e}_3 &= \underline{e}_1 \times \underline{e}_2 \\ \underline{e}_1 &= \underline{e}_2 \times \underline{e}_3 \\ \underline{e}_2 &= \underline{e}_3 \times \underline{e}_1 \end{aligned}$$

$$\boxed{(\underline{e}_1, \underline{e}_2, \underline{e}_3) = 1}$$

Left handed



$$\begin{aligned} \underline{e}_3 &= \underline{e}_2 \times \underline{e}_1 \\ \underline{e}_1 &= \underline{e}_3 \times \underline{e}_2 \\ \underline{e}_2 &= \underline{e}_1 \times \underline{e}_3 \end{aligned}$$

$$\boxed{(\underline{e}_1, \underline{e}_2, \underline{e}_3) = -1}$$

4.3 The Vector Product in a right-handed basis

$$\underline{a} \times \underline{b} \;=\; \left(\sum_{i=1}^{3} a_i \,\underline{e}_i\right) \times \left(\sum_{j=1}^{3} b_j \,\underline{e}_j\right) \;=\; \sum_{i=1}^{3}\sum_{j=1}^{3} a_i \, b_j \, (\underline{e}_i \times \underline{e}_j) \,.$$

Since $\underline{e}_1 \times \underline{e}_1 = \underline{e}_2 \times \underline{e}_2 = \underline{e}_3 \times \underline{e}_3 = 0$, and $\underline{e}_1 \times \underline{e}_2 = -\underline{e}_2 \times \underline{e}_1 = \underline{e}_3$, *etc*, we have

$$\underline{a} \times \underline{b} = \underline{e}_1(a_2b_3 - a_3b_2) + \underline{e}_2(a_3b_1 - a_1b_3) + \underline{e}_3(a_1b_2 - a_2b_1) \tag{18}$$

from which we deduce that

$$(\underline{a} \times \underline{b})_1 = a_2b_3 - a_3b_2 \;,\; \textit{etc.}$$

Notice that the right-hand side of equation (18) corresponds to the expansion of the determinant

$$\begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

by the first row (see the next section for some properties of determinants.)

4.4 Determinants and the scalar triple product

We may label the elements of a 3×3 array of numbers or *matrix* A by a_{ij} (or alternatively by A_{ij}) where i labels the row and j labels the column in which a_{ij} appears

$$A \;=\; \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Then the *determinant* of the matrix A is defined as

$$\begin{aligned} \det A \;&\equiv\; \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &=\; a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

It is now easy to write down an expression for the scalar triple product

$$\begin{aligned} \underline{a} \cdot (\underline{b} \times \underline{c}) \;&=\; \sum_{i=1}^{3} a_i \, (\underline{b} \times \underline{c})_i \\ &=\; a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2) \\ &=\; \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{aligned}$$

Some properties of the determinant

An alternative expression for the determinant is given by noting that

$$\begin{aligned}
\det A &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}) \\
&= \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}
\end{aligned}$$

Evidently, the rows and columns of the matrix can be interchanged or *transposed* without changing the determinant. This may be written more elegantly by defining the *transpose* A^T of a matrix A as the matrix with elements $(A^T)_{ij} = a_{ji}$. Then

$$\det A = \det A^T \, .$$

The symmetry properties of the determinant may be deduced from the scalar triple product (STP) by noting that interchanging two adjacent vectors in the STP is equivalent to interchanging two adjacent rows (or columns) of the determinant and changes its value by a factor -1. Also adding a multiple of one row (or column) to another does not change the value of $\det A$.

4.5 Summary of the algebraic approach to vectors

We are now able to define vectors and the various products of vectors in an algebraic way (as opposed to the geometrical approach of lectures 1 and 2).

A **vector** is *represented* (in some orthonormal basis $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$) by an ordered set of 3 numbers with certain laws of addition. For example

$$\begin{aligned}
\underline{a} &\quad \text{is represented by} \quad (a_1,\, a_2,\, a_3) \\
\underline{a} + \underline{b} &\quad \text{is represented by} \quad (a_1 + b_1,\, a_2 + b_2\,, a_3 + b_3)\, .
\end{aligned}$$

The various 'products' of vectors are now defined as follows:

The Scalar Product is denoted by $\underline{a} \cdot \underline{b}$ and *defined* as

$$\begin{aligned}
\underline{a} \cdot \underline{b} &\equiv \sum_i a_i b_i \\
\underline{a} \cdot \underline{a} &= a^2 \quad \text{defines the magnitude } a \text{ of the vector.}
\end{aligned}$$

The Vector Product is denoted by $\underline{a} \times \underline{b}$ and *defined* in a right-handed basis as

$$\underline{a} \times \underline{b} \equiv \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The Scalar Triple Product

$$(\underline{a}, \underline{b}, \underline{c}) \equiv \sum_i a_i \, (\underline{b} \times \underline{c})_i = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

In all the above formula the summations imply sums over each index taking values 1, 2, 3.

4.6 The Kronecker delta symbol δ_{ij}

We *define* the symbol δ_{ij} (pronounced *"delta i j"*), where i and j can take on the values 1, 2, 3, as follows

$$\boxed{\begin{aligned} \delta_{ij} &= 1 \quad \text{when } i = j \\ &= 0 \quad \text{when } i \neq j \end{aligned}}$$

i.e. $\delta_{11} = \delta_{22} = \delta_{33} = 1$ and $\delta_{12} = \delta_{13} = \delta_{23} = \cdots = 0$.

The equations satisfied by the three *orthonormal basis vectors* $\underline{e}_i$ can now be written as

$$\boxed{\underline{e}_i \cdot \underline{e}_j = \delta_{ij}}$$

e.g. $\underline{e}_1 \cdot \underline{e}_2 = \delta_{12} = 0\,, \quad \underline{e}_1 \cdot \underline{e}_1 = \delta_{11} = 1$

Notes

(i) Since there are two free indices i and j, $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$ is equivalent to 9 equations.

(ii) $\delta_{ij} = \delta_{ji}$. We say δ_{ij} is *symmetric* in its indices.

(iii) $\displaystyle\sum_{i=1}^{3} \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

(iv) $\displaystyle\sum_{j=1}^{3} a_j \delta_{jk} = a_1\delta_{1k} + a_2\delta_{2k} + a_3\delta_{3k}$

To go further, first note that k is a free index.

If $k = 1$, then only the first term on the RHS contributes and the RHS $= a_1$. Similarly, if $k = 2$ then the RHS $= a_2$, and if $k = 3$ the RHS $= a_3$. Hence

$$\boxed{\sum_{j=1}^{3} a_j \delta_{jk} = a_k}$$

In other words, Kronecker delta δ_{jk} picks out the k^{th} term in the sum over j.

Generalising the reasoning in 4 implies the so-called *sifting property* of Kronecker delta

$$\boxed{\sum_{j=1}^{3} (\text{anything})_j \, \delta_{jk} = (\text{anything }\,)_k}$$

where $(\text{anything})_j$ denotes any expression that has a single free index j.

Matrix representation: δ_{ij} may be thought of as the elements of a 3×3 unit matrix

$$\begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{33} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

In other words, δ_{ij} is the ij^{th} element of the unit matrix I, *i.e.* $I_{ij} = \delta_{ij}$.

Examples of the use of Kronecker delta

1.
$$\begin{aligned} \underline{a} \cdot \underline{e}_j &= \left(\sum_{i=1}^{3} a_i \, \underline{e}_i \right) \cdot \underline{e}_j = \sum_{i=1}^{3} a_i \, (\underline{e}_i \cdot \underline{e}_j) \\ &= \sum_{i=1}^{3} a_i \delta_{ij} = a_j \quad \text{because terms with } i \neq j \text{ vanish.} \end{aligned}$$

2.
$$\begin{aligned} \underline{a} \cdot \underline{b} &= \left(\sum_{i=1}^{3} a_i \, \underline{e}_i \right) \cdot \left(\sum_{j=1}^{3} b_j \, \underline{e}_j \right) \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \, (\underline{e}_i \cdot \underline{e}_j) = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \delta_{ij} \\ &= \sum_{i=1}^{3} a_i b_i \quad \left(\text{or } \sum_{j=1}^{3} a_j b_j \right) \end{aligned}$$

5 More About Suffix Notation

5.1 The Einstein Summation Convention

The novelty of writing out summations soon wears thin. The standard way to avoid this tedium is to adopt the Einstein summation convention. By adhering **strictly** to the following conventions or "rules" the summation signs are suppressed completely.

Rules of the summation convention

(i) Omit *all* summation signs.

(ii) If a suffix appears *twice*, a summation is implied, *e.g.* $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$.
Here i is a *dummy* or *repeated* index.

(iii) If a suffix appears only *once* it can take any value *e.g.* $a_i = b_i$ holds for $i = 1, 2, 3$.
Here i is a *free* index. Note that there may be more than one free index.
Always check that the free indices match on both sides of an equation.
For example, $a_j = b_i$ is WRONG.

(iv) A given suffix must **not** appear more than **twice** in any term in an expression.
Always check that there aren't more than two identical indices *e.g.* $a_i b_i c_i$ is simply WRONG.

Examples

$\underline{a} = a_i\,\underline{e}_i$ (*i* is a dummy index)

$\underline{a}\cdot\underline{e}_j = a_i\,\underline{e}_i\cdot\underline{e}_j = a_i\delta_{ij} = a_j$ (*i* is a dummy index, but *j* is a free index)

$\underline{a}\cdot\underline{b} = (a_i\,\underline{e}_i)\cdot(b_j\,\underline{e}_j) = a_ib_j\delta_{ij} = a_jb_j$ (i, j are both dummy indices)

$(\underline{a}\cdot\underline{b})(\underline{a}\cdot\underline{c}) = a_ib_ia_jc_j$ (again i, j are dummy indices)

Armed with the summation convention one can rewrite many of the equations from the previous sections without summation signs, *e.g.* the sifting property of δ_{ij} now becomes

$$\boxed{[\ldots]_j\,\delta_{jk} = [\ldots]_k}$$

The repeated index j is implicitly summed over, so that, for example, $\delta_{ij}\delta_{jk} = \delta_{ik}$.

> From now on, except where indicated, the summation convention will be assumed. You should make sure that you are completely at ease with it.

5.2 Levi-Civita Symbol ϵ_{ijk}

We have seen how δ_{ij} can be used to express the orthonormality of basis vectors succinctly.

We now seek to make a similar simplification for the vector products of basis vectors (taken here to be right handed), *i.e.* we seek a simple, uniform way of writing the equations

$$\begin{array}{lllllllll} \underline{e}_1\times\underline{e}_2 & = & \underline{e}_3 & \quad \underline{e}_2\times\underline{e}_3 & = & \underline{e}_1 & \quad \underline{e}_3\times\underline{e}_1 & = & \underline{e}_2 \\ \underline{e}_1\times\underline{e}_1 & = & 0 & \quad \underline{e}_2\times\underline{e}_2 & = & 0 & \quad \underline{e}_3\times\underline{e}_3 & = & 0 \end{array}$$

To do so we define the Levi-Cevita or 'epsilon symbol' ϵ_{ijk} (pronounced *'epsilon i j k'*), where i, j and k can take on the values 1 to 3, such that

$$\boxed{\begin{array}{lll} \epsilon_{ijk} & = & +1 \text{ if } ijk \text{ is an } \textit{even} \text{ permutation of } 123 \\ & = & -1 \text{ if } ijk \text{ is an } \textit{odd} \text{ permutation of } 123 \\ & = & \;\;\,0 \text{ otherwise } (\textit{i.e.} \text{ 2 or more indices are the same}) \end{array}}$$

An *even* permutation consists of an *even* number of transpositions of two indices;
An *odd* permutation consists of an *odd* number of transpositions of two indices.

Examples:

$$\begin{array}{lll} \epsilon_{123} & = & +1 \\ \epsilon_{213} & = & -1 \ \{\text{since } (123)\to(213) \text{ under } \textit{one} \text{ transposition } [1\leftrightarrow 2]\} \\ \epsilon_{312} & = & +1 \ \{(123)\to(132)\to(312);\ 2 \text{ transpositions; } [2\leftrightarrow 3][1\leftrightarrow 3]\} \\ \epsilon_{113} & = & 0\,;\quad \epsilon_{111} = 0\,;\ \textit{etc.} \end{array}$$

$$\boxed{\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1 \qquad \epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1 \qquad \text{all others } = 0}$$

Note the symmetry of ϵ_{ijk} under *cyclic permutations*

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji} \tag{19}$$

This holds for all values of i, j and k. To understand it, note that

(i) If any two of the free indices i, j, k are the same, all terms vanish.

(ii) If (ijk) is an even (odd) permutation of (123), then so are (jki) and (kij), but (jik), (ikj) and (kji) are odd (even) permutations of (123).

Each of equations (19) has three free indices so they each represent 27 equations.
E.g. in $\epsilon_{ijk} = \epsilon_{kij}$, 3 equations say '$1 = 1$', 3 equations say '$-1 = -1$', and 21 equations say '$0 = 0$'.

5.3 Vector product

The equations satisfied by the vector products of the (right-handed) orthonormal basis vectors $\underline{e}_i$ can now be written uniformly as

$$\boxed{\underline{e}_i \times \underline{e}_j = \epsilon_{ijk}\, \underline{e}_k} \qquad \forall\, i, j = 1,2,3$$

where there is an implicit sum over the 'dummy' or 'repeated' index k. For example,

$$\underline{e}_1 \times \underline{e}_2 = \epsilon_{121}\, \underline{e}_1 + \epsilon_{122}\, \underline{e}_2 + \epsilon_{123}\, \underline{e}_3 = \underline{e}_3 \qquad \underline{e}_1 \times \underline{e}_1 = \epsilon_{111}\, \underline{e}_1 + \epsilon_{112}\, \underline{e}_2 + \epsilon_{113}\, \underline{e}_3 = 0$$

Now consider

$$\underline{a} \times \underline{b} = a_i\, b_j\, \underline{e}_i \times \underline{e}_j = \epsilon_{ijk}\, a_i b_j\, \underline{e}_k$$

but, by definition, we also have

$$\underline{a} \times \underline{b} = (\underline{a} \times \underline{b})_k\, \underline{e}_k$$

therefore

$$\boxed{(\underline{a} \times \underline{b})_k = \epsilon_{ijk}\, a_i b_j}$$

Note that we are using the summation convention. For example, writing out the sums

$$\begin{aligned}(\underline{a} \times \underline{b})_3 &= \epsilon_{113}\, a_1 b_1 + \epsilon_{123}\, a_2 b_3 + \epsilon_{133}\, a_3 b_3 + \epsilon_{213}\, a_2 b_1 + \cdots \\ &= \epsilon_{123}\, a_1 b_2 + \epsilon_{213}\, a_2 b_1 \qquad \text{(plus terms that are zero)} \\ &= a_1 b_2 - a_2 b_1\end{aligned}$$

We can use the cyclic symmetry of the ϵ symbol to find an alternative form for the components of the vector product

$$(\underline{a} \times \underline{b})_k = \epsilon_{ijk}\, a_i b_j = \epsilon_{kij}\, a_i b_j\,,$$

or *relabelling* the dummy indices $k \to i, \quad i \to j, \quad j \to k$

$$\boxed{(\underline{a} \times \underline{b})_i = \epsilon_{ijk}\, a_j b_k}$$

which is (probably) the most useful form.

The scalar triple product can also be written using ϵ_{ijk}

$$(\underline{a}, \underline{b}, \underline{c}) = \underline{a} \cdot \; (\underline{b} \times \underline{c}) = a_i (\underline{b} \times \underline{c})_i$$

giving

$$\boxed{(\underline{a}, \underline{b}, \underline{c}) = \epsilon_{ijk}\, a_i b_j c_k}$$

As an exercise in index manipulation we can prove the cyclic symmetry of the scalar product

$$\begin{aligned}
(\underline{a},\, \underline{b},\, \underline{c}) &= \;\;\epsilon_{ijk}\, a_i b_j c_k \\
&= -\epsilon_{ikj}\, a_i b_j c_k && \text{(interchanging two indices of } \epsilon_{ijk}\text{)} \\
&= +\epsilon_{kij}\, a_i b_j c_k && \text{(interchanging two indices again)} \\
&= \;\;\epsilon_{ijk}\, a_j b_k c_i && \text{(relabelling indices } k \to i,\; i \to j,\; j \to k\text{)} \\
&= \;\;\epsilon_{ijk}\, c_i a_j b_k \\
&= \;\;(\underline{c},\, \underline{a},\, \underline{b})
\end{aligned}$$

5.4 Product of two Levi-Civita symbols

We have already shown geometrically that

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$$

This be derived independently using components. For example,

$$\begin{aligned}
[\underline{a} \times (\underline{b} \times \underline{c})]_1 &= a_2\, (\underline{b} \times \underline{c})_3 - a_3\, (\underline{b} \times \underline{c})_2 \\
&= a_2\, (b_1 c_2 - b_2 c_1) - a_3\, (b_3 c_1 - b_1 c_3) \\
&= b_1\, (a_2 c_2 + a_3 c_3) - c_1\, (a_2 b_2 + a_3 b_3) \\
&= b_1\, (a_1 c_1 + a_2 c_2 + a_3 c_3) - c_1\, (a_1 b_1 + a_2 b_2 + a_3 b_3) \\
&= b_1\, (\underline{a} \cdot \underline{c}) - c_1\, (\underline{a} \cdot \underline{b})
\end{aligned}$$

From this equality we deduce that there must be a relation between two ϵ symbols (because there are two cross products) and some number of δ symbols. Consider

$$\begin{aligned}
[\underline{a} \times (\underline{b} \times \underline{c})]_i &= \epsilon_{ijk} a_j\, (\underline{b} \times \underline{c})_k \\
&= \epsilon_{ijk} a_j\, \epsilon_{klm}\, b_l\, c_m \\
&= \epsilon_{ijk}\, \epsilon_{klm}\, a_j\, b_l\, c_m
\end{aligned}$$

Alternatively

$$\begin{aligned}
[(\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}]_i &= (\underline{a} \cdot \underline{c})\, b_i - (\underline{a} \cdot \underline{b})\, c_i \\
&= (a_j\, c_m\, \delta_{jm})\, \delta_{il}\, b_l - (a_j\, b_l\, \delta_{jl})\, \delta_{im}\, c_m \\
&= (\delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl})\, a_j\, b_l\, c_m\,.
\end{aligned}$$

These equations must be equal for *all* components a_j, b_l, c_m independently, so we must have

$$\boxed{\epsilon_{ijk}\, \epsilon_{klm} = \delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl}}$$

This is a **very** important result and must be learnt by heart.

To verify it, one can check all possible cases. For example

$$\epsilon_{12k}\,\epsilon_{k12} = \epsilon_{121}\,\epsilon_{112} + \epsilon_{122}\,\epsilon_{212} + \epsilon_{123}\,\epsilon_{312} = 1 = \delta_{11}\delta_{22} - \delta_{12}\delta_{21}$$

However as we have $3^4 = 81$ equations, 6 saying '$1 = 1$', 6 saying '$-1 = -1$', and 69 saying $0 = 0$', this will take some time. More generally, note that the left hand side of the boxed equation may be written out as

- $\epsilon_{ij1}\,\epsilon_{1lm} + \epsilon_{ij2}\,\epsilon_{2lm} + \epsilon_{ij3}\,\epsilon_{3lm}$ where i, j, l, m are free indices;
- for this to be non-zero we must have $i \neq j$ and $l \neq m$;
- only one term of the three in the sum can be non-zero;
- if $i = l$ and $j = m$ we have $+1$, if $i = m$ and $j = l$ we have -1.

Example: Simplify $(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d})$ using suffix notation.

$$\begin{aligned}
(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d}) &= (\underline{a} \times \underline{b})_i\,(\underline{c} \times \underline{d})_i \;=\; \epsilon_{ijk}\,a_j\,b_k\,\epsilon_{ilm}\,c_l\,d_m \\
&= (\delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl})\,a_j\,b_k\,c_l\,d_m \;=\; a_j\,b_k\,c_j\,d_k - a_j\,b_k\,c_k\,d_j \\
&= (\underline{a} \cdot \underline{c})\,(\underline{b} \cdot \underline{d}) \;-\; (\underline{a} \cdot \underline{d})\,(\underline{b} \cdot \underline{c})
\end{aligned}$$

where we used the cyclic property $\epsilon_{ijk} = \epsilon_{jki}$ to obtain the second line.

5.5 Determinants using the Levi-Civita symbol

The result for the scalar triple product gives another expression for the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \big(\underline{a}, \underline{b}, \underline{c}\big) = \epsilon_{ijk}\,a_i\,b_j\,c_k\,. \tag{20}$$

Consider the 3×3 matrix A, with elements a_{ij}

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Relabelling the rows in the matrix in equation (20): $a_i \to a_{1i},\; b_i \to a_{2i},\; c_i \to a_{3i}$ gives

$$\boxed{\det A = \epsilon_{ijk}\,a_{1i}\,a_{2j}\,a_{3k}}$$

which may be taken as the *definition* of a determinant.

An alternative expression is given by noting that previously we showed that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \qquad \text{or} \qquad \det A = \det A^T$$

where $A^T_{ij} = a_{ji}$ so now $\det A = \det A^T = \epsilon_{ijk} A^T_{1i}\,A^T_{2j}\,A^T_{3k}$ which may be rewritten

$$\boxed{\det A = \epsilon_{ijk}\, a_{i1}\, a_{j2}\, a_{k3}}$$

The other properties of determinants can be proved easily: namely interchanging two rows or columns changes the sign of the determinant and adding a multiple of one row/column to another row/column respectively does not change the value of the determinant (exercises for the student).

For completeness, we quote here one further important result

$$\boxed{\det AB = \det A\ \det B}$$

[The definition of matix multiplication is given in Section (6.3).] The proof of this result is discussed in an example sheet.

6 Change of Basis

6.1 Linear Transformation of Basis

Suppose $\{\underline{e}_i\}$ and $\{\underline{e}_i'\}$ are two different orthonormal bases. How do we relate them?

Clearly $\underline{e}_1'$ can be written as a linear combination of the vectors $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$. Let us write the linear combination as

$$\underline{e}_1' \;=\; \ell_{11}\,\underline{e}_1 + \ell_{12}\,\underline{e}_2 + \ell_{13}\,\underline{e}_3$$

with similar expressions for $\underline{e}_2'$ and $\underline{e}_3'$. Hence we may write

$$\boxed{\underline{e}_i' \;=\; \ell_{ij}\,\underline{e}_j} \tag{21}$$

where we are using the summation convention. The nine numbers ℓ_{ij}, with i, j = 1, 2, 3, relate the basis vectors $\underline{e}_1'$, $\underline{e}_2'$, $\underline{e}_3'$ to the basis vectors $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$.

Notes

(i) The nine numbers ℓ_{ij} define the change of basis or 'linear transformation'.

(ii) To determine ℓ_{ij}, consider the quantity

$$\underline{e}_i' \cdot \underline{e}_j \;=\; (\ell_{ik}\,\underline{e}_k)\cdot \underline{e}_j = \ell_{ik}\,\delta_{kj} = \ell_{ij}\,.$$

Therefore

$$\boxed{\underline{e}_i' \cdot \underline{e}_j = \ell_{ij}} \tag{22}$$

so ℓ_{ij} are the projections (or direction cosines) of $\underline{e}_i'$ (i = 1, 2, 3) onto the $\underline{e}_i$ basis.

(iii) The basis vectors $\underline{e}_i'$ are orthonormal, therefore

$$\underline{e}_i' \cdot \underline{e}_j' \;=\; \delta_{ij}$$

The LHS of this equation may be written as

$$\underline{e}_i' \cdot \underline{e}_j' = (\ell_{ik}\,\underline{e}_k) \cdot (\ell_{jl}\,\underline{e}_l) = \ell_{ik}\,\ell_{jl}\,(\underline{e}_k \cdot \underline{e}_l) = \ell_{ik}\,\ell_{jl}\,\delta_{kl} = \ell_{ik}\,\ell_{jk}$$

where we used the sifting property of δ_{kl} in the final step. Hence

$$\boxed{\ell_{ik}\ell_{jk} \;=\; \delta_{ij}} \tag{23}$$

6.2 Inverse Relations

Let us now express the unprimed basis in terms of the primed basis. If we write

$$\underline{e}_i = m_{ij}\,\underline{e}_j'$$

then

$$\ell_{ij} = \underline{e}_i' \cdot \underline{e}_j = \underline{e}_i' \cdot \left(m_{jk}\,\underline{e}_k'\right) = m_{jk}\,\delta_{ik} = m_{ji}$$

and we deduce that

$$m_{ij} = \ell_{ji} \tag{24}$$

The $\underline{e}_i$ are orthonormal so $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$. The LHS of this equation may be re-written

$$\underline{e}_i \cdot \underline{e}_j = \left(m_{ik}\,\underline{e}_k'\right) \cdot \left(m_{jl}\,\underline{e}_l'\right) = m_{ik}\,m_{jl}\,\delta_{kl} = m_{ik}\,m_{jk} = \ell_{ki}\,\ell_{kj}$$

and we obtain a second relation

$$\boxed{\ell_{ki}\ell_{kj} \;=\; \delta_{ij}} \tag{25}$$

6.3 The Transformation Matrix

Let us re-write the above results using a matrix notation.

First note that the summation convention can be used to describe matrix multiplication. The ij^{th} component of the product of two 3×3 matrices A and B is obtained by 'multiplying the i^{th} row of A into the j^{th} column of B', namely

$$(AB)_{ij} = a_{i1}\,b_{1j} + a_{i2}\,b_{2j} + a_{i3}\,b_{3j} = a_{ik}\,b_{kj} \tag{26}$$

Likewise, recalling the definition of the transpose of a matrix $(A^T)_{ij} = A_{ji}$ (or a_{ji}),

$$(A^T B)_{ij} = (A^T)_{ik}\,(B)_{kj} = a_{ki}b_{kj} \tag{27}$$

We may identify the nine numbers ℓ_{ij} as the elements of a square matrix, denoted by L, and known as the *transformation matrix*

$$L \;=\; \begin{pmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix}$$

Equation (24) then tells us that $M = L^T$ is the transformation matrix for the *inverse* transformation.

Comparing equation (23) with equation (26), and equation (25) with equation (27), and recalling that δ_{ij} is the ij^{th} element of the *unit matrix* I, we see that the relations $\ell_{ik}\,\ell_{jk} = \ell_{ki}\,\ell_{kj} = \delta_{ij}$ can be written in matrix notation as

$$\boxed{LL^T \;=\; L^T L \;=\; I} \qquad \text{and hence} \qquad \boxed{L^{-1} \;=\; L^T}$$

where L^{-1} is the matrix inverse of L.

A matrix that satisfies these conditions is called an *orthogonal matrix*, and the transformation (from the $\underline{e}_i$ basis to the $\underline{e}_i'$ basis) is called an *orthogonal transformation*.

Now from $\underline{e}_i' = \ell_{ij}\,\underline{e}_j$, we have for the scalar triple product (assuming $\underline{e}_i$ is a RH basis)

$$\begin{aligned}
\left(\underline{e}_1', \underline{e}_2', \underline{e}_3'\right) &= \underline{e}_1' \cdot \left(\underline{e}_2' \times \underline{e}_3'\right) \\
&= \ell_{1i}\,\underline{e}_i \cdot (\ell_{2j}\,\underline{e}_j \times \ell_{3k}\,\underline{e}_k) \\
&= \ell_{1i}\,\ell_{2j}\,\ell_{3k}\;\underline{e}_i \cdot (\underline{e}_j \times \underline{e}_k) \\
&= \ell_{1i}\,\ell_{2j}\,\ell_{3k}\;\underline{e}_i \cdot (\epsilon_{ljk}\,\underline{e}_\ell) \\
&= \ell_{1i}\,\ell_{2j}\,\ell_{3k}\,\epsilon_{ljk}\,\delta_{il} \\
&= \epsilon_{ijk}\,\ell_{1i}\,\ell_{2j}\,\ell_{3k} \;=\; \det L
\end{aligned}$$

So

$$\det L \;=\; \left(\underline{e}_1', \underline{e}_2', \underline{e}_3'\right) \;=\; \begin{cases} +1 & \text{if primed basis is RH} \\ -1 & \text{if primed basis is LH} \end{cases}$$

We say

If $\det L = +1$ the orthogonal transformation is 'proper'
If $\det L = -1$ the orthogonal transformation is 'improper'

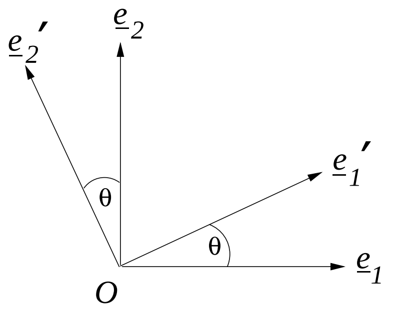
An alternative proof uses the following properties of determinants: $\det AB = \det A \det B$ and $\det A^T = \det A$. These, together with $\det I = 1$, give

$$\det LL^T = \det L\,\det L^T = (\det L)^2 = 1\,,$$

hence $\det L = \pm 1$.

6.4 Examples of Orthogonal Transformations

Rotation about the $\underline{e}_3$ axis: We have $\underline{e}_3{}' = \underline{e}_3$ and thus for a rotation through θ,



$$
\begin{aligned}
\underline{e}_3{}' \cdot \underline{e}_1 &= \underline{e}_1{}' \cdot \underline{e}_3 = \underline{e}_3{}' \cdot \underline{e}_2 = \underline{e}_2{}' \cdot \underline{e}_3 = 0\,, \quad \underline{e}_3{}' \cdot \underline{e}_3 = 1 \\
\underline{e}_1{}' \cdot \underline{e}_1 &= \cos\theta \\
\underline{e}_1{}' \cdot \underline{e}_2 &= \cos(\pi/2 - \theta) \;=\; \sin\theta \\
\underline{e}_2{}' \cdot \underline{e}_2 &= \cos\theta \\
\underline{e}_2{}' \cdot \underline{e}_1 &= \cos(\pi/2 + \theta) \;=\; -\sin\theta
\end{aligned}
$$

Thus

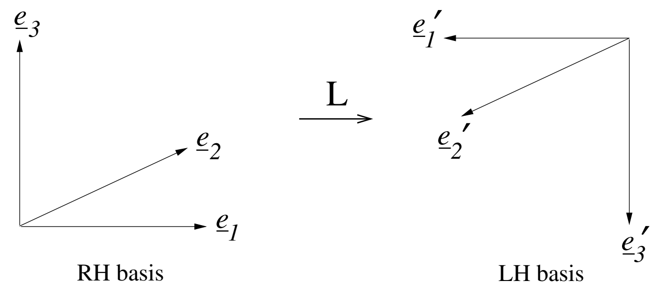
$$
L \;=\; \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

It is easy to check that $LL^T = I$. Since $\det L = \cos^2\theta + \sin^2\theta = 1$, this is a *proper* transformation. Note that rotations cannot change the handedness of the basis vectors.

Inversion or Parity transformation: This is defined by $\underline{e}_i' = -\underline{e}_i$, $i = 1, 2, 3$.

$$
i.e. \quad \ell_{ij} \;=\; -\delta_{ij} \qquad \text{or} \qquad L \;=\; \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \;=\; -I\,.
$$

Clearly $LL^T = I$. Since $\det L = -1$, this is an *improper* transformation. Note that the handedness of the basis is reversed: $\underline{e}_1{}' \times \underline{e}_2{}' = -\underline{e}_3{}'$



Reflection: Consider reflection of the axes in $\underline{e}_2$–$\underline{e}_3$ plane so that $\underline{e}_1{}' = -\underline{e}_1$, $\underline{e}_2{}' = \underline{e}_2$ and $\underline{e}_3{}' = \underline{e}_3$. The transformation matrix is

$$
L \;=\; \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

Since $\det L = -1$, this is an *improper* transformation, therefore the handedness of the basis changes.

6.5 Products of Transformations

Consider a transformation L to the basis $\{\underline{e}_i'\}$ followed by a transformation M to another basis $\{\underline{e}_i''\}$

$$\underline{e}_i \overset{L}{\to} \underline{e}_i' \overset{M}{\to} \underline{e}_i''$$

Clearly there must be an orthogonal transformation $\underline{e}_i \overset{N}{\to} \underline{e}_i''$. To find it, we write

$$\underline{e}_i'' = m_{ij}\, \underline{e}_j' = m_{ij}\, \ell_{jk}\, \underline{e}_k = (ML)_{ik}\, \underline{e}_k \quad \text{so} \quad \boxed{N = ML}$$

Notes

(i) Note the order of the product: the matrix corresponding to the first change of basis stands to the right of that for the second change of basis. In general, transformations do not commute, *i.e.* $ML \neq LM$.

Example: a rotation of θ about $\underline{e}_3$ followed by a reflection in the $\underline{e}_2$–$\underline{e}_3$ plane.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

whereas if we reverse the order

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(ii) The inversion and the identity transformations commute with all transformations.

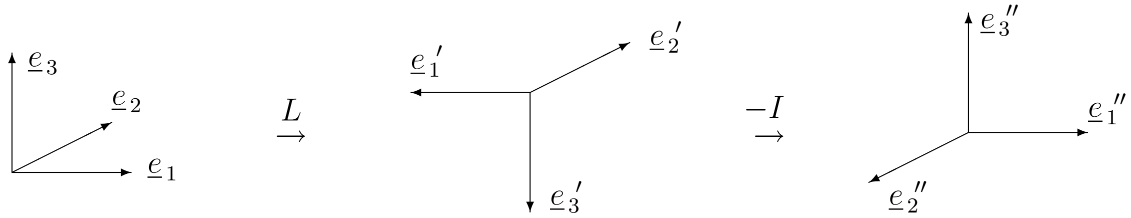
6.6 Improper Transformations

We may write any improper transformation M (for which $\det M = -1$) as $M = (-I)L$, where $L = -M$ and $\det L = +1$. Thus an improper transformation can always be expressed as a proper transformation followed by an inversion.

Example: The matrix M for a reflection in the $\underline{e}_1$–$\underline{e}_3$ plane is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Identifying L from $M = (-I)\,L$ we see that L is a rotation of π about $\underline{e}_2$.



6.7 Summary

If $\det L = +1$ we have a *proper* orthogonal transformation which is equivalent to rotation of axes. It can be proven that any rotation is a proper orthogonal transformation and vice-versa. The essence of the proof is that any rotation through a finite angle θ can be *continuously* connected to an infinitesimal or zero rotation for which $\det L = \det I = 1$ trivially, whereas $\det L = 1 \mapsto \det L = -1$ is discontinuous.

If $\det L = -1$ we have an *improper* orthogonal transformation which is equivalent to rotation of axes then inversion. This is known as an improper rotation since it *changes the handedness of the basis.*

7 Transformation Properties of Vectors and Scalars

7.1 Transformation of vector components

Let $\underline{a}$ be any vector, with components a_i in the basis $\{\underline{e}_i\}$ and a'_i in the basis $\{\underline{e}'_i\}$ *i.e.*

$$\boxed{\underline{a} = a_i\,\underline{e}_i = a'_i\,\underline{e}'_i} \,.$$

The components are related as follows, taking care with dummy indices

$$a'_i = \underline{a}\cdot\underline{e}'_i = (a_j\,\underline{e}_j)\cdot\underline{e}'_i = (\underline{e}'_i\cdot\underline{e}_j)\,a_j = \ell_{ij}\,a_j$$

$$\boxed{a'_i = \ell_{ij}\,a_j}$$

$$a_i = \underline{a}\cdot\underline{e}_i = (a'_k\,\underline{e}'_k)\cdot\underline{e}_i = \ell_{ki}\,a'_k = (L^T)_{ik}\,a'_k.$$

Note carefully that the **vector** $\underline{a}$ does *not* change

Therefore we do *not* put a prime on the vector itself. However, the *components* of this vector *are* different in different bases, and so are denoted by a_i in the basis $\{\underline{e}_i\}$, and by a'_i in the basis $\{\underline{e}'_i\}$, and so on.

These transformations are called *passive transformations*: the basis is transformed, but the vector remains *fixed*. Alternatively we can keep the basis fixed and transform the vector, this is an *active transformation*. They are equivalent (and indeed one is just the inverse of the other). In this course we shall only consider the passive viewpoint (to avoid confusion).

In matrix form we can write the transformation of components as

$$\boxed{\begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix} = \begin{pmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = L \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}}$$

and since $L^{-1} = L^T$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = L^T \begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix}$$

Example: Consider a rotation of the axes about $\underline{e}_3$

$$\begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \cos\theta\, a_1 + \sin\theta\, a_2 \\ \cos\theta\, a_2 - \sin\theta\, a_1 \\ a_3 \end{pmatrix}$$

A direct check of this using trigonometric considerations is significantly harder!

7.2 Transformation of the scalar product

Let $\underline{a}$ and $\underline{b}$ be vectors with components a_i and b_i in the $\{\underline{e}_i\}$ basis, and components a'_i and b'_i in the $\{\underline{e}'_i\}$ basis. In the $\{\underline{e}_i\}$ basis, the scalar product, denoted by $\underline{a} \cdot \underline{b}$, is

$$\underline{a} \cdot \underline{b} = a_i\, b_i\,.$$

In the basis $\{\underline{e}'_i\}$, we denote the scalar product by $(\underline{a} \cdot \underline{b})'$, and we have

$$\begin{aligned} (\underline{a} \cdot \underline{b})' &= a'_i\, b'_i \;=\; \ell_{ij}\, a_j\, \ell_{ik}\, b_k \;=\; \delta_{jk} a_j b_k \\ &= a_j\, b_j \;=\; \underline{a} \cdot \underline{b}\,. \end{aligned}$$

Thus the scalar product is the same when evaluated in any basis. This is of course expected from the geometrical definition of scalar product which is independent of basis. We say that the scalar product is *invariant* under a change of basis.

Summary: We have now obtained an algebraic definition of scalar and vector quantities.

Under the orthogonal transformation from the basis $\{\underline{e}_i\}$ to the basis $\{\underline{e}'_i\}$, defined by the transformation matrix L such that $\underline{e}'_i = \ell_{ij}\, \underline{e}_j$, we have that:

- A **scalar** is a single number ϕ which is invariant:

  $$\boxed{\phi' = \phi}$$

  Of course, not all scalar quantities in physics are expressible as the scalar product of two vectors *e.g.* mass, temperature.

- A **vector** is an 'ordered triple' of numbers a_i which transforms to a'_i such that

  $$\boxed{a'_i = \ell_{ij} a_j}$$

7.3 Transformation of the vector product

Great care is needed with the vector product under improper transformations.

Inversion: Let $\underline{e}_i' = -\underline{e}_i$, so $\ell_{ij} = -\delta_{ij}$ and hence $a_i' = -a_i$ and $b_i' = -b_i$. Therefore

$$a_i'\,\underline{e}_i' \;=\; (-a_i)\,(-\underline{e}_i) \;=\; a_i\,\underline{e}_i \;=\; \underline{a} \quad \text{and} \quad b_i'\,\underline{e}_i' \;=\; (-b_i)\,(-\underline{e}_i) \;=\; b_i\,\underline{e}_i \;=\; \underline{b}$$

The vectors $\underline{a}$ and $\underline{b}$ are unchanged by the transformation – as they should be.

However if we calculate the vector product $\underline{c} = \underline{a} \times \underline{b}$ in the new basis using the determinant formula, we obtain

$$\begin{vmatrix} \underline{e}_1' & \underline{e}_2' & \underline{e}_3' \\ a_1' & a_2' & a_3' \\ b_1' & b_2' & b_3' \end{vmatrix} \;=\; (-1)^3 \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \;=\; - \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

which is $-\underline{c}$ as calculated in the original basis!

The explanation is that if $\{\underline{e}_i\}$ is a *RH basis*, then $\{\underline{e}_i'\}$ is a *LH basis* because L is an *improper* transformation. The formula we used for the vector product holds in a right-handed basis. If we use this formula in a left-handed basis, the direction of the vector product is reversed (it is equivalent to using a left-hand rule rather than a right-hand rule to calculate the vector product).

Let us then *define* the components of $\underline{c} = \underline{a} \times \underline{b}$ as

$$\boxed{c_i \equiv (\underline{a} \times \underline{b})_i = \epsilon_{ijk} a_j b_k}$$

in *any* orthonormal basis (LH or RH). This is equivalent to using the determinant formula. With this definition $\underline{a}$, $\underline{b}$ and $\underline{c}$ have the *same* 'handedness' as the underlying basis.

General case: To derive the transformation law for the vector product for arbitrary L requires several steps, and is not quite trivial.

In section (5.5), we showed that the determinant of a 3×3 matrix A can be written as

$$\det A = \epsilon_{ijk}\, a_{i1}\, a_{j2}\, a_{k3}$$

This can be generalised to [see tutorial question (4.5)]

$$\boxed{\epsilon_{rst}\ \det A = \epsilon_{ijk}\, a_{ir}\, a_{js}\, a_{kt}}$$

The extra ϵ on the LHS of this equation tells us that the determinant changes sign when we swap two columns of the matrix. Applying this to the transformation matrix L gives

$$\epsilon_{rst}\ \det L = \epsilon_{ijk}\, \ell_{ir}\, \ell_{js}\, \ell_{kt}$$

Multiplying this equation by ℓ_{lt} and using the orthogonality relation $\ell_{lt}\, \ell_{kt} = \delta_{lk}$, gives

$$(\det L)\, \epsilon_{rst}\, \ell_{lt} = \epsilon_{ijl}\, \ell_{ir}\, \ell_{js}.$$

Relabelling the free index $l \mapsto k$ gives

$$\boxed{(\det L)\,\epsilon_{rst}\,\ell_{kt} = \epsilon_{ijk}\,\ell_{ir}\,\ell_{js}}$$

We can now calculate the transformation law for the components of the vector product. Recalling that $a'_j = \ell_{jr}\,a_r$ and $b'_k = \ell_{ks}\,b_s$, we find

$$\begin{aligned}(\underline{a}\times\underline{b})'_i &= \epsilon_{ijk}\,a'_j\,b'_k \;=\; \epsilon_{ijk}\,\ell_{jr}\,\ell_{ks}\,a_r\,b_s \;=\; \epsilon_{jki}\,\ell_{jr}\,\ell_{ks}\,a_r\,b_s \\ &= (\det L)\,\epsilon_{rst}\,\ell_{it}\,a_r\,b_s \;=\; (\det L)\,\ell_{it}\,(\epsilon_{trs}\,a_r\,b_s)\end{aligned}$$

where we used the last boxed identity (relabelling a lot of indices - exercise!) to get the first expression in the second line. Finally, we have

$$\boxed{(\underline{a}\times\underline{b})'_i \;=\; (\det L)\,\ell_{it}\,(\underline{a}\times\underline{b})_t}$$

So the vector product transforms just like a vector under proper transformations, for which $\det L = +1$, but it picks up an extra minus sign under improper transformations, for which $\det L = -1$.

The vector product is an example of what is known as a *pseudovector* or *axial vector*.

In general, a pseudovector $\underline{c}$ is defined by the transformation law

$$\boxed{c'_i = (\det L)\,\ell_{ij}\,c_j}$$

You should know this result, but the detailed derivation is a bit tough, so you wouldn't be expected to reproduce it in an examination.

Physical Examples

The following are *true* or *polar* vectors:

Position	$\underline{r}$		
Velocity	$\underline{v}$	$=$	$\dot{\underline{r}}$ where $\underline{r} = \underline{r}(t)$, and $\dot{\underline{r}} \equiv \dfrac{d\underline{r}}{dt}$ (t is a scalar)
Acceleration	$\underline{a}$	$=$	$\dot{\underline{v}}$
Force	$\underline{F}$	$=$	$m\,\underline{a}$ (defined by Newton's law)
Electric field	$\underline{E}$	$=$	$\dfrac{1}{q}\underline{F}$ (where $\underline{F}$ is the force on a particle of charge q)

The following are *pseudo* or *axial* vectors:

Angular momentum	$\underline{L}$	$=$	$\underline{r}\times m\underline{v}$
Torque	$\underline{T}$	$=$	$\underline{r}\times\underline{F}$
Angular velocity ($\underline{\omega}$)	$\underline{v}$	$=$	$\underline{\omega}\times\underline{r}$
Magnetic field ($\underline{B}$)	$\underline{F}$	$=$	$q\,\underline{v}\times\underline{B}$ (where $\underline{F}$ is the force on a particle of charge q and velocity $\underline{v}$ due to $\underline{B}$)

7.4 Summary of the story so far

We now take the opportunity to summarise some key-points of the course thus far.

NB this is NOT a list of everything you need to know!

Key points from the geometrical approach

You should recognise on sight that

$$\begin{aligned} \underline{r} \times \underline{u} = \underline{c} \quad & \text{is a line} \quad (\underline{r} \text{ lies on a line}) \\ \underline{r} \cdot \underline{n} = p \quad & \text{is a plane } (\underline{r} \text{ lies in a plane}) \end{aligned}$$

Useful properties of scalar and vector products to remember

$$\begin{aligned} \underline{a} \cdot \underline{b} = 0 \quad & \Leftrightarrow \quad \text{vectors orthogonal} \\ \underline{a} \times \underline{b} = 0 \quad & \Leftrightarrow \quad \text{vectors collinear} \\ \underline{a} \cdot (\underline{b} \times \underline{c}) = 0 \quad & \Leftrightarrow \quad \text{vectors co-planar or linearly dependent} \\ \underline{a} \times (\underline{b} \times \underline{c}) \quad & = \quad (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c} \end{aligned}$$

Key points of suffix notation and the summation convention

We label orthonormal basis vectors by $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ (or just $\{\underline{e}_i\}$), and write the expansion of a vector $\underline{a}$ as

$$\underline{a} = a_i \, \underline{e}_i \qquad \left(\equiv \sum_{i=1}^{3} a_i \, \underline{e}_i \right)$$

There is *always* an implicit sum over any *repeated* or *dummy* index, i in this case.

The Kronecker delta symbol δ_{ij} can be used to express the orthonormality of the basis

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$$

Kronecker delta has a very useful sifting property

$$[\cdots]_j \delta_{jk} = [\cdots]_k$$

Whether the basis is right- or left-handed is determined by

$$(\underline{e}_1, \underline{e}_2, \underline{e}_3) = \pm 1$$

We introduce ϵ_{ijk} to enable us to write the vector products of basis vectors in a RH basis in a uniform way

$$\underline{e}_i \times \underline{e}_j = \epsilon_{ijk} \, \underline{e}_k \, .$$

The vector and scalar triple products in any orthonormal basis are

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{or equivalently} \quad (\underline{a} \times \underline{b})_i = \epsilon_{ijk} a_j b_k$$

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{or equivalently} \quad \underline{a} \cdot (\underline{b} \times \underline{c}) = \epsilon_{ijk} a_i b_j c_k$$

The *most important* identity in the game is

$$\epsilon_{ijk}\,\epsilon_{klm} \;=\; \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

Key points of the algebraic approach to change of basis

The new basis is written in terms of the old through

$$\underline{e}_i' \;=\; \ell_{ij}\,\underline{e}_j \quad \text{where } \ell_{ij} \text{ are elements of the } 3\times 3 \text{ transformation matrix } L$$

L is an orthogonal matrix, the defining property of which is $L^{-1} = L^T$, and this can be written as

$$LL^T = L^T L = I \qquad \text{or} \qquad \ell_{ik}\ell_{jk} = \ell_{ki}\ell_{kj} = \delta_{ij}$$

The determinant $\det L = \pm 1$ tells us whether the transformation is proper or improper, *i.e.* whether the handedness of the basis is changed.

A *scalar* is *defined* as a number that is invariant under an orthogonal transformation.

A *vector* is *defined* as an object $\underline{a}$ represented in a basis by three numbers a_i which transform to a_i' through

$$a_i' \;= \ell_{ij} a_j.$$

or in matrix form

$$\begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix} \;=\; L \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Regarding a and a' as 3×1 column matrices, this may be written succinctly as

$$a' = L\,a\,.$$

8 Tensors of Second Rank

8.1 Nature of Physical Laws

The simplest physical laws are expressed in terms of scalar quantities which are independent of our choice of basis, *e.g.* the ideal-gas law

$$pV = nRT$$

relating pressure, volume and temperature.

At the next level of complexity are laws relating vector quantities

$$\begin{aligned} \underline{F} &= m\,\underline{a} && \text{Newton's Law} \\ \underline{J} &= \sigma\,\underline{E} && \text{Ohm's Law, } \underline{J} \text{ is the current density vector,} \\ & && \sigma \text{ is conductivity } (\sigma \propto 1/R) \end{aligned}$$

Notes

(i) These laws take the form *vector* = *scalar* × *vector*;

(ii) They relate two vectors in the *same* direction.

Example: Consider Newton's Law in a particular Cartesian basis $\{\underline{e}_i\}$. The acceleration vector $\underline{a}$ is represented by its components $\{a_i\}$, and the force $\underline{F}$ by its components $\{F_i\}$, so we write

$$F_i = m\,a_i$$

In another basis $\{\underline{e}_i'\}$ defined by $\underline{e}_i' = \ell_{ij}\,\underline{e}_j$ we have

$$F_i' = m\,a_i'$$

where the set of numbers, $\{a_i'\}$, is in general different from the set $\{a_i\}$. Likewise, the set $\{F_i'\}$ differs from the set $\{F_i\}$, but they are of course related by

$$a_i' = \ell_{ij}\,a_j \quad \text{and} \quad F_i' = \ell_{ij}\,F_j\,.$$

We can think of $\underline{F} = m\underline{a}$ as representing an infinite set of relations between measured components in various bases. Because all vectors transform in the same way under orthogonal transformations, the relations have the *same form* in all bases. We say that Newton's Law, expressed in component form, is *form invariant* or *covariant*.

8.2 Examples of more complicated laws

Ohm's law in an anisotropic medium

The simple form of Ohm's Law stated above, in which an applied electric field $\underline{E}$ produces a current in the same direction, only holds for conducting media which are isotropic, that is, the same in all directions. This is certainly not the case in crystalline media, where the regular lattice will favour conduction in some directions more than in others.

The most general relation between $\underline{J}$ and $\underline{E}$ which is linear and is such that $\underline{J}$ vanishes when $\underline{E}$ vanishes is of the form

$$J_i = \sigma_{ij} E_j$$

where σ_{ij} are the components of the *conductivity tensor* in the chosen basis; they characterise the conduction properties when $\underline{J}$ and $\underline{E}$ are measured in that basis. Thus we need nine numbers, σ_{ij}, to characterise the conductivity of an anisotropic medium. The conductivity tensor is an example of a *second rank tensor*.

Now consider an orthogonal transformation of basis. Simply changing basis cannot alter the form of the physical law and so we conclude that

$$J'_i = \sigma'_{ij} E'_j \quad \text{where} \quad J'_i = \ell_{ij} J_j \quad \text{and} \quad E'_j = \ell_{jk} E_k$$

Thus we deduce that

$$\ell_{ij}\, J_j \;=\; \ell_{ij}\, \sigma_{jk}\, E_k \;=\; \sigma'_{ij}\, \ell_{jk}\, E_k$$

which we can rewrite as

$$(\sigma'_{ij}\, \ell_{jk} - \ell_{ij}\, \sigma_{jk})\, E_k = 0$$

This must be true for arbitrary electric fields E_k and hence

$$\sigma'_{ij}\, \ell_{jk} \;=\; \ell_{ij}\, \sigma_{jk}$$

Multiplying both sides by ℓ_{lk}, noting that $\ell_{lk}\ell_{jk} = \delta_{lj}$ and using the sifting property we find that $\sigma'_{il} = \ell_{ij}\, \ell_{lk}\, \sigma_{jk}$ or re-labelling

$$\boxed{\sigma'_{ij} \;=\; \ell_{ip}\, \ell_{jq}\, \sigma_{pq}}$$

This exemplifies how the components of a *second rank tensor* change or *transform* under an orthogonal transformation, and indeed will be taken as our *definition* of a second rank tensor.

This discussion of the covariance of Ohm's law is an example of the very general *quotient theorem* (see Junior Honours *Tensors and Fields*.)

Kronecker delta as a tensor

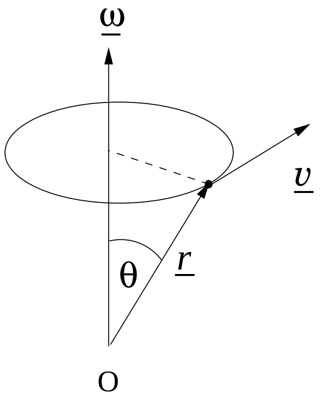
Since Kronecker delta has two indices it is natural to ask whether it is a second rank tensor.

We defined δ_{ij} to be 1 if $i = j$, and 0 otherwise, in *all* bases, and thus we have that $\delta'_{ij} = \delta_{ij}$. Recalling that $\delta_{ij} = \ell_{ip}\ell_{jp}$, we may write

$$\delta'_{ij} = \ell_{ip}\, \ell_{jq}\, \delta_{pq}$$

Since the nine-numbers δ_{ij} transform as the components a second rank tensor, by our definition above it is indeed a second-rank tensor. Since $\delta'_{ij} = \delta_{ij}$, we call it an *invariant tensor*.

Rotating rigid body



Consider a particle of mass m at a point $\underline{r}$ in a rigid body rotating with *angular velocity* $\underline{\omega}$. Recall that $\underline{v} = \underline{\omega} \times \underline{r}$. You were asked in Lecture 1 to check that this gives the right direction for $\underline{v}$; that it is perpendicular to the plane of $\underline{\omega}$ and $\underline{r}$; that the magnitude $|\underline{v}| = \omega r \sin\theta = \omega \times$ (radius of the circle in which the point is travelling.)

Now consider the *angular momentum* of the particle about the origin O; this is defined by $\underline{L} = \underline{r} \times \underline{p} = \underline{r} \times (m\underline{v})$ where m is the mass of the particle.

Using the above expression for $\underline{v}$ we obtain

$$\underline{L} = m\underline{r} \times (\underline{\omega} \times \underline{r}) = m\left[\underline{\omega}(\underline{r}\cdot\underline{r}) - \underline{r}(\underline{r}\cdot\underline{\omega})\right] \tag{28}$$

where we have used the identity for the vector triple product. Note that only if $\underline{r}$ is perpendicular to $\underline{\omega}$ do we obtain $\underline{L} = mr^2\underline{\omega}$, which means that only then are $\underline{L}$ and $\underline{\omega}$ in the same direction.

Taking components of equation (28) in an orthonormal basis $\{\underline{e}_i\}$, we find that

$$\begin{aligned} L_i &= m\left[\omega_i\,(\underline{r}\cdot\underline{r}) - x_i\,(\underline{r}\cdot\underline{\omega})\right] \\ &= m\left[r^2\,\omega_i - x_i\,x_j\,\omega_j\right] \qquad \text{noting that} \quad \underline{r}\cdot\underline{\omega} = x_j\,\omega_j \\ &= m\left[r^2\,\delta_{ij} - x_i\,x_j\right]\omega_j \qquad \text{using} \quad \omega_i = \delta_{ij}\,\omega_j \end{aligned}$$

Thus

$$\boxed{L_i = I_{ij}(O)\,\omega_j \qquad \text{where} \qquad I_{ij}(O) = m\,[r^2\,\delta_{ij} - x_i\,x_j]}$$

By the quotient theorem $I_{ij}(O)$ are the components of the *(moment of) inertia tensor*, relative to the origin, O, in the $\underline{e}_i$ basis. The inertia tensor is another example of a second rank tensor. This may also be shown directly

$$\begin{aligned} I'_{ij}(O) &= m[r'^2\,\delta'_{ij} - x'_i\,x'_j] \\ &= m[r^2\,\ell_{ip}\,\ell_{jq}\,\delta_{pq} - \ell_{ip}\,x_p\,\ell_{jq}\,x_q] \\ &= \ell_{ip}\,\ell_{jq}\,I_{pq}(O) \end{aligned}$$

i.e. I transforms as a second rank tensor.

Summary of why we need tensors

(i) Physical laws often relate two vectors.

(ii) A second rank tensor provides a linear relation between two vectors which may be in different directions.

(iii) Tensors allow the generalisation of isotropic laws ('physics is the same in all directions') to anisotropic laws ('physics is different in different directions')

In general, a second rank tensor maps a given vector onto a vector in a different direction. If a vector $\underline{n}$ has components n_i then

$$T_{ij} n_j \;=\; m_i \;,$$

where m_i are components of $\underline{m}$, the vector that $\underline{n}$ is mapped onto.

However, some special vectors called *eigenvectors* may exist such that $m_i = \lambda\, n_i$, *i.e.* the new vector is in the *same* direction as the original vector. Eigenvectors usually have special physical significance (see later).

8.3 General properties

Scalars and vectors are called tensors of rank zero and rank one respectively, where *rank = number of indices in a Cartesian basis.* Thus we have

$$\begin{aligned} \phi' &= \phi && \text{scalar} \\ a'_i &= \ell_{ip}\, a_p && \text{vector} \\ T'_{ij} &= \ell_{ip}\ell_{jq}\, T_{pq} && \text{rank-two tensor} \end{aligned}$$

We can also have pseudoscalars, pseudovectors and pseudotensors,

$$\begin{aligned} \phi' &= (\det L)\, \phi && \text{pseudoscalar} \\ a'_i &= (\det L)\, \ell_{ip} a_p && \text{pseudovector} \\ T'_{ij} &= (\det L)\, \ell_{ip}\ell_{jq} T_{pq} && \text{rank-two pseudotensor} \end{aligned}$$

Let $\underline{a}$, $\underline{b}$ and $\underline{c}$ be (true) vectors. An example of a pseudovector is $\underline{a} \times \underline{b}$ (see section 7.3), and an example of a pseudoscalar is the scalar triple product:

$$\begin{aligned} (\underline{a}, \underline{b}, \underline{c})' &= a'_i\, (\underline{b} \times \underline{c})'_i \\ &= \ell_{ip}\, a_p\, (\det L)\, \ell_{iq}\, (\underline{b} \times \underline{c})_q \\ &= (\det L)\; a_p (\underline{b} \times \underline{c})_q\; \delta_{pq} \\ &= (\det L)\; \underline{a} \cdot (\underline{b} \times \underline{c}) \;=\; (\det L)\; (\underline{a}, \underline{b}, \underline{c}) \end{aligned}$$

Note that if $\underline{a}$ is a vector and $\underline{b}$ is a pseudovector, then $\underline{a} \times \underline{b}$ is a vector because $(\det L)^2 = 1$.

We can also define tensors of rank greater than two by introducing more indices, together with more ℓs. Indeed, ϵ is a pseudotensor of rank three.

Rank-two tensors have some additional special properties: the set of nine numbers, T_{ij}, representing the tensor T, can be written as a 3×3 matrix

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

This is of course *not* true for higher rank tensors (which have more than 9 components).

We can rewrite the transformation law for a general second rank tensor as

$$T'_{ij} \;=\; \ell_{ip}\,\ell_{jq}\,T_{pq} \;=\; (L)_{ip}\,T_{pq}\,(L^T)_{qj}$$

So the transformation law in matrix form is

$$\boxed{T' = LTL^T}$$

Notes

(i) It is *wrong* to say that a second rank tensor is a matrix; rather the tensor is the fundamental object and it is *represented in a given basis* by a matrix.

(ii) It is *wrong* to say a matrix is a tensor, *e.g.* the transformation matrix L is not a tensor but nine numbers defining the transformation between *two different bases.*

8.4 Invariants

Trace of a tensor: the trace of a matrix A (whose elements are a_{ij}) is defined as the sum of its diagonal elements

$$\mathrm{Tr}\, A = a_{ii}$$

Recalling that $(AB)_{ik} = a_{ij}\,b_{jk}$, we can derive the useful *cyclic* property of the trace

$$\mathrm{Tr}\,(AB) = a_{ij}b_{ji} = b_{ji}a_{ij} = \mathrm{Tr}\,(BA)$$

Now consider the trace of the the tensor in the transformed basis:

$$\mathrm{Tr}\,T' \;=\; T'_{ii} \;=\; \ell_{ip}\ell_{iq}T_{pq} \;=\; \delta_{pq}T_{pq} \;=\; T_{pp} \;=\; \mathrm{Tr}\,T$$

Thus evaluating the trace gives the same result in any basis, it is a scalar invariant. Alternatively using a matrix notation

$$\mathrm{Tr}\,T' = \mathrm{Tr}\,(LTL^T) = \mathrm{Tr}\,(TL^TL) = \mathrm{Tr}\,T$$

Determinant: It is easiest to use the matrix form of the transformation law

$$\det T' = \det(LTL^T) = \det L\,\det T\,\det L^T = \det T$$

(since $\det L = \det L^T = \pm 1$.) Thus the determinant of the tensor is invariant, it's a scalar.

Symmetry of a tensor: if the matrix T_{ij} representing the tensor T is *symmetric* then

$$T_{ij} = T_{ji}$$

Under a change of basis

$$\begin{aligned} T'_{ij} &= \ell_{ip}\,\ell_{jq}\,T_{pq} \\ &= \ell_{ip}\,\ell_{jq}\,T_{qp} \quad \text{(using symmetry)} \\ &= \ell_{iq}\,\ell_{jp}\,T_{pq} \quad \text{(relabelling } p \leftrightarrow q\text{)} \\ &= T'_{ji} \end{aligned}$$

Therefore a symmetric tensor remains symmetric under a change of basis. (The inertia tensor is an example of a symmetric tensor.) Similarly an antisymmetric tensor $T_{ij} = -T_{ji}$ remains antisymmetric (exercise).

In fact one can decompose an arbitrary second rank tensor T_{ij} into a symmetric part S_{ij} and an antisymmetric part A_{ij} through

$$\boxed{T_{ij} = \tfrac{1}{2}\left[T_{ij} + T_{ji}\right] + \tfrac{1}{2}\left[T_{ij} - T_{ji}\right] = S_{ij} + A_{ij}}$$

where $S_{ij} = \frac{1}{2}(T_{ij} + T_{ji})$ is symmetric, and $A_{ij} = \frac{1}{2}(T_{ij} - T_{ji})$ is antisymmetric.

Relabelling the components of A by $a_{12} = -a_{21} = f_3$, $a_{23} = -a_{32} = f_1$, $a_{31} = -a_{13} = f_2$ gives

$$A_{ij} = \begin{pmatrix} 0 & f_3 & -f_2 \\ -f_3 & 0 & f_1 \\ f_2 & -f_1 & 0 \end{pmatrix}_{ij} = \epsilon_{ijk} f_k \,,$$

This relation can be inverted

$$\epsilon_{ijk} A_{jk} \;=\; \epsilon_{ijk}\,\epsilon_{jkl} f_l \;=\; -\left(\delta_{ij}\,\delta_{jl} - \delta_{il}\,\delta_{jj}\right) f_l \;=\; 2 f_i \,.$$

i.e. the components A_{ij} of an antisymmetric tensor can always be written in terms of the components f_i of a pseudovector, and vice versa.

9 The Inertia Tensor

9.1 Kinetic energy and the Inertia Tensor

We saw previously that for a single particle of mass m, located at position $\underline{r}$ with respect to an origin O on the axis of rotation of a rigid body

$$\boxed{L_i \;=\; I_{ij}(O)\,\omega_j \qquad \text{where} \qquad I_{ij}(O) \;=\; m\,\{r^2\,\delta_{ij} - x_i\,x_j\}}$$

where $I_{ij}(O)$ are the components of the inertia tensor, relative to O, in the basis $\{\underline{e}_i\}$.

For a *collection of N particles* of mass $m^{(\alpha)}$ at $\underline{r}^{(\alpha)}$, where $\alpha = 1 \ldots N$,

$$\boxed{I_{ij}(O) \;=\; \textstyle\sum_{\alpha=1}^{N} m^{(\alpha)} \left\{ \left(\underline{r}^{(\alpha)} \cdot \underline{r}^{(\alpha)}\right) \delta_{ij} - x_i^{(\alpha)}\, x_j^{(\alpha)} \right\}} \qquad (29)$$

For a *continuous body*, the sums become integrals, giving

$$\boxed{I_{ij}(O) \;=\; \int_V \rho(\underline{r}) \left\{ \left(\underline{r} \cdot \underline{r}\right) \delta_{ij} - x_i\, x_j \right\} dV}$$

where $\rho(\underline{r})$ is the density (mass per unit volume) at position $\underline{r}$, and therefore $\rho(\underline{r})\,dV$ is the mass of the volume element dV at $\underline{r}$.

For laminae (flat objects) and solid bodies, these are 2- and 3-dimensional integrals respectively.

If the basis is *fixed relative to the body*, the $I_{ij}(O)$ are *constants* in time.

The inertia tensor is also useful in computing the kinetic energy K

$$\begin{aligned}
K &= \sum_{\alpha=1}^{N} \frac{1}{2} m^{(\alpha)}\, v^{(\alpha)2} \\
&= \frac{1}{2}\sum m^{(\alpha)} \left(\underline{\omega} \times \underline{r}^{(\alpha)}\right)^2 \\
&= \frac{1}{2}\sum m^{(\alpha)} \epsilon_{ijk}\, \omega_j\, x_k^{(\alpha)}\, \epsilon_{ilm}\, \omega_l\, x_m^{(\alpha)} \\
&= \frac{1}{2}\sum m^{(\alpha)} \left(\delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}\right) x_k^{(\alpha)} x_m^{(\alpha)}\, \omega_j\, \omega_l \\
&= \frac{1}{2}\sum m^{(\alpha)} \left(r^{(\alpha)2}\, \omega^2 - x_k^{(\alpha)} x_j^{(\alpha)} \omega_j\, \omega_k\right) \\
&= \frac{1}{2}\sum m^{(\alpha)} \left(r^{(\alpha)2}\, \delta_{ij} - x_i^{(\alpha)} x_j^{(\alpha)}\right) \omega_i\, \omega_j
\end{aligned}$$

Thus we have

$$\boxed{K = \frac{1}{2} I_{ij}(O)\, \omega_i\, \omega_j \equiv \frac{1}{2} \omega^T I(O)\, \omega \equiv \frac{1}{2} \underline{\omega} \cdot \underline{L}}$$

where $I(O)$ is a 3×3 matrix, and ω^T and ω are row and column matrices respectively.

It's sometimes sufficient to combine the diagonal and off-diagonal terms by writing $\underline{\omega} = \omega\, \underline{n}$ and then computing the component of $\underline{L}$ along the axis of rotation

$$\begin{aligned}
h \equiv \underline{L} \cdot \underline{n} = I_{ij}(O)\, \omega_j\, n_i &= \sum_\alpha m^{(\alpha)} \left\{ r^{(\alpha)2}\, \delta_{ij} - x_i^{(\alpha)}\, x_j^{(\alpha)} \right\} n_i\, n_j\, \omega \\
&= \sum_\alpha m^{(\alpha)} \left\{ r^{(\alpha)2} - (\underline{r}^{(\alpha)} \cdot \underline{n})^2 \right\} \omega
\end{aligned}$$

or

$$h = \tilde{I}\omega \quad \text{with} \quad \tilde{I} = \sum_\alpha m^{(\alpha)}\, r_\perp^{(\alpha)2}$$

where $r_\perp^{(\alpha)}$ is the perpendicular distance of the α^{th} particle from the axis of rotation and $\tilde{I}$ is the *moment of inertia about* $\underline{n}$. Similarly we have

$$K = \frac{1}{2} \tilde{I} \omega^2$$

Note that the inertia tensor $I_{ij}(O)$ is a 'geometric' factor and once calculated can be used for any axis of rotation; the quantity $\tilde{I}$ depends on the axis $\underline{n}$ and must be re-calculated if a different axis of rotation is chosen.

9.2 Computing the Inertia Tensor

Consider the *diagonal* term

$$\begin{aligned} I_{33}(O) &= \sum_\alpha m^{(\alpha)} \left\{ (\underline{r}^{(\alpha)} \cdot \underline{r}^{(\alpha)}) - (x_3^{(\alpha)})^2 \right\} \\ &= \sum_\alpha m^{(\alpha)} \left\{ (x_1^{(\alpha)})^2 + (x_2^{(\alpha)})^2 \right\} \\ &= \sum_\alpha m^{(\alpha)} \, (r_\perp^{(\alpha)})^2 \,, \end{aligned}$$

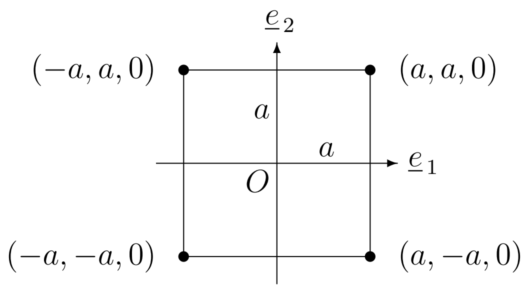
where $r_\perp^{(\alpha)}$ is the *perpendicular distance* of $m^{(\alpha)}$ from the $\underline{e}_3$ axis through O.

So $I_{33}(O)$ is the usual *moment of inertia* about the $\underline{n} = \underline{e}_3$ axis. It is simply the mass of each particle in the body, multiplied by the square of its distance from the $\underline{e}_3$ axis, summed over all of the particles. Similarly, the other diagonal terms I_{11} and I_{22} are the moments of inertia about the $\underline{e}_1$ and $\underline{e}_2$ axes respectively.

The *off-diagonal* terms are called the *products of inertia*, and have the form, for example

$$I_{12}(O) = -\sum_\alpha m^{(\alpha)} \, x_1^{(\alpha)} \, x_2^{(\alpha)} \,.$$

Example: Consider 4 masses m at the vertices of a square of side $2a$, with the origin O at the centre of the square.



For $m^{(1)} = m$ at $(a, a, 0)$, $\underline{r}^{(1)} = a\underline{e}_1 + a\underline{e}_2$, so $\underline{r}^{(1)} \cdot \underline{r}^{(1)} = 2a^2$, $x_1^{(1)} = a$, $x_2^{(1)} = a$ and $x_3^{(1)} = 0$

$$I(O) = m \left\{ 2a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - a^2 \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} = ma^2 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} .$$

For $m^{(2)} = m$ at $(a, -a, 0)$, $\underline{r}^{(2)} = a\underline{e}_1 - a\underline{e}_2$, so $\underline{r}^{(2)} \cdot \underline{r}^{(2)} = 2a^2$, $x_1^{(2)} = a$ and $x_2^{(2)} = -a$

$$I(O) = m \left\{ 2a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - a^2 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} = ma^2 \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} .$$

For $m^{(3)} = m$ at $(-a, -a, 0)$, $\underline{r}^{(3)} = -a\underline{e}_1 - a\underline{e}_2$, so $\underline{r}^{(3)} \cdot \underline{r}^{(3)} = 2a^2$, $x_1^{(3)} = -a$ and $x_2^{(3)} = -a$

$$I(O) = m \left\{ 2a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - a^2 \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} = ma^2 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} .$$

For $m^{(4)} = m$ at $(-a, a, 0)$, $\underline{r}^{(4)} = -a\underline{e}_1 + a\underline{e}_2$, so $\underline{r}^{(4)} \cdot \underline{r}^{(4)} = 2a^2$, $x_1^{(4)} = -a$ and $x_2^{(4)} = a$

$$I(O) \;=\; m\left\{2a^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - a^2\begin{pmatrix}1&-1&0\\-1&1&0\\0&0&0\end{pmatrix}\right\} \;=\; ma^2\begin{pmatrix}1&1&0\\1&1&0\\0&0&2\end{pmatrix}.$$

Adding all four contributions gives the inertia tensor for the system of 4 particles,

$$\boxed{I(O) \;=\; 4ma^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}}$$

Note that the resulting inertia tensor is diagonal in this basis (even though the individual inertia tensors for each mass are not diagonal); the products of inertia are all zero. This implies that the basis vectors are eigenvectors of the inertia tensor (see later for details.) For example, if $\underline{\omega} = \omega(0, 0, 1)$, then $\underline{L}(O) = 8m\omega a^2(0, 0, 1) \propto \underline{\omega}$.

In general, $\underline{L}(O)$ is not parallel to $\underline{\omega}$. For example, if $\underline{\omega} = \omega(0, 1, 1)$ then $\underline{L}(O) = 4m\omega a^2(0, 1, 2)$, which is not parallel to $\underline{\omega}$.

There are of course other bases in which the inertia tensor is not diagonal.

9.3 Two Useful Theorems

Perpendicular Axes Theorem

For a lamina (two dimensional sheet), or a collection of particles confined to a plane (choosing $\underline{e}_3$ to be normal to the plane), with O in the plane

$$\boxed{I_{11}(O) + I_{22}(O) \;=\; I_{33}(O)}$$

This is simply checked by using equation (29) and noting that $x_3^{(\alpha)} = 0$.

Parallel Axes Theorem

The position vector $\underline{R}$ of the *centre of mass* G of the body is defined to be

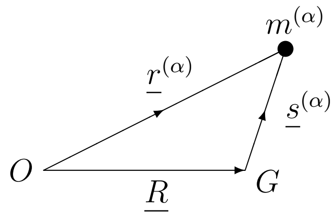
$$\underline{R} \;=\; \frac{1}{M}\sum_\alpha m^{(\alpha)}\,\underline{r}^{(\alpha)},$$

where $\underline{r}^{(\alpha)}$ are the position vectors relative to the origin O, and $M = \sum_\alpha m^{(\alpha)}$, is the *total mass* of the system.

The parallel axes theorem states that

$$\boxed{I_{ij}(O) - I_{ij}(G) \;=\; M\left\{(\underline{R}\cdot\underline{R})\,\delta_{ij} - R_i R_j\right\}}$$

Proof: Let $\underline{s}^{(\alpha)}$ be the position of $m^{(\alpha)}$ *with respect to* G, then



$$I_{ij}(G) \;=\; \sum_\alpha m^{(\alpha)} \left\{ (\underline{s}^{(\alpha)} \cdot \underline{s}^{(\alpha)})\, \delta_{ij} - s_i^{(\alpha)} s_j^{(\alpha)} \right\}$$

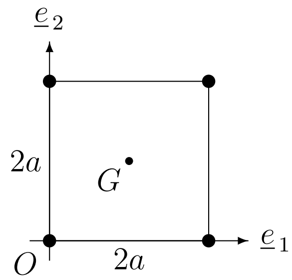
and

$$\begin{aligned}
I_{ij}(O) &= \sum_\alpha m^{(\alpha)} \left\{ (\underline{r}^{(\alpha)} \cdot \underline{r}^{(\alpha)})\, \delta_{ij} - x_i^{(\alpha)}\, x_j^{(\alpha)} \right\} \\
&= \sum_\alpha m^{(\alpha)} \left\{ (\underline{R} + \underline{s}^{(\alpha)})^2\, \delta_{ij} - (\underline{R} + \underline{s}^{(\alpha)})_i\, (\underline{R} + \underline{s}^{(\alpha)})_j \right\} \\
&= M \left\{ R^2 \delta_{ij} - R_i R_j \right\} + \sum_\alpha m^{(\alpha)} \left\{ (\underline{s}^{(\alpha)} \cdot \underline{s}^{(\alpha)})\, \delta_{ij} - s_i^{(\alpha)} s_j^{(\alpha)} \right\} \\
&\qquad + 2\, \delta_{ij}\, \underline{R} \cdot \sum_{(\alpha)} m^{(\alpha)} \underline{s}^{(\alpha)} - R_i \sum_\alpha m^{(\alpha)} s_j^{(\alpha)} - R_j \sum_\alpha m^{(\alpha)} s_i^{(\alpha)} \\
&= M \left\{ R^2 \delta_{ij} - R_i R_j \right\} + I_{ij}(G)
\end{aligned}$$

The cross terms vanish because

$$\sum_\alpha m^{(\alpha)} s_i^{(\alpha)} \;=\; \sum_\alpha m^{(\alpha)} \left(x_i^{(\alpha)} - R_i \right) \;=\; 0\,.$$

Example of the use of the Parallel Axes Theorem



Consider the same arrangement of masses as before but with O at one corner of the square, *i.e.* a (massless) lamina of side $2a$, with masses m at each corner and the origin O at the bottom left, so that the masses are at $(0,0,0)$, $(2a,0,0)$, $(0,2a,0)$ and $(2a,2a,0)$

We have $M = 4m$ and

$$\begin{aligned}
\overrightarrow{OG} \;=\; \underline{R} &= \frac{1}{4m} \left\{ m(0,0,0) + m(2a,0,0) + m(0,2a,0) + m(2a,2a,0) \right\} \\
&= (a, a, 0)
\end{aligned}$$

So G is at the centre of the square and $R^2 = 2a^2$. We can now use the parallel axis theorem to relate the inertia tensor of the previous example to that of the present one

$$I(O) - I(G) \;=\; 4m \left\{ 2a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - a^2 \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \;=\; 4ma^2 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

From the previous example,

$$I(G) \;=\; 4ma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and hence}$$

$$I(O) \;=\; 4ma^2 \begin{pmatrix} 1+1 & 0-1 & 0 \\ 0-1 & 1+1 & 0 \\ 0 & 0 & 2+2 \end{pmatrix} \;=\; 4ma^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

10 Eigenvectors of Real, Symmetric Tensors

Let T be a 2nd-rank tensor with components T_{ij}, then an eigenvector $\underline{n}$ of T obeys (in any basis)

$$\boxed{T_{ij} n_j = \lambda\, n_i}$$

where λ is the eigenvalue of the eigenvector.

The tensor acts on the eigenvector to produce a vector in the *same* direction.

The *direction* of $\underline{n}$ doesn't depend on the basis although its components do (because $\underline{n}$ is a vector), and is referred to as a *principal axis*; λ is a scalar (it doesn't depend on the basis) and is referred to as a *principal value.*

10.1 Construction of the Eigenvectors

Much of this section should be revision – so we'll try to go through it quickly.

Since $n_i = \delta_{ij} n_j$, we can write the equation for an eigenvector as

$$(T_{ij} - \lambda\, \delta_{ij})\, n_j \;=\; 0$$

This set of three linear equations has a non-trivial solution (*i.e.* a solution with $\underline{n} \neq 0$) iff

$$\boxed{\det\,(T - \lambda\, I) \;\equiv\; 0} \;.$$

i.e.

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} \;=\; 0\;.$$

This is known as the 'characteristic' or 'secular' equation and is *cubic* in λ, giving 3 real (to be proven) solutions $\lambda^{(1)}$, $\lambda^{(2)}$ and $\lambda^{(3)}$, with corresponding eigenvectors $\underline{n}^{(1)}$, $\underline{n}^{(2)}$ and $\underline{n}^{(3)}$.

Example:

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} \;=\; 0.$$

Thus

$$(1-\lambda)\{\lambda(\lambda-1)-1\} - \{(1-\lambda)-0\} \;=\; 0$$

and so

$$(1-\lambda)\{\lambda^2-\lambda-2\} \;=\; (1-\lambda)(\lambda-2)(\lambda+1) \;=\; 0.$$

Thus the solutions are $\lambda = 1$, $\lambda = 2$ and $\lambda = -1$.

Check: The sum of the eigenvalues is 2, and is equal to the *trace* of the tensor; the product of the eigenvalues is -2, and is equal to the *determinant*. The reason for this will shortly become apparent.

We now find the eigenvector for each of these eigenvalues, by solving $T_{ij}\, n_j = \lambda n_i$

$$\begin{array}{rcrcrcl} (1-\lambda)\, n_1 & + & n_2 & & & = & 0 \\ n_1 & - & \lambda\, n_2 & + & n_3 & = & 0 \\ & & n_2 & + & (1-\lambda)\, n_3 & = & 0. \end{array}$$

for $\lambda = 1$, $\lambda = 2$ and $\lambda = -1$ in turn.

For $\lambda = \lambda^{(1)} = 1$, we denote the corresponding eigenvector by $\underline{n}^{(1)}$ and the equations for the components of $\underline{n}^{(1)}$ are (dropping the label (1) for brevity):

$$\left.\begin{array}{rcrcrcl} & & n_2 & & & = & 0 \\ n_1 & - & n_2 & + & n_3 & = & 0 \\ & & n_2 & & & = & 0 \end{array}\right\} \quad \Rightarrow \quad n_2 = 0\,, \quad n_3 = -n_1$$

Thus $n_1 : n_2 : n_3 \;=\; 1 : 0 : -1$, and a *unit* vector in the direction of $\underline{n}^{(1)}$ is

$$\underline{n}^{(1)} \;=\; \frac{1}{\sqrt{2}}(1, 0, -1)\;.$$

Note that we could equally well have chosen $\underline{n}^{(1)} = \frac{-1}{\sqrt{2}}(1, 0, -1)\,.$

For $\lambda = \lambda^{(2)} = 2$, the equations for the components of $\underline{n}^{(2)}$ are:

$$\left.\begin{array}{rcrcrcl} -n_1 & + & n_2 & & & = & 0 \\ n_1 & - & 2n_2 & + & n_3 & = & 0 \\ & & n_2 & - & n_3 & = & 0 \end{array}\right\} \quad \Rightarrow \quad n_2 = n_3 = n_1\,.$$

Thus $n_1 : n_2 : n_3 \;=\; 1 : 1 : 1$ and a *unit* vector in the direction of $\underline{n}^{(2)}$ is

$$\underline{n}^{(2)} \;=\; \frac{1}{\sqrt{3}}(1, 1, 1)\;.$$

For $\lambda = \lambda^{(3)} = -1$, a similar calculation (exercise) gives

$$\underline{n}^{(3)} \;=\; \frac{1}{\sqrt{6}}(1, -2, 1)\;.$$

Note that $\underline{n}^{(1)} \cdot \underline{n}^{(2)} \;=\; \underline{n}^{(1)} \cdot \underline{n}^{(3)} \;=\; \underline{n}^{(2)} \cdot \underline{n}^{(3)} \;=\; 0$ and so the eigenvectors are mutually orthogonal.

The scalar triple product of the triad $\underline{n}^{(1)}$, $\underline{n}^{(2)}$ and $\underline{n}^{(3)}$, with the above choice of signs, is $+1$, (*i.e.* $\underline{n}^{(3)} = \underline{n}^{(1)} \times \underline{n}^{(2)}$) and so they form a *right-handed* basis. Changing the sign of *any one* (or all three) of the vectors would produce a left-handed basis.

10.2 Eigenvalues and Eigenvectors of a real symmetric tensor

Important Theorem: If T_{ij} is *real* and *symmetric*, its eigenvalues are *real*. The eigenvectors corresponding to *distinct* eigenvalues are *orthogonal*.

Proof: Let $\underline{n}$ be an eigenvector, with eigenvalue λ, then by definition

$$T_{ij}\, n_j \;=\; \lambda\, n_i \tag{30}$$

This has a non-zero solution if $\det(T - \lambda I) = 0$. Now multiply equation (30) by n_i^*, and sum over i, giving

$$n_i^*\, T_{ij}\, n_j = \lambda\, \underline{n}^* \cdot \underline{n} \tag{31}$$

Next, take the complex conjugate of equation (30), multiply by n_i and sum over i, to give

$$n_i\, T_{ij}^*\, n_j^* = \lambda^*\, \underline{n}^* \cdot \underline{n} \tag{32}$$

But T_{ij} is *real* and *symmetric*, so $T_{ij}^* = T_{ji}$, and therefore

$$\text{LHS of equation (32)} \;=\; n_j^*\, T_{ji}\, n_i \;=\; \text{LHS of equation (31)}$$

So subtracting (32) from (31) gives

$$\boxed{(\lambda - \lambda^*)\ (\underline{n}^* \cdot \underline{n}) \;=\; 0}$$

and hence as $\underline{n}^* \cdot \underline{n} > 0$ then

$$\boxed{\lambda \;=\; \lambda^*}$$

Therefore *the eigenvalues of a real symmetric matrix are real.*

Since λ is real and T_{ij} are real, and the eigenvalue equation (30) is real and linear (in n_i), then real eigenvectors can be found, and they can be *normalised to unity.*

Now consider two *distinct* eigenvalues, $\lambda^{(\alpha)} \neq \lambda^{(\beta)}$, with corresponding eigenvectors $\underline{n}^{(\alpha)}$, $\underline{n}^{(\beta)}$. Then

$$\begin{aligned} n_i^{(\beta)}\, T_{ij}\, n_j^{(\alpha)} \;&=\; \lambda^{(\alpha)}\, \underline{n}^{(\beta)} \cdot \underline{n}^{(\alpha)} \\ n_i^{(\alpha)}\, T_{ij}\, n_j^{(\beta)} \;&=\; \lambda^{(\beta)}\, \underline{n}^{(\alpha)} \cdot \underline{n}^{(\beta)} \end{aligned}$$

Again due to the symmetry of T_{ij} the LHS of these equations are equal so we have

$$(\lambda^{(\alpha)} - \lambda^{(\beta)})\, \underline{n}^{(\alpha)} \cdot \underline{n}^{(\beta)} = 0$$

But $\lambda^{(\alpha)} \neq \lambda^{(\beta)}$, so this implies

$$\boxed{\underline{n}^{(\alpha)} \cdot \underline{n}^{(\beta)} \;=\; 0}$$

Therefore *eigenvectors are orthogonal if their eigenvalues are distinct.*

10.3 Degenerate eigenvalues

If the characteristic equation is of the form

$$(\lambda^{(1)} - \lambda)\,(\lambda^{(2)} - \lambda)^2 = 0$$

there is a *repeated root* and we have a *doubly degenerate* eigenvalue $\lambda^{(2)}$. But

(i) The theorem states that the eigenvector subspace corresponding to a degenerate eigenvalue is orthogonal to the eigenvector(s) corresponding to the other eigenvalue(s) $\lambda^{(i)} \neq \lambda^{(2)}$. [1]

(ii) Within this subspace, the eigenvectors can always be *chosen* to be orthogonal.

So we can *always* find a complete set of *mutually orthogonal* eigenvectors, and then normalise them to unity.

Example:

$$T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad |T - \lambda I| = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda = 2\,, \lambda = -1\,(\text{twice})\,.$$

For $\lambda = \lambda^{(1)} = 2$ with eigenvector $\underline{n}^{(1)}$

$$\left.\begin{array}{rcrcrcl} -2n_1 & + & n_2 & + & n_3 & = & 0 \\ n_1 & - & 2n_2 & + & n_3 & = & 0 \\ n_1 & + & n_2 & - & 2n_3 & = & 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{rcl} n_1 & = & n_2 \;=\; n_3 \\ \underline{n}^{(1)} & = & \frac{1}{\sqrt{3}}(1,1,1)\,. \end{array}\right.$$

For $\lambda = \lambda^{(2)} = -1$ with eigenvector $\underline{n}^{(2)}$

$$n_1^{(2)} + n_2^{(2)} + n_3^{(2)} = 0 \tag{33}$$

is the only independent equation. This can be written as $\underline{n}^{(1)} \cdot \underline{n}^{(2)} = 0$ which is the equation for a plane normal to $\underline{n}^{(1)}$. Thus *any* vector orthogonal to $\underline{n}^{(1)}$ is an eigenvector with eigenvalue -1.

If we *choose* $n_3^{(2)} = 0$, then $n_2^{(2)} = -n_1^{(2)}$, and a possible normalised eigenvector is

$$\underline{n}^{(2)} \quad = \quad \frac{1}{\sqrt{2}}(1, -1, 0)\,.$$

If we demand that the third eigenvector $\underline{n}^{(3)}$ is orthogonal to $\underline{n}^{(2)}$, then we must have $n_2^{(3)} = n_1^{(3)}$. Equation (33) then gives $n_3^{(3)} = -2n_1^{(3)}$ and so

$$\underline{n}^{(3)} \quad = \quad \frac{1}{\sqrt{6}}(1, 1, -2)\,.$$

[1]For a 3×3 symmetric matrix with a doubly generate eigenvalue, there is of course only one distinct eigenvalue. But the proof of the theorem is valid for $N \times N$ matrices for any N.

Alternatively, the third eigenvector can be calculated using

$$\underline{n}^{(3)} \;=\; \pm\, \underline{n}^{(1)} \times \underline{n}^{(2)}$$

The sign chosen determines the handedness of the triad $\underline{n}^{(1)}$, $\underline{n}^{(2)}$, $\underline{n}^{(3)}$. This particular pair, $\underline{n}^{(2)}$ and $\underline{n}^{(3)}$, is just one of an *infinite number* of orthogonal pairs that are eigenvectors of T_{ij}, all lying in the plane normal to $\underline{n}^{(1)}$.

Finally, if the characteristic equation is of the form

$$(\lambda^{(1)} - \lambda)^3 = 0$$

then we have a triply degenerate eigenvalue $\lambda^{(1)}$. This can only occur if the tensor is equal to $\lambda^{(1)}\, \delta_{ij}$ which means it is 'isotropic' and *any* vector is an eigenvector with eigenvalue $\lambda^{(1)}$.

11 Diagonalisation of a Real, Symmetric Tensor

In an arbitrary basis $\{\underline{e}_i\}$, the tensor T_{ij} is in general not diagonal, *i.e.* $T_{ij} \neq 0$ for some or all $i \neq j$.

However if we transform to a basis constructed from the normalised eigenvectors – the 'principal axes' basis – the tensor becomes diagonal.

Let us transform to the basis $\{\underline{e}_i'\}$ chosen such that

$$\boxed{\underline{e}_i' \;=\; \underline{n}^{(i)}}$$

where $\underline{n}^{(i)}$ are the three *orthogonal* and *normalised* eigenvectors of T_{ij} with eigenvalues $\lambda^{(i)}$ respectively.

The transformation matrix L is then given by

$$\ell_{ij} \;=\; \underline{e}_i' \cdot \underline{e}_j \;=\; \underline{n}^{(i)} \cdot \underline{e}_j \;=\; n_j^{(i)} \;.$$

so the *rows* of L are the components of the *normalised eigenvectors* of T.

First check that this L is indeed orthogonal:

$$\begin{aligned} LL^T \;&=\; \begin{pmatrix} n_1^{(1)} & n_2^{(1)} & n_3^{(1)} \\ n_1^{(2)} & n_2^{(2)} & n_3^{(2)} \\ n_1^{(3)} & n_2^{(3)} & n_3^{(3)} \end{pmatrix} \begin{pmatrix} n_1^{(1)} & n_1^{(2)} & n_1^{(3)} \\ n_2^{(1)} & n_2^{(2)} & n_2^{(3)} \\ n_3^{(1)} & n_3^{(2)} & n_3^{(3)} \end{pmatrix} \\ &=\; \begin{pmatrix} \underline{n}^{(1)}\cdot\underline{n}^{(1)} & \underline{n}^{(1)}\cdot\underline{n}^{(2)} & \underline{n}^{(1)}\cdot\underline{n}^{(3)} \\ \underline{n}^{(2)}\cdot\underline{n}^{(1)} & \underline{n}^{(2)}\cdot\underline{n}^{(2)} & \underline{n}^{(2)}\cdot\underline{n}^{(3)} \\ \underline{n}^{(3)}\cdot\underline{n}^{(1)} & \underline{n}^{(3)}\cdot\underline{n}^{(2)} & \underline{n}^{(3)}\cdot\underline{n}^{(3)} \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;=\; I \end{aligned}$$

Alternatively (and much more quickly!)

$$(LL^T)_{ij} = l_{ik}\, l_{jk} = n_k^{(i)}\, n_k^{(j)} = \underline{n}^{(i)} \cdot \underline{n}^{(j)} = \delta_{ij}$$

As a by-product, we have just shown that the rows of an orthonormal matrix are indeed orthonormal. Similarly for the columns.

In the basis $\{\underline{e}_i'\}$

$$T'_{ij} = (LTL^T)_{ij}$$

Now since the *columns* of L^T are the *normalised eigenvectors* of T, we have

$$TL^T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} n_1^{(1)} & n_1^{(2)} & n_1^{(3)} \\ n_2^{(1)} & n_2^{(2)} & n_2^{(3)} \\ n_3^{(1)} & n_3^{(2)} & n_3^{(3)} \end{pmatrix} = \begin{pmatrix} \lambda^{(1)}n_1^{(1)} & \lambda^{(2)}n_1^{(2)} & \lambda^{(3)}n_1^{(3)} \\ \lambda^{(1)}n_2^{(1)} & \lambda^{(2)}n_2^{(2)} & \lambda^{(3)}n_2^{(3)} \\ \lambda^{(1)}n_3^{(1)} & \lambda^{(2)}n_3^{(2)} & \lambda^{(3)}n_3^{(3)} \end{pmatrix}$$

$$LTL^T = \begin{pmatrix} n_1^{(1)} & n_2^{(1)} & n_3^{(1)} \\ n_1^{(2)} & n_2^{(2)} & n_3^{(2)} \\ n_1^{(3)} & n_2^{(3)} & n_3^{(3)} \end{pmatrix} \begin{pmatrix} \lambda^{(1)}n_1^{(1)} & \lambda^{(2)}n_1^{(2)} & \lambda^{(3)}n_1^{(3)} \\ \lambda^{(1)}n_2^{(1)} & \lambda^{(2)}n_2^{(2)} & \lambda^{(3)}n_2^{(3)} \\ \lambda^{(1)}n_3^{(1)} & \lambda^{(2)}n_3^{(2)} & \lambda^{(3)}n_3^{(3)} \end{pmatrix} = \begin{pmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{pmatrix}$$

where the last results follow from the orthonormality of the $\underline{n}^{(i)}$ (rows of L, columns of L^T).

Alternatively (and for once dispensing with the summation convention)

$$\begin{aligned} T'_{ij} &= (LTL^T)_{ij} = \sum_{pq} \ell_{ip}\,\ell_{jq}\,T_{pq} = \sum_{pq} n_p^{(i)}\,T_{pq}\,n_q^{(j)} = \sum_p n_p^{(i)}\,\lambda^{(j)}\,n_p^{(j)} = \lambda^{(j)}\,\underline{n}^{(i)}\cdot\underline{n}^{(j)} \\ &= \lambda^{(i)}\,\delta_{ij} \end{aligned}$$

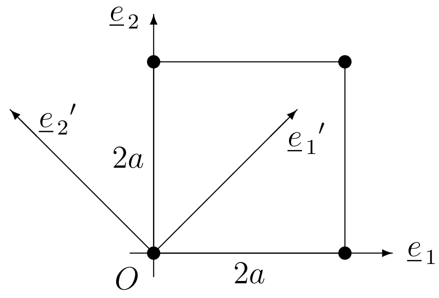
Thus, with respect to a basis defined by the eigenvectors or principal axes of the tensor, the tensor is *diagonal*, and is sometimes written $T' = \mathrm{diag}\,\{\lambda^{(1)},\, \lambda^{(2)},\, \lambda^{(3)}\}$. The diagonal basis is referred to as the *principal axes basis*.

Note: In the diagonal basis the trace of a tensor is the sum of the eigenvalues; the determinant of the tensor is the product of the eigenvalues. Since the trace and determinant are invariants this means that in any basis the trace and determinant are the sum and product of the eigenvalues respectively.

The Inertia Tensor: Consider the four masses arranged in a square with the origin at the left hand corner, as in our previous example. The inertia tensor is

$$I(O) = 4ma^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

It's easy to check (exercise) that the normalised eigenvectors (principal axes of inertia) are $\frac{1}{\sqrt{2}}(\underline{e}_1+\underline{e}_2)$ (eigenvalue $4ma^2$), $\frac{1}{\sqrt{2}}(-\underline{e}_1+\underline{e}_2)$ (eigenvalue $12ma^2$) and $\underline{e}_3$ (eigenvalue $16ma^2$).



Defining the $\underline{e}_i'$ basis: $\underline{e}_1' = \frac{1}{\sqrt{2}}(\underline{e}_1 + \underline{e}_2)$, $\underline{e}_2' = \frac{1}{\sqrt{2}}(-\underline{e}_1 + \underline{e}_2)$, $\underline{e}_3' = \underline{e}_3$, so $\underline{e}_1' \times \underline{e}_2' = \underline{e}_3'$ to give a RH basis, one obtains

$$L = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{(which from section (6.4) is a rotation of } \pi/4 \text{ about } \underline{e}_3 \text{ axis)}$$

and the inertia tensor in the $\{\underline{e}_i'\}$ basis has components $I'_{ij}(O) = \left(L\, I(O)\, L^T\right)_{ij}$ so that

$$\begin{aligned} I'(O) &= 4ma^2 \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= 4ma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \end{aligned}$$

We see that the tensor is indeed diagonal with diagonal elements which are the eigenvalues (principal moments of inertia).

Remark: Diagonalisability is a very special and useful property of real, symmetric tensors. It is a property shared by the more general class of Hermitian operators which you will meet in quantum mechanics. A general tensor does not share this property. For example, a real non-symmetric tensor cannot in general be diagonalised by an orthogonal transformation.

12 Fields

Many physical quantities vary in some region of space, *e.g.* the temperature $T(\underline{r})$ of a body. To study this we require the concept of a field.

If to each point $\underline{r}$ in some region of space there corresponds a *scalar* $\phi(x_1, x_2, x_3)$, then $\phi(\underline{r})$ is a *scalar field.*

Examples: temperature distribution in a body $T(\underline{r})$, pressure in the atmosphere $p(\underline{r})$, electric charge density or mass density $\rho(\underline{r})$, electrostatic potential $\phi(\underline{r})$.

Similarly a *vector field* assigns a vector $\underline{v}(\underline{r})$ to each point $\underline{r}$ of some region.

Examples: velocity in a fluid $\underline{v}(\underline{r})$, electric current density $\underline{j}(\underline{r})$, electric field $\underline{E}(\underline{r})$, magnetic field $\underline{B}(\underline{r})$ (actually a pseudovector field).

A vector field in 2-d can be represented graphically, at a carefully selected set of points $\underline{r}$, by an arrow whose length and direction is proportional to $\underline{v}(\underline{r})$ *e.g.* wind velocity on a weather forecast chart.

12.1 Level Surfaces or Equipotentials of a Scalar Field

If $\phi(\underline{r})$ is a non-constant scalar field, then the equation $\phi(\underline{r}) = c$ where c is a constant, defines a *level surface* or *equipotential* of the field. Different level surfaces do not intersect, or ϕ would be multi-valued at the point of intersection.

Familiar examples in two dimensions are contours of constant height on a geographical map, $h(x_1, x_2) = c$, which are of course level curves rather than level surfaces. Isobars on a weather map are level curves of pressure $p(x_1, x_2) = c$.

Examples in three dimensions:

(i) Let $\phi(\underline{r}) = r^2 \equiv x_1^2 + x_2^2 + x_3^2 = x^2 + y^2 + z^2$

The level surface $\phi(\underline{r}) = r_0^2$ is a sphere of radius r_0 centred on the origin.

As r_0 is varied, we obtain a family of level surfaces or equipotentials which are concentric spheres.

(ii) The electrostatic potential at $\underline{r}$ due to a point charge q situated at the point $\underline{a}$ is

$$\phi(\underline{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\underline{r} - \underline{a}|}$$

The equipotentials or level surfaces are concentric spheres centred on the point $\underline{a}$.

(iii) Let $\phi(\underline{r}) = \underline{k} \cdot \underline{r}$.

The level surfaces are planes $\underline{k} \cdot \underline{r} = constant$, with $\underline{k}$ normal to the planes.

(iv) Let $\phi(\underline{r}) = \exp(i\underline{k} \cdot \underline{r})$, which is a complex scalar field.

Since $\underline{k} \cdot \underline{r} = constant$ is the equation for a plane, the level surfaces are again planes.

12.2 Gradient of a Scalar Field

How do we describe mathematically the variation of a scalar field as a function of position?

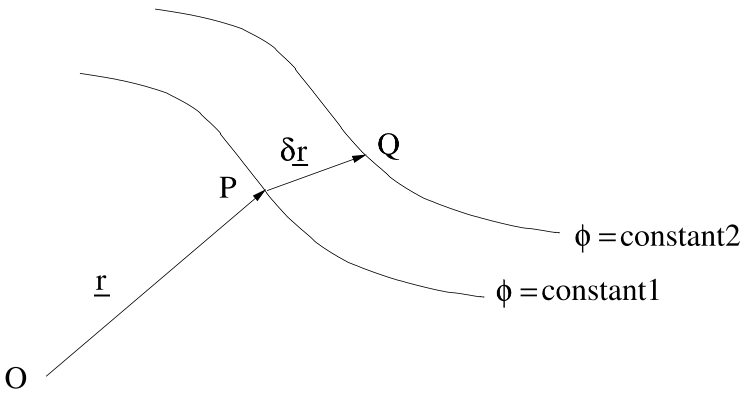
As an example, think of a 2-d contour map of the height $h = h(x_1, x_2)$ of a hill. $h(x_1, x_2)$ is a scalar field. If we are on the hill and move in the x_1–x_2 plane then the change in height will depend on the direction in which we move (unless the hill is completely flat!) For example there will be a direction in which the height increases most steeply: 'straight up the hill.' We now introduce a formalism to describe how a scalar field $\phi(\underline{r})$ changes as a function of $\underline{r}$.

Mathematical Note: A scalar field $\phi(\underline{r}) = \phi(x_1, x_2, x_3)$ is said to be *continuously differentiable* in a region R if its first order partial derivatives

$$\frac{\partial\phi(\underline{r})}{\partial x_1}, \qquad \frac{\partial\phi(\underline{r})}{\partial x_2} \quad \text{and} \quad \frac{\partial\phi(\underline{r})}{\partial x_3}$$

exist, and are *continuous* at every point $\underline{r} \in R$. We will generally assume scalar fields are continuously differentiable.

Let $\phi(\underline{r})$ be a scalar field, and consider 2 nearby points: P with position vector $\underline{r}$, and Q with position vector $\underline{r} + \delta\underline{r}$. Assume P and Q lie on *different* level surfaces as shown:



Now use the definition of the derivative (or lowest-order Taylor's theorem) for a function of 3 variables to evaluate the change in ϕ as we move from P to Q

$$\begin{aligned}
\delta\phi &\equiv \phi(\underline{r} + \delta\underline{r}) - \phi(\underline{r}) \\
&= \phi(x_1 + \delta x_1,\, x_2 + \delta x_2,\, x_3 + \delta x_3) - \phi(x_1,\, x_2,\, x_3) \\
&= \phi(x_1,\, x_2 + \delta x_2,\, x_3 + \delta x_3) + \frac{\partial\phi(\underline{r})}{\partial x_1}\,\delta x_1 - \phi(x_1,\, x_2,\, x_3) + \ldots \\
&= \frac{\partial\phi(\underline{r})}{\partial x_1}\,\delta x_1 + \frac{\partial\phi(\underline{r})}{\partial x_2}\,\delta x_2 + \frac{\partial\phi(\underline{r})}{\partial x_3}\,\delta x_3 + O(\delta x_i\,\delta x_j)
\end{aligned}$$

where we assumed that the higher order partial derivatives exist. Neglecting these higher order terms, we write

$$\boxed{\delta\phi = \underline{\nabla}\,\phi \cdot \delta\underline{r}}$$

where the 3 quantities

$$\boxed{\left(\underline{\nabla}\,\phi\right)_i \;\equiv\; \frac{\partial\phi}{\partial x_i} \;\equiv\; \partial_i\phi}$$

form the Cartesian components of a *vector field*

$$\boxed{\underline{\nabla}\,\phi(\underline{r}) \equiv \frac{\partial\phi}{\partial x_1}\,\underline{e}_1 + \frac{\partial\phi}{\partial x_2}\,\underline{e}_2 + \frac{\partial\phi}{\partial x_3}\,\underline{e}_3 \;=\; \frac{\partial\phi}{\partial x_i}\,\underline{e}_i}$$

where we used the summation convention in the last line, there is an implicit sum over the dummy index i. (See later for a derivation of the transformation properties of $\underline{\nabla}\,\phi$.)

Note that the *partial derivative* $\partial\phi/\partial x_1$ is the derivative of $\phi(\underline{r})$ with respect to x_1, keeping x_2 and x_3 fixed, *etc.*

In 'xyz' notation

$$\underline{\nabla}\,\phi = \frac{\partial\phi}{\partial x}\,\underline{e}_x + \frac{\partial\phi}{\partial y}\,\underline{e}_y + \frac{\partial\phi}{\partial z}\,\underline{e}_z$$

The *vector field* $\underline{\nabla}\,\phi(\underline{r})$ is called the *gradient* of $\phi(\underline{r})$, and is pronounced *'grad phi'*.

Example: Calculate the gradient of $\phi(\underline{r}) = r^2 = x_1^2 + x_2^2 + x_3^2$.

$$(\underline{\nabla}\,r^2)_1 = \frac{\partial}{\partial x_1}(x_1^2 + x_2^2 + x_3^2) = 2x_1$$

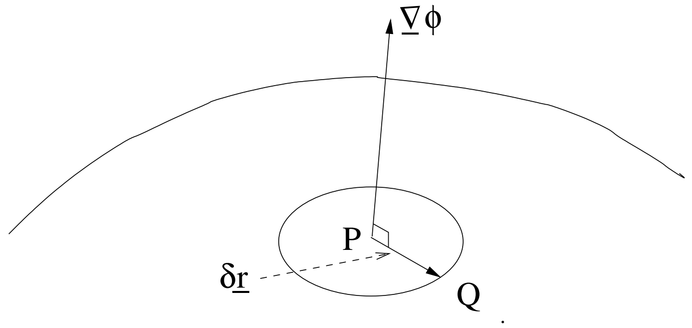
Similarly for x_2, x_3, and hence

$$\underline{\nabla}\,r^2 = 2\underline{r}$$

12.3 Interpretation of the gradient

In deriving the expression for $\delta\phi$ above, we assumed that the points P and Q lie on *different* level surfaces. Now consider the situation where P and Q are nearby points on the *same* level surface. In that case, $\delta\phi = 0$ and so

$$\delta\phi = \underline{\nabla}\,\phi \cdot \delta\underline{r} = 0$$



The infinitesimal vector $\delta\underline{r}$ lies in the level surface at $\underline{r}$, and the above equation holds for all such $\delta\underline{r}$, hence

$$\boxed{\underline{\nabla}\,\phi(\underline{r}) \text{ is } \textit{normal to the level surface at } \underline{r}.}$$

To construct a *unit normal* $\underline{n}(\underline{r})$ to the level surface at $\underline{r}$, we divide $\underline{\nabla}\,\phi$ by its length

$$\underline{n}(\underline{r}) \;=\; \frac{\underline{\nabla}\,\phi}{|\underline{\nabla}\,\phi|} \qquad \big(\text{for} \quad |\underline{\nabla}\,\phi| \neq 0\big)$$

12.4 Directional Derivative

Consider the change, $\delta\phi$, produced in $\phi(\underline{r})$ by moving a distance δs in the direction of the unit vector $\hat{\underline{s}}$, so that $\delta\underline{r} = \delta s\, \hat{\underline{s}}$. Then

$$\delta\phi = \underline{\nabla}\,\phi \cdot \delta\underline{r} = (\underline{\nabla}\,\phi) \cdot \hat{\underline{s}}\;\delta s$$

As $\delta s \to 0$, the rate of change of ϕ as we move in the direction of $\hat{\underline{s}}$ is

$$\frac{d\phi}{ds} = \hat{\underline{s}} \cdot \underline{\nabla}\,\phi = |\underline{\nabla}\,\phi|\;\cos\theta \tag{34}$$

where θ is the angle between $\hat{\underline{s}}$ and the normal to the level surface at $\underline{r}$.

$\hat{\underline{s}} \cdot \underline{\nabla}\,\phi$ is called the *directional derivative* of the scalar field ϕ in the direction of $\hat{\underline{s}}$

The directional derivative has its *maximum* value when $\hat{\underline{s}}$ is *parallel* to $\underline{\nabla}\,\phi$, and is *zero* when $\delta s\,\hat{\underline{s}}$ lies in the level surface. Therefore

$\underline{\nabla}\,\phi$ points in the direction of the *maximum* rate of increase in ϕ

Recall that this direction is normal to the level surface. A familiar example is that of contour lines on a map: the steepest direction is perpendicular to the contour lines.

Example: Find the directional derivative of $\phi(\underline{r}) = xy(x+z)$ at the point $(1,2,-1)$ in the direction of $(\underline{e}_x + \underline{e}_y)/\sqrt{2}$.

$$\underline{\nabla}\,\phi = (2xy + yz)\,\underline{e}_x + x(x+z)\,\underline{e}_y + xy\,\underline{e}_z = 2\underline{e}_x + 2\underline{e}_z \qquad \text{at} \quad (1,2,-1)$$

Thus at this point

$$\frac{1}{\sqrt{2}}(\underline{e}_x + \underline{e}_y) \cdot \underline{\nabla}\,\phi = \sqrt{2}$$

Physical example: Let $T(\underline{r})$ be the temperature of the atmosphere at the point $\underline{r}$. An object flies through the atmosphere with velocity $\underline{v}$. Obtain an expression for the rate of change of temperature experienced by the object.

As the object moves from $\underline{r}$ to $\underline{r} + \delta\underline{r}$ in time δt, it sees a change in temperature

$$\delta T \;=\; \underline{\nabla}\,T \cdot \delta\underline{r} \;=\; \left(\underline{\nabla}\,T \cdot \frac{\delta\underline{r}}{\delta t}\right)\delta t$$

Taking the limit $\delta t \to 0$, we obtain

$$\frac{dT(\underline{r})}{dt} = \underline{v} \cdot \underline{\nabla}\,T(\underline{r})$$

13 More on gradient, the operator *del*

13.1 Examples of the Gradient in Physical Laws

Gravitational force due to the Earth: The potential energy of a particle of mass m at a height z above the Earth's surface is $\phi = mgz$. The force due to gravity can be written as

$$\underline{F} = -\underline{\nabla}\,\phi = -mg\,\underline{e}_z$$

Newton's Law of Gravitation: Now consider the gravitational force on a mass m at $\underline{r}$ due to a mass m_0 at the origin. We can write this as

$$\underline{F} \;=\; -\,\frac{Gmm_0}{r^2}\,\hat{\underline{r}} \;=\; -\,\underline{\nabla}\,\phi$$

where the potential energy $\phi = -Gmm_0/r$ and $\hat{\underline{r}}$ is a unit vector in the direction of $\underline{r}$. We shall show below that $\underline{\nabla}\,(1/r) = -\underline{r}/r^3$.

In these two examples we see that the force acts down the potential energy gradient.

13.2 Examples on gradient

The previous example on directional derivatives used the *'xyz'* notation. This gets unwieldy for more complicated examples, and suffix notation is more convenient.

(i) Let $\phi(\underline{r}) = r^2 = x_1^2 + x_2^2 + x_3^2$, then

$$(\underline{\nabla}\,r^2)_i = \frac{\partial}{\partial x_i}(x_1^2 + x_2^2 + x_3^2) = 2x_i \qquad \text{or} \qquad \underline{\nabla}\,r^2 = 2\underline{r}$$

Or, using the shorthand $\partial/\partial x_i \equiv \partial_i$ and the summation convention to write $r^2 = x_j x_j$

$$(\underline{\nabla} r^2)_i = \partial_i\, r^2 = \partial_i\,(x_j x_j) = \delta_{ij} x_j + x_j \delta_{ij} = 2x_i$$

where we used the important property of partial derivatives

$$\boxed{\frac{\partial x_j}{\partial x_i} \;=\; \delta_{ij} \;=\; \partial_i x_j}$$

The level surfaces of r^2 are spheres centred on the origin, and the gradient of r^2 at $\underline{r}$ points radially outward with magnitude $2r$.

(ii) Let $\phi = \underline{a} \cdot \underline{r}$ where $\underline{a}$ is a *constant* vector.

$$(\underline{\nabla}\,(\underline{a} \cdot \underline{r}))_i = \partial_i(a_j x_j) = a_j \delta_{ij} = a_i$$

This is not surprising, since the equipotentials $\underline{a} \cdot \underline{r} = c$ are planes orthogonal to $\underline{a}$.

(iii) Let $\phi(\underline{r}) \;=\; r \;=\; \sqrt{x_1^2 + x_2^2 + x_3^2} \;=\; (x_j x_j)^{1/2}$

$$
\begin{aligned}
(\underline{\nabla}\, r)_i &= \partial_i\, (x_j x_j)^{1/2} \\
&= \frac{1}{2}\, (x_j x_j)^{-1/2}\, \partial_i (x_k x_k) \qquad \text{(chain rule)} \\
&= \frac{1}{2r}\, 2\, x_i \\
&= (\underline{\hat{r}})_i \qquad \text{so} \qquad \boxed{\underline{\nabla}\, r \;=\; \frac{1}{r}\,\underline{r} \;=\; \underline{\hat{r}}}
\end{aligned}
$$

The gradient of the length of the position vector is the unit vector pointing radially outwards from the origin. It is normal to the level surfaces which are spheres centered on the origin.

13.3 Identities for gradients

If $\phi(\underline{r})$ and $\psi(\underline{r})$ are real scalar fields, then:

(i) **Distributive law**

$$
\boxed{\underline{\nabla}\,(\phi + \psi) \;=\; \underline{\nabla}\,\phi + \underline{\nabla}\,\psi}
$$

Proof:

$$
\big(\underline{\nabla}\,(\phi\,+\,\psi)\big)_i \;\equiv\; \partial_i(\phi + \psi) \;=\; \partial_i\phi + \partial_i\psi \;\equiv\; \big(\underline{\nabla}\,\phi\big)_i + \big(\underline{\nabla}\,\psi\big)_i
$$

(ii) **Product rule**

$$
\boxed{\underline{\nabla}\,(\phi\,\psi) \;=\; \big(\underline{\nabla}\,\phi\big)\,\psi + \phi\,\big(\underline{\nabla}\,\psi\big)}
$$

Proof:

$$
\big(\underline{\nabla}\,(\phi\psi)\big)_i \;\equiv\; \partial_i\,(\phi\psi) \;=\; (\partial_i\phi)\psi + \phi(\partial_i\psi) \;\equiv\; \big(\underline{\nabla}\,\phi\big)_i\,\psi + \phi\,\big(\underline{\nabla}\,\psi\big)_i
$$

(iii) **Chain rule:** If $F(\phi(\underline{r}))$ is a scalar field, then

$$
\boxed{\underline{\nabla}\,F\,(\phi) \;=\; \frac{dF(\phi)}{d\phi}\;\underline{\nabla}\,\phi}
$$

Proof:

$$
\big(\underline{\nabla}\,F(\phi)\big)_i = \frac{\partial}{\partial x_i}(F(\phi)) = \frac{dF(\phi)}{d\phi}\,\frac{\partial\phi}{\partial x_i} = \frac{dF(\phi)}{d\phi}\;\big(\underline{\nabla}\,\phi\big)_i
$$

Example of Chain Rule: If $\phi(\underline{r}) = r$ we can use result (iii) from section (13.2) to give

$$\underline{\nabla}\, F(r) \;=\; \frac{dF(r)}{dr}\,\underline{\nabla}\, r \;=\; \frac{F'(r)}{r}\;\underline{r}$$

If $F\left(\phi(\underline{r})\right) = r^n$, we have $\phi(\underline{r}) = r$ as in the previous example, and so

$$\boxed{\underline{\nabla}\,(r^n) \;=\; \frac{d\,r^n}{dr}\,\left(\underline{\nabla}\, r\right) \;=\; \left(n\,r^{n-1}\right)\,\frac{1}{r}\,\underline{r} \;=\; \left(n\,r^{n-2}\right)\,\underline{r}}$$

In particular

$$\underline{\nabla}\left(\frac{1}{r}\right) \;=\; -\,\frac{\underline{r}}{r^3}$$

We can also do this directly in suffix notation

$$\left(\underline{\nabla}\left(\frac{1}{r}\right)\right)_i \;=\; \partial_i\,(x_j x_j)^{-\frac{1}{2}} \;=\; -\,\frac{1}{2}\,(x_k x_k)^{-3/2}\,2\,\delta_{ij}\,x_j \;=\; -\,\frac{x_i}{r^3}$$

13.4 Transformation of the gradient

We now prove that the gradient of a scalar field is indeed a vector field - thus far we merely assumed it was!

Let the point P have coordinates x_i in the $\{\underline{e}_i\}$ basis, and the *same* point P have coordinates x'_i in the $\{\underline{e}_i'\}$ basis, *i.e.* we consider the vector transformation law $x_i \to x'_i = \ell_{ij}\, x_j$.

$\phi(\underline{r})$ is a *scalar field* if it depends only on the physical point P and not on the coordinates x_i or x'_i used to specify P. The *value* of ϕ at P is *invariant* under a change of basis, but the function may look different, *i.e.*

$$\boxed{\phi'(x'_1,\, x'_2,\, x'_3) \;=\; \phi(x_1,\, x_2,\, x_3)}$$

Similarly $\underline{a}$ is a *vector field* if its components transform as

$$\boxed{a'_i(x'_1,\, x'_2,\, x'_3) \;=\; \ell_{ip}\, a_p(x_1,\, x_2,\, x_3)}$$

Now consider $\underline{\nabla}\,\phi$ in the new (primed) basis. Its components transform as

$$\partial'_i\,\phi'(\underline{r}') \;\equiv\; \frac{\partial}{\partial x'_i}\,\phi'(x'_1,\, x'_2,\, x'_3) \;=\; \frac{\partial x_j}{\partial x'_i}\,\frac{\partial}{\partial x_j}\,\phi(x_1,\, x_2,\, x_3)$$

(using the chain rule). Now since $x_j = \ell_{kj}\, x'_k$ (inverse transformation for vector components)

$$\frac{\partial x_j}{\partial x'_i} \;=\; \ell_{kj}\,\frac{\partial x'_k}{\partial x'_i} \;=\; \ell_{kj}\,\delta_{ik} \;=\; \ell_{ij}\,.$$

Hence

$$\partial'_i\,\phi'(\underline{r}') \;=\; \ell_{ij}\,\frac{\partial}{\partial x_j}\,\phi(x_1,\, x_2,\, x_3) \;\equiv\; \ell_{ij}\,\partial_j\phi(\underline{r})$$

which shows that the components of $\underline{\nabla}\,\phi$ transform in the same way as the components of a vector. Thus $\underline{\nabla}\,\phi(\underline{r})$ transforms as a *vector field* as claimed.

13.5 The operator *del*

We can think of $\underline{\nabla}$ as a *vector operator*, called *del*, which acts on the *scalar field* $\phi(\underline{r})$ to produce the *vector field* $\underline{\nabla}\,\phi(\underline{r})$.

In Cartesians:
$$\boxed{\underline{\nabla} \;\equiv\; \underline{e}_1 \frac{\partial}{\partial x_1} + \underline{e}_2 \frac{\partial}{\partial x_2} + \underline{e}_3 \frac{\partial}{\partial x_3} \equiv \underline{e}_i\,\partial_i}$$

We call $\underline{\nabla}$ an 'operator' since it operates on something to its *right*. It is a vector operator because it has vector transformation properties,

$$\boxed{\partial_i' \;=\; \ell_{ip}\,\partial_p}$$

We have seen how $\underline{\nabla}$ acts on a scalar field to produce a vector field. We can make products of the vector operator $\underline{\nabla}$ with other vector quantities to produce new operators and fields in the same way we could make scalar and vector products of two vectors.

For example, the directional derivative of ϕ in the direction $\hat{\underline{s}}$, was given by $\hat{\underline{s}}\cdot\underline{\nabla}\,\phi$. More generally, we can interpret $\underline{a}\cdot\underline{\nabla}$ as a *scalar operator*

$$\underline{a}\cdot\underline{\nabla} \;=\; a_i\,\partial_i$$

i.e. $\underline{a}\cdot\underline{\nabla}$ acts on a scalar field to its *right* to produce another scalar field

$$(\underline{a}\cdot\underline{\nabla})\,\phi \;=\; a_i\,\partial_i\phi \;=\; a_1\frac{\partial\phi}{\partial x_1} + a_2\frac{\partial\phi}{\partial x_2} + a_3\frac{\partial\phi}{\partial x_3}$$

We can also act with this operator on a vector field $\underline{b}(\underline{r})$ to get another vector field,

$$(\underline{a}\cdot\underline{\nabla})\,\underline{b} = \underline{e}_1\,(\underline{a}\cdot\underline{\nabla})\,b_1 + \underline{e}_2\,(\underline{a}\cdot\underline{\nabla})\,b_2 + \underline{e}_3\,(\underline{a}\cdot\underline{\nabla})\,b_3$$

or, equivalently, in components

$$\big((\underline{a}\cdot\underline{\nabla})\,\underline{b}\big)_i \;=\; (\underline{a}\cdot\underline{\nabla})\,b_i \;=\; a_j\,\partial_j\,b_i$$

The alternative expression $\underline{a}\cdot\big(\underline{\nabla}\,\underline{b}\big)$ is *undefined* because $\underline{\nabla}\,\underline{b}$ doesn't make sense.

(For this reason, the parentheses are sometimes omitted, and $\underline{a}\cdot\underline{\nabla}\;\underline{b}$ is taken to mean $(\underline{a}\cdot\underline{\nabla})\;\underline{b}$, but I wouldn't recommend doing this.)

NB Great care is required with the order in products since, in general, products involving operators are not commutative. For example

$$\underline{a}\cdot\underline{\nabla} \;\neq\; \underline{\nabla}\cdot\underline{a}$$

The quantity $\underline{a}\cdot\underline{\nabla}$ is a scalar differential *operator* whereas $\underline{\nabla}\cdot\underline{a} \equiv \partial_i\,a_i$ gives a scalar field called the *divergence* of $\underline{a}$.

14 Vector Operators, *Div*, *Curl* and the *Laplacian*

We now combine the vector operator $\underline{\nabla}$ *(del)* with a vector field to define two new operations *div* and *curl*. Then we define the *Laplacian*.

14.1 Divergence

We define the *divergence* of a vector field $\underline{a}(\underline{r})$ (pronounced *'div a'*) by

$$\boxed{\text{div}\,\underline{a}(\underline{r}) \;\equiv\; \underline{\nabla}\cdot\underline{a}(\underline{r})}$$

In Cartesian coordinates

$$\begin{aligned}\underline{\nabla}\cdot\underline{a} \;&\equiv\; \frac{\partial a_1}{\partial x_1}+\frac{\partial a_2}{\partial x_2}+\frac{\partial a_3}{\partial x_3} \;=\; \frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\\ &\equiv\; \frac{\partial}{\partial x_i}\,a_i \;\equiv\; \partial_i\,a_i\end{aligned}$$

It's easy to show that $\underline{\nabla}\cdot\underline{a}$ is a scalar field when $\underline{a}$ is a vector field:
Under a transformation $x_i' = \ell_{ij}\,x_j$, we have $\partial_i' = \ell_{ij}\,\partial_j$ so

$$(\underline{\nabla}\cdot\underline{a})' = \partial_i'\,a_i' = (\ell_{ij}\,\partial_j)\,(\ell_{ik}\,a_k) = \delta_{jk}\,\partial_j\,a_k = \partial_j\,a_j = \underline{\nabla}\cdot\underline{a}$$

Hence $\underline{\nabla}\cdot\underline{a}$ is invariant under a change of basis and is thus a **scalar field**.

Example: If $\underline{a}(\underline{r}) = \underline{r}$ then $\boxed{\underline{\nabla}\cdot\underline{r} \;=\; 3}$ which is a useful and important result.

Explicitly:

$$\underline{\nabla}\cdot\underline{r} = \frac{\partial x_1}{\partial x_1}+\frac{\partial x_2}{\partial x_2}+\frac{\partial x_3}{\partial x_3} = 1+1+1 = 3$$

In suffix notation:

$$\underline{\nabla}\cdot\underline{r} = \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3$$

Example: In *'xyz'* notation, let $\underline{a} = x^2z\,\underline{e}_x - 2y^3z^2\,\underline{e}_y + xy^2z\,\underline{e}_z$

$$\begin{aligned}\underline{\nabla}\cdot\underline{a} \;&=\; \frac{\partial}{\partial x}\,(x^2z) - \frac{\partial}{\partial y}\,(2y^3z^2) + \frac{\partial}{\partial z}\,(xy^2z)\\ &=\; 2xz - 6y^2z^2 + xy^2\end{aligned}$$

Then, at the point $(1,1,1)$ for instance, $\underline{\nabla}\cdot\underline{a} = 2-6+1 = -3$.

14.2 Curl

We define the *curl* of a vector field $\underline{a}(\underline{r})$ by

$$\text{curl}\, \underline{a}(\underline{r}) \;\equiv\; \underline{\nabla} \times \underline{a}(\underline{r})$$

Note that $\underline{\nabla} \times \underline{a}$ is a *vector* field (more precisely, a pseudo-vector field, if $\underline{a}$ is a vector field). In Cartesian coordinates

$$\underline{\nabla} \times \underline{a} \;=\; \underline{e}_i \left(\underline{\nabla} \times \underline{a}\right)_i \;=\; \underline{e}_i\, \epsilon_{ijk} \frac{\partial}{\partial x_j}\, a_k$$

Equivalently, the i^{th} component of $\underline{\nabla} \times \underline{a}$ is

$$\left(\underline{\nabla} \times \underline{a}\right)_i \;=\; \epsilon_{ijk}\, \partial_j\, a_k$$

More explicitly

$$\left(\underline{\nabla} \times \underline{a}\right)_1 \;=\; \frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \quad \textit{etc}$$

We can also write the curl in determinant form, as for the ordinary vector product:

$$\underline{\nabla} \times \underline{a} \;=\; \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ a_1 & a_2 & a_3 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} \underline{e}_x & \underline{e}_y & \underline{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

Example: If $\underline{a}(\underline{r}) = \underline{r}$ then $\boxed{\underline{\nabla} \times \underline{r} \;=\; 0}$ another useful and important result

Explicitly: $$(\underline{\nabla} \times \underline{r})_i \;=\; \epsilon_{ijk}\, \partial_j x_k \;=\; \epsilon_{ijk}\, \delta_{jk} = \epsilon_{ijj} = 0$$

or, using the determinant formula, $$\underline{\nabla} \times \underline{r} \;=\; \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ x_1 & x_2 & x_3 \end{vmatrix} \;\equiv\; 0$$

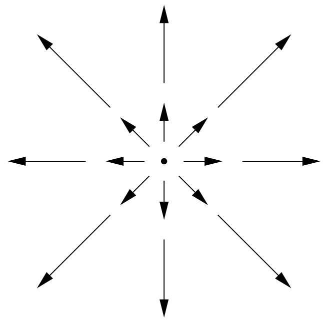
Example: Compute the curl of $\underline{a} = x^2 y\, \underline{e}_1 + y^2 x\, \underline{e}_2 + xyz\, \underline{e}_3$

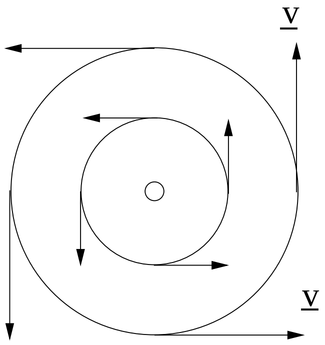
$$\underline{\nabla} \times \underline{a} \;=\; \begin{vmatrix} \underline{e}_x & \underline{e}_y & \underline{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & y^2 x & xyz \end{vmatrix} = (xz - 0)\, \underline{e}_x - (yz - 0)\, \underline{e}_y + (y^2 - x^2)\, \underline{e}_z$$

14.3 Physical Interpretation of *div* and *curl*

A full interpretation of the divergence and curl of a vector field is best left until after we have studied the Divergence Theorem and Stokes' Theorem respectively. However, we can gain some intuitive understanding by looking at simple examples where div and/or curl vanish.

First consider the radial field $\underline{a} = \underline{r}$. We have just shown that $\underline{\nabla} \cdot \underline{r} = 3$ and $\underline{\nabla} \times \underline{r} = 0$. We may sketch the vector field $\underline{a}(\underline{r})$ by drawing vectors of the appropriate direction and magnitude at selected points. These give the tangents of 'flow lines'. Roughly speaking, in this example the divergence is positive because bigger arrows come out of any point than go into it. So the field 'diverges'. (Once the concept of flux of a vector field is understood this will make more sense.)





Now consider the field $\underline{v} = \underline{\omega} \times \underline{r}$ where $\underline{\omega}$ is a *constant* vector. One can think of $\underline{v}$ as the velocity of a point in a rigid rotating body. The sketch shows a cross-section of the field $\underline{v}$ with $\underline{\omega}$ chosen to point out of the page. We can calculate $\underline{\nabla} \times \underline{v}$ as follows:

$$\begin{aligned}
\left(\underline{\nabla} \times \left(\underline{\omega} \times \underline{r}\right)\right)_i &= \epsilon_{ijk}\, \partial_j \left(\underline{\omega} \times \underline{r}\right)_k \;=\; \epsilon_{ijk}\, \partial_j\, \epsilon_{klm}\, \omega_l\, x_m \\
&= \left(\delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl}\right)\, \omega_l\, \delta_{jm} \qquad \left(\text{because } \frac{\partial \omega_l}{\partial x_j} = 0\right) \\
&= \left(\omega_i\, \delta_{jj} - \delta_{ij}\, \omega_j\right) \;=\; 2\, \omega_i
\end{aligned}$$

Thus we obtain yet another useful and important result:

$$\boxed{\underline{\nabla} \times \left(\underline{\omega} \times \underline{r}\right) = 2\underline{\omega}}$$

We also have $\underline{\nabla} \cdot (\underline{\omega} \times \underline{r}) = 0$:

$$\underline{\nabla} \cdot (\underline{\omega} \times \underline{r}) = \partial_i\, \epsilon_{ijk}\, \omega_j\, x_k = \epsilon_{ijk}\, \omega_j\, \delta_{ik} = \epsilon_{iji}\, \omega_j = 0$$

To understand intuitively the non-zero curl imagine that the flow lines are those of a rotating fluid with a ball centred on a flow line of the field. The centre of the ball will follow the flow line. However the effect of the neighbouring flow lines is to make the ball rotate. Therefore the field has non-zero 'curl' and the axis of rotation gives the direction of the curl. In the previous example ($\underline{a} = \underline{r}$) the ball would just move away from origin without rotating therefore the field $\underline{r}$ has zero curl.

Terminology:

(i) If $\underline{\nabla} \cdot \underline{a} = 0$ in some region R, we say $\underline{a}$ is *solenoidal* in R.

(ii) If $\underline{\nabla} \times \underline{a} = 0$ in some region R, we say $\underline{a}$ *irrotational* in R.

14.4 The Laplacian Operator ∇^2

Consider taking the *divergence* of the *gradient* of a scalar field $\phi(\underline{r})$

$$\underline{\nabla} \cdot (\underline{\nabla}\, \phi) \;=\; \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\phi \;=\; \partial^2 \phi \;\equiv\; \nabla^2 \phi$$

∇^2 is the *Laplacian operator*, pronounced *'del-squared'*. In Cartesian coordinates

$$\nabla^2 \;=\; \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} \;\equiv\; \partial_i\,\partial_i \;\equiv\; \partial^2$$

More explicitly

$$\boxed{\nabla^2 \phi \;=\; \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} \quad \text{or} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2}}$$

The Laplacian of a scalar field $\nabla^2 \phi$ is a scalar field, *i.e.* the Laplacian is a *scalar operator*. Under the transformation $x'_i = \ell_{ij}\, x_j$ we have $\partial'_i = \ell_{ij}\, \partial_j$ so

$$(\nabla^2)' \;=\; \partial'_i\, \partial'_i \;=\; \ell_{ij}\, \partial_j\, \ell_{ik}\, \partial_k \;=\; \delta_{jk}\, \partial_j\, \partial_k \;=\; \partial_j\, \partial_j \;=\; \nabla^2$$

Example: Using indices

$$\nabla^2 r^2 \;=\; \partial_i\, \partial_i\, (x_j x_j) \;=\; \partial_i\, (2x_i) \;=\; 2\,\delta_{ii} \;=\; 6$$

Or, directly, using simple results derived previously,

$$\nabla^2 r^2 \;=\; \underline{\nabla} \cdot (\underline{\nabla}\, r^2) \;=\; \underline{\nabla} \cdot (2\underline{r}) \;=\; 2 \times 3 \;=\; 6\,.$$

In *Cartesian coordinates only*, the effect of the Laplacian on a vector field $\underline{a}$ is *defined* to be

$$\nabla^2 \underline{a} \;=\; \partial_i\, \partial_i\, \underline{a} \;=\; \frac{\partial^2}{\partial x_1^2}\,\underline{a} \;+\; \frac{\partial^2}{\partial x_2^2}\,\underline{a} \;+\; \frac{\partial^2}{\partial x_3^2}\,\underline{a}$$

The Laplacian acts on a vector field to produce another vector field.

15 Vector Operator Identities

There are many identities involving div, grad, and curl. It is not necessary to know *all* of these, but you should know and be able to use the product and chain rules for gradients (see Section (13.3), together with the product laws for div and curl given below. These are almost obvious anyway!

You should be *familiar* with the rest and to be able to *derive* and *use* them when necessary.

It is also extremely useful to *know* and be able to derive the results for ubiquitous quantities such as $\underline{\nabla}\, r$, $\underline{\nabla}\, r^n$, $\underline{\nabla} \cdot \underline{r}$, $\underline{\nabla} \times \underline{r}$, $(\underline{a} \cdot \underline{\nabla})\underline{r}$, $\underline{\nabla}\, (\underline{a} \cdot \underline{r})$, $\underline{\nabla} \times (\underline{a} \times \underline{r})$ where $\underline{a}$ is a constant vector.

This is like learning and understanding multiplication tables, or knowing the derivatives of elementary functions such as $\sin x$.

Most importantly you should be at ease with div, grad and curl. This only comes through practice and deriving the various identities gives you just that. In these derivations the advantages of suffix notation, the summation convention and ϵ_{ijk} will (hopefully) become apparent.

In what follows $\phi(\underline{r})$, $\underline{a}(\underline{r})$ and $\underline{b}(\underline{r})$ are continuously-differentiable scalar and vector fields.

15.1 Distributive Laws

1. $\underline{\nabla} \cdot (\underline{a} + \underline{b}) \;=\; \underline{\nabla} \cdot \underline{a} + \underline{\nabla} \cdot \underline{b}$
2. $\underline{\nabla} \times (\underline{a} + \underline{b}) \;=\; \underline{\nabla} \times \underline{a} + \underline{\nabla} \times \underline{b}$

The proofs of these are straightforward using suffix or *'xyz'* notation and follow from the fact that div and curl are linear operations.

15.2 Product Laws

The results of taking the div or curl of *products* of vector and scalar fields are the most useful:

3. $\underline{\nabla} \cdot (\phi\, \underline{a}) \;=\; \left(\underline{\nabla}\, \phi\right) \cdot \underline{a} \,+\, \phi \left(\underline{\nabla} \cdot \underline{a}\right)$
4. $\underline{\nabla} \times (\phi\, \underline{a}) \;=\; \left(\underline{\nabla}\, \phi\right) \times \underline{a} \,+\, \phi \left(\underline{\nabla} \times \underline{a}\right)$

Proof of (4):

$$\begin{aligned}
\left(\underline{\nabla} \times \left(\phi\, \underline{a}\right)\right)_i &= \epsilon_{ijk}\, \partial_j \left(\phi\, a_k\right) \\
&= \epsilon_{ijk} \left(\left(\partial_j \phi\right) a_k + \phi \left(\partial_j a_k\right)\right) \\
&= \left(\underline{\nabla}\, \phi \times \underline{a}\right)_i + \phi \left(\underline{\nabla} \times \underline{a}\right)_i
\end{aligned}$$

One can also obtain this using *'xyz'* notation: $$\underline{\nabla} \times \left(\phi\, \underline{a}\right) = \begin{vmatrix} \underline{e}_x & \underline{e}_y & \underline{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \phi\, a_x & \phi\, a_y & \phi\, a_z \end{vmatrix}$$

The x component is

$$\begin{aligned}
\frac{\partial(\phi a_z)}{\partial y} - \frac{\partial(\phi a_y)}{\partial z} &= \left(\frac{\partial \phi}{\partial y}\right) a_z - \left(\frac{\partial \phi}{\partial z}\right) a_y + \phi \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) \\
&= \left(\underline{\nabla}\, \phi \times \underline{a}\right)_x + \phi \left(\underline{\nabla} \times \underline{a}\right)_x
\end{aligned}$$

A similar proof holds for the y and z components, but suffix notation is so much quicker…

Although we used Cartesian coordinates in our proofs, the identities hold in all coordinate systems (the concept of a vector is coordinate-independent).

15.3 Products of Two Vector Fields

The following identities are useful but less obvious:

5. $\underline{\nabla}\ \big(\underline{a}\cdot\underline{b}\big) \;=\; \big(\underline{a}\cdot\underline{\nabla}\big)\ \underline{b} + \big(\underline{b}\cdot\underline{\nabla}\big)\,\underline{a} + \underline{a}\times\big(\underline{\nabla}\times\underline{b}\big) + \underline{b}\times\big(\underline{\nabla}\times\underline{a}\big)$
6. $\underline{\nabla}\cdot\big(\underline{a}\times\underline{b}\big) \;=\; \underline{b}\cdot\big(\underline{\nabla}\times\underline{a}\big) - \underline{a}\cdot\big(\underline{\nabla}\times\underline{b}\big)$
7. $\underline{\nabla}\times\big(\underline{a}\times\underline{b}\big) \;=\; \underline{a}\ \big(\underline{\nabla}\cdot\underline{b}\big) - \underline{b}\ \big(\underline{\nabla}\cdot\underline{a}\big) + \big(\underline{b}\cdot\underline{\nabla}\big)\ \underline{a} - \big(\underline{a}\cdot\underline{\nabla}\big)\ \underline{b}$

Proof of (6):

$$
\begin{aligned}
\underline{\nabla}\cdot\big(\underline{a}\times\underline{b}\big) \;=\; \partial_i\,(\epsilon_{ijk}\,a_j\,b_k) \;&=\; \epsilon_{ijk}\,(\partial_i a_j)\,b_k + \epsilon_{ijk}\,a_j\,\partial_i b_k \\
&=\; b_k\,\epsilon_{kij}\,\partial_i\,a_j \quad - a_j\,\epsilon_{jik}\,\partial_i\,b_k \\
&=\; b_k\,\big(\underline{\nabla}\times\underline{a}\big)_k - a_k\,\big(\underline{\nabla}\times\underline{b}\big)_k
\end{aligned}
$$

Proof of (7):

$$
\begin{aligned}
\big(\underline{\nabla}\times\big(\underline{a}\times\underline{b}\big)\big)_i \;&=\; \epsilon_{ijk}\,\partial_j\,\big(\underline{a}\times\underline{b}\big)_k \\
&=\; \epsilon_{ijk}\,\partial_j\,(\epsilon_{klm}\,a_l b_m) \\
&=\; (\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl})\,\partial_j\,(a_l\,b_m) \\
&=\; \partial_j\,(a_i\,b_j\,) - \partial_j\,(a_j\,b_i) \\
&=\; (\partial_j\,a_i)\ b_j + a_i\ (\partial_j\,b_j) - (\partial_j\,a_j)\ b_i - a_j\ (\partial_j\,b_i) \\
&=\; \big(\underline{b}\cdot\underline{\nabla}\big)\ a_i + \big(\underline{\nabla}\cdot\underline{b}\big)\ a_i - \big(\underline{\nabla}\cdot\underline{a}\big)\ b_i - \big(\underline{a}\cdot\underline{\nabla}\big)\ b_i
\end{aligned}
$$

Other results involving one $\underline{\nabla}$ can be derived similarly.

Although identities 5, 6 & 7 may be used in explicit calculations, it's usually just as easy to apply the standard index notation rules (as we did in deriving them.)

Example: Show that $\underline{\nabla}\cdot\big(r^{-3}\underline{r}\big) = 0$, for $r \neq 0$ (where $r = |\underline{r}|$ as usual).

Method 1: Using identities and simple results: Using identity (3), we have

$$\underline{\nabla}\cdot\big(r^{-3}\underline{r}\big) \;=\; \big(\underline{\nabla}\,r^{-3}\big)\cdot\underline{r} \;+\; r^{-3}\ \big(\underline{\nabla}\cdot\underline{r}\big)$$

But we've shown previously that $\underline{\nabla}\,r^n = n\,r^{n-2}\,\underline{r}$ and $\underline{\nabla}\cdot\underline{r} = 3$. Hence

$$
\begin{aligned}
\underline{\nabla}\cdot\big(r^{-3}\underline{r}\big) \;&=\; \left(\frac{-3}{r^5}\,\underline{r}\right)\cdot\underline{r} \;+\; \frac{3}{r^3} \\
&=\; \frac{-3}{r^5}r^2 \;+\; \frac{3}{r^3} \;=\; 0 \qquad \text{(except at } r = 0\text{)}
\end{aligned}
$$

Method 2: Direct calculation using index notation:

$$\underline{\nabla} \cdot \left(r^{-3}\underline{r}\right) = \partial_i \left(x_i/r^3\right) = \left(\partial_i x_i\right)/r^3 + x_i\,\partial_i\,r^{-3}$$
$$= \delta_{ii}/r^3 + x_i\left(-3x_i/r^5\right)$$
$$= 3/r^3 - 3/r^3 = 0 \qquad \text{(except at } r = 0\text{)}$$

15.4 Identities involving two ∇s

8. $\underline{\nabla} \times \left(\underline{\nabla}\,\phi\right) = 0$ — curl grad ϕ is always zero.

9. $\underline{\nabla} \cdot \left(\underline{\nabla} \times \underline{a}\right) = 0$ — div curl $\underline{a}$ is always zero.

10. $\underline{\nabla} \times \left(\underline{\nabla} \times \underline{a}\right) = \underline{\nabla}\left(\underline{\nabla} \cdot \underline{a}\right) - \nabla^2 \underline{a}$

Proofs are obtained readily in Cartesian coordinates using suffix notation. You should know the first two, and knowing the second is useful – but you can always derive it from scratch.

Proof of (8):

$$\left(\underline{\nabla} \times \left(\underline{\nabla}\,\phi\right)\right)_i = \epsilon_{ijk}\,\partial_j\left(\underline{\nabla}\,\phi\right)_k = \epsilon_{ijk}\,\partial_j\,\partial_k\,\phi$$
$$= \epsilon_{ijk}\,\partial_k\,\partial_j\,\phi \qquad \left(\text{since } \frac{\partial^2\phi}{\partial x_1\,\partial x_2} = \frac{\partial^2\phi}{\partial x_2\,\partial x_1}\ etc\right)$$
$$= \epsilon_{ikj}\,\partial_j\,\partial_k\,\phi \qquad \text{(interchanging labels } j \text{ and } k\text{)}$$
$$= -\epsilon_{ijk}\,\partial_j\,\partial_k\,\phi \qquad (ikj \rightarrow ijk \text{ gives minus sign})$$
$$= -\left(\underline{\nabla} \times \left(\underline{\nabla}\,\phi\right)\right)_i = 0$$

since any vector equal to minus itself is must be zero. The proof of (9) is similar. It is important to understand how these two identities stem from the anti-symmetry of ϵ_{ijk}.

Identity (10) can be proven using the identity for the product of two epsilon symbols – tutorial. Again, the proof is far simpler than trying to use *'xyz'* – try both and see for yourself. It is an extremely important result and is used frequently in electromagnetism, fluid mechanics, and other 'field theories'.

Identity (10) is also used in curvilinear coordinate systems to *define* the action of the Laplacian on a vector field as

$$\nabla^2\,\underline{a} \equiv \underline{\nabla}\left(\underline{\nabla} \cdot \underline{a}\right) - \underline{\nabla} \times \left(\underline{\nabla} \times \underline{a}\right)$$

(See Junior Honours courses.) A mnemonic for the Laplacian acting on a vector field is *GDMCC – Grad-Div Minus Curl-Curl.*

Finally, when a scalar field ϕ depends only on the magnitude of the position vector $r = |\underline{r}|$, we have

$$\nabla^2\,\phi(r) = \phi''(r) + \frac{2\phi'(r)}{r} = \frac{1}{r^2}\left(r^2\phi'(r)\right)'$$

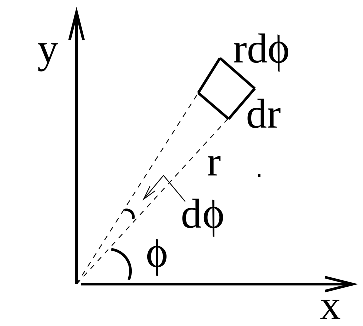
where the prime denotes differentiation with respect to r. Proof of this relation is left to the tutorial.

16 Scalar and Vector Integration and Line Integrals

16.1 Polar Coordinate Systems

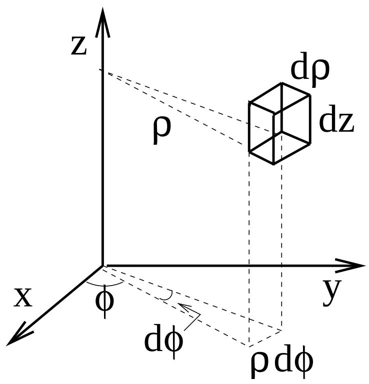
Before starting integral vector calculus we give a brief reminder of polar coordinate systems. In the figures below, dA indicates an area element and dV a volume element. Note that different conventions, *e.g.* for the angles ϕ and θ, are sometimes used.

Plane polar coordinates: (r, ϕ)



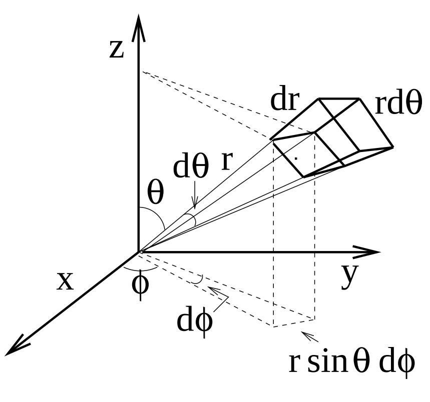
$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$dA = r\,dr\,d\phi$$

Cylindrical coordinates: (ρ, ϕ, z)



$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$
$$dV = \rho\,d\rho\,d\phi\,dz$$

Spherical polar coordinates: (r, θ, ϕ)



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$
$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

16.2 Scalar & Vector Integration

You should already be familar with integration in $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$. Here we review integration of a scalar field with an example.

Consider a hemisphere of radius a centered on the $\underline{e}_3$ axis, with its bottom face at $x_3 = 0$. If the mass density (a scalar field) is $\rho(r) = \sigma/r$ where σ is a constant, what is the total mass, M, of the hemisphere?

It is most convenient to use spherical polar coordinates. Then $dV = r^2 \sin\theta dr\, d\theta\, d\phi$ and

$$M \;=\; \int_{\text{hemisphere}} \rho(\underline{r})\, dV \;=\; \int_0^a r^2 \rho(r) dr \int_0^{\pi/2} \sin\theta d\theta \int_0^{2\pi} d\phi \;=\; 2\pi\sigma \int_0^a r dr \;=\; \pi\sigma a^2$$

Now consider the centre of mass vector

$$M\underline{R} = \int_V \underline{r}\, \rho(\underline{r})\, dV$$

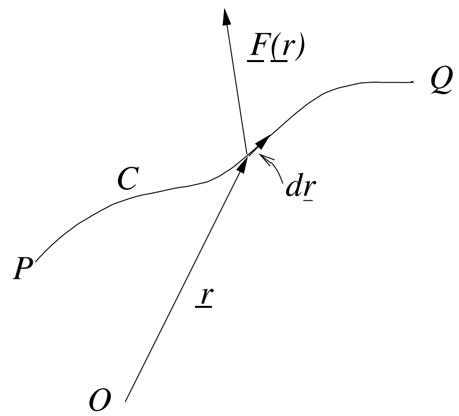
This is our first example of integrating a vector field, $\underline{r}\, \rho(\underline{r})$, in this example. To do this, we integrate each component in turn using $\underline{r} = r \sin\theta \cos\phi\, \underline{e}_1 + r \sin\theta \sin\phi\, \underline{e}_2 + r\cos\theta\, \underline{e}_3$

$$\begin{aligned}
MX &= \int_0^a r^3 \rho(r)\, dr \int_0^{\pi/2} \sin^2\theta d\theta \int_0^{2\pi} \cos\phi\ d\phi = 0 \quad \text{(since the } \phi \text{ integral gives 0)} \\
MY &= \int_0^a r^3 \rho(r) dr \int_0^{\pi/2} \sin^2\theta d\theta \int_0^{2\pi} \sin\phi\ d\phi = 0 \quad \text{(since the } \phi \text{ integral gives 0)} \\
MZ &= \int_0^a r^3 \rho(r) dr \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^{2\pi} d\phi \;=\; 2\pi\sigma \int_0^a r^2 dr \int_0^{\pi/2} \frac{\sin 2\theta}{2}\, d\theta \\
&= \frac{2\pi\sigma a^3}{3} \left[\frac{-\cos 2\theta}{4} \right]_0^{\pi/2} \;=\; \frac{\pi\sigma a^3}{3} \qquad \Rightarrow \quad \underline{R} = \frac{a}{3}\, \underline{e}_3
\end{aligned}$$

16.3 Line Integrals

As an example, consider a particle constrained to move on a wire. Only the component of the force along the wire does any work. Therefore the work, dW, by the force in moving the particle from $\underline{r}$ to $\underline{r} + d\underline{r}$ is

$$dW = \underline{F} \cdot d\underline{r}$$



The *total* work done in moving the particle along a wire which follows some curve C between two points P, Q is

$$W_C = \int_P^Q dW = \int_C \underline{F}(\underline{r}) \cdot d\underline{r}$$

This is a line integral along the curve C.

More generally, let $\underline{a}(\underline{r})$ be a vector field defined in the region R, and let C be a curve in R joining two points P and Q. As usual, $\underline{r}$ is the position vector at some point on the curve, and $d\underline{r}$ is an infinitesimal vector *along* the curve at $\underline{r}$.

The magnitude of $d\underline{r}$ is the infinitesimal *arc length*: $ds = \sqrt{d\underline{r} \cdot d\underline{r}}$

We define $\underline{t}$ to be the *unit vector* tangent to the curve at $\underline{r}$ (points in the direction of $d\underline{r}$)

$$\underline{t} \;=\; \frac{d\underline{r}}{ds}$$

Formally, we define the line integral $\int_C \underline{a} \cdot d\underline{r}$ as a (Riemann) sum by dividing the curve into intervals:

$$\int_C \underline{a} \cdot d\underline{r} \;=\; \int_C \underline{a} \cdot \underline{t}\, ds \;=\; \lim_{\substack{\delta s \to 0 \\ n \to \infty}} \sum_{i=0}^{n-1} \left(\underline{a}\left(\underline{r}^{(i)}\right) \cdot \underline{t}^{(i)} \right) \delta s^{(i)}$$

the i^{th} interval having length $\delta s^{(i)}$, unit tangent vector $\underline{t}^{(i)}$, *etc.* It can be shown that the limit is unique.

In general, $\int_C \underline{a} \cdot d\underline{r}$ *depends on the path* joining P and Q.

In Cartesian coordinates, we have

$$\int_C \underline{a} \cdot d\underline{r} \;=\; \int_C a_i\, dx_i \;=\; \int_C (a_1 dx_1 + a_2 dx_2 + a_3 dx_3)$$

16.4 Parametric Representation of a line integral

Often a curve in 3d can be parameterised by a single parameter, *e.g.* if the curve were the trajectory of a particle then the parameter would be the time t. Sometimes the parameter of a line integral is chosen to be the arc-length s along the curve C.

If we parameterise by λ (varying from λ_P to λ_Q) then

$$x_i \;=\; x_i(\lambda), \quad \text{with } \lambda_P \le \lambda \le \lambda_Q$$

and

$$\int_C \underline{a} \cdot d\underline{r} \;=\; \int_{\lambda_P}^{\lambda_Q} \left(\underline{a} \cdot \frac{d\underline{r}}{d\lambda} \right) d\lambda \;=\; \int_{\lambda_P}^{\lambda_Q} \left(a_1 \frac{dx_1}{d\lambda} + a_2 \frac{dx_2}{d\lambda} + a_3 \frac{dx_3}{d\lambda} \right) d\lambda$$

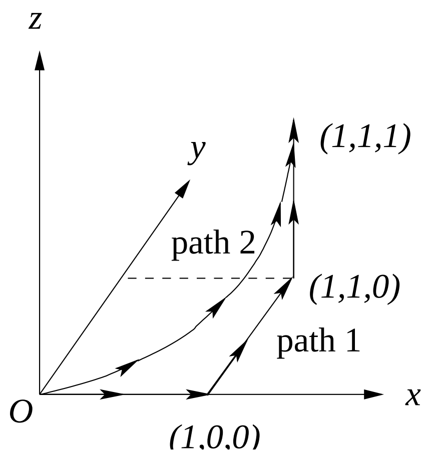
If necessary, the curve C may be subdivided into sections, each with a different parameterisation (piecewise smooth curve).

Example: Let $\underline{a} = (3x^2 + 6y)\, \underline{e}_x - 14yz\underline{e}_y + 20xz^2\underline{e}_z$. Evaluate $\int_C \underline{a} \cdot d\underline{r}$ between the points with Cartesian coordinates $(0, 0, 0)$ and $(1, 1, 1)$, along the two paths C:

(i) $(0, 0, 0) \to (1, 0, 0) \to (1, 1, 0) \to (1, 1, 1)$

(3 contiguous straight lines parallel to the x, y & z axes respectively.)

(ii) $x = \lambda$, $y = \lambda^2$, $z = \lambda^3$; from $\lambda = 0$ to $\lambda = 1$.



(i) (a) Along the line from $(0,0,0)$ to $(1,0,0)$, we have $y = z = 0$, so $dy = dz = 0$, hence $d\underline{r} = \underline{e}_x\, dx$ and $\underline{a} = 3x^2\, \underline{e}_x$ (the parameter is x here), and

$$\int_{(0,0,0)}^{(1,0,0)} \underline{a}\cdot d\underline{r} \;=\; \int_{x=0}^{x=1} 3x^2\, dx \;=\; \left[x^3\right]_0^1 \;=\; 1$$

(b) Along the line from $(1,0,0)$ to $(1,1,0)$, we have $x = 1$, $dx = 0$, $z = dz = 0$, so $d\underline{r} = \underline{e}_y\, dy$ (the parameter is y here), and

$$\underline{a} \;=\; \left(3x^2 + 6y\right)\Big|_{x=1}\, \underline{e}_x \;=\; (3+6y)\, \underline{e}_x$$

$$\Rightarrow \quad \int_{(1,0,0)}^{(1,1,0)} \underline{a}\cdot d\underline{r} \;=\; \int_{y=0}^{y=1} (3+6y)\; \underline{e}_x \cdot \underline{e}_y\, dy \;=\; 0$$

(c) Along the line from $(1,1,0)$ to $(1,1,1)$, we have $x = y = 1$, $dx = dy = 0$, and hence $d\underline{r} = \underline{e}_z\, dz$ and $\underline{a} = 9\,\underline{e}_x - 14z\,\underline{e}_y + 20z^2\,\underline{e}_z$, therefore

$$\int_{(1,1,0)}^{(1,1,1)} \underline{a}\cdot d\underline{r} \;=\; \int_{z=0}^{z=1} 20z^2\, dz \;=\; \left[\frac{20}{3} z^3\right]_0^1 \;=\; \frac{20}{3}$$

Adding the 3 contributions we get

$$\int_C \underline{a}\cdot d\underline{r} \;=\; 1 + 0 + \frac{20}{3} \;=\; \frac{23}{3} \qquad \text{along path (1)}$$

(ii) To integrate $\underline{a} = (3x^2 + 6y)\, \underline{e}_x - 14yz\underline{e}_y + 20xz^2\underline{e}_z$ along path (2), we parameterise

$$\begin{aligned} \underline{r} &= \lambda\, \underline{e}_x + \lambda^2\, \underline{e}_y + \lambda^3\, \underline{e}_z \\ \frac{d\underline{r}}{d\lambda} &= \underline{e}_x + 2\lambda\, \underline{e}_y + 3\lambda^2\, \underline{e}_z \\ \underline{a} &= \left(3\lambda^2 + 6\lambda^2\right)\, \underline{e}_x - 14\lambda^5\, \underline{e}_y + 20\lambda^7\, \underline{e}_z \quad \text{so that} \end{aligned}$$

$$\int_C \left(\underline{a}\cdot\frac{d\underline{r}}{d\lambda}\right) d\lambda \;=\; \int_{\lambda=0}^{\lambda=1} \left(9\lambda^2 - 28\lambda^6 + 60\lambda^9\right)\, d\lambda \;=\; \left[3\lambda^3 - 4\lambda^7 + 6\lambda^{10}\right]_0^1 \;=\; 5$$

Hence $\displaystyle\int_C \underline{a}\cdot d\underline{r} \;=\; 5$ along path (2)

In this case, the integral of $\underline{a}$ from $(0,0,0)$ to $(1,1,1)$ depends on the path taken.

The line integral $\int_C \underline{a} \cdot d\underline{r}$ is a *scalar* quantity. Another *scalar* line integral is $\int_C f\, ds$ where $f(\underline{r})$ is a scalar field and ds is the infinitesimal arc-length introduced earlier.

Line integrals around a *simple* (doesn't intersect itself) *closed* curve C are denoted by $\oint_C$

$$e.g. \qquad \oint_C \underline{a} \cdot d\underline{r} \quad \equiv \quad \text{the } \textit{circulation} \text{ of } \underline{a} \text{ around } C$$

Example: Let $f(\underline{r}) = ax^2 + by^2$. Evaluate $\oint_C f\, ds$ around the unit circle C centred on the origin in the $x-y$ plane:

$$x = \cos\phi,\ y = \sin\phi,\ z = 0; \quad 0 \le \phi \le 2\pi.$$

We have

$$\begin{aligned} f(\underline{r}) &= ax^2 + by^2 = a\cos^2\phi + b\sin^2\phi \\ \underline{r} &= \cos\phi\, \underline{e}_x + \sin\phi\, \underline{e}_y \\ d\underline{r} &= \left(-\sin\phi\, \underline{e}_x + \cos\phi\, \underline{e}_y\right) d\phi \\ \text{so} \quad ds &= \sqrt{d\underline{r} \cdot d\underline{r}} = \left(\sin^2\phi + \cos^2\phi\right)^{1/2} d\phi = d\phi \end{aligned}$$

Therefore, for this example,

$$\oint_C f\, ds = \int_0^{2\pi} \left(a\cos^2\phi + b\sin^2\phi\right) d\phi = \pi\,(a+b)$$

The *length* s of a curve C is given by $s = \int_C ds$. In this example $s = 2\pi$.

We can also define *vector* line integrals, *e.g.*

(i) $\int_C \underline{a}\, ds = \underline{e}_i \int_C a_i\, ds$ in Cartesian coordinates.

(ii) $\int_C \underline{a} \times d\underline{r} = \underline{e}_i\, \epsilon_{ijk} \int_C a_j\, dx_k$ in Cartesian coordinates.

(iii) $\int_C f\, d\underline{r} = \underline{e}_i \int_C f\, dx_i$ in Cartesian coordinates.

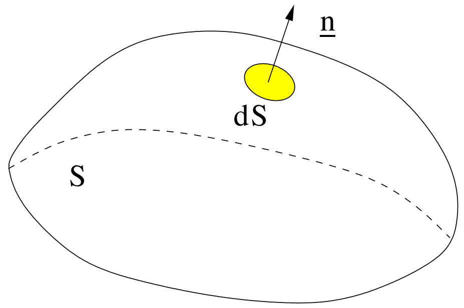
Example: Consider a current of magnitude I flowing along a wire following a closed path C. The magnetic force on an element $d\underline{r}$ of the wire is $I d\underline{r} \times \underline{B}$ where $\underline{B}$ is the magnetic field at $\underline{r}$. Let $\underline{B}(\underline{r}) = x\, \underline{e}_x + y\, \underline{e}_y$. Evaluate $\oint_C \underline{B} \times d\underline{r}$ for a circular current loop of radius a in the $x-y$ plane, centred on the origin.

$$\begin{aligned} \underline{B} &= a\cos\phi\, \underline{e}_x + a\sin\phi\, \underline{e}_y \\ d\underline{r} &= \left(-a\sin\phi\, \underline{e}_x + a\cos\phi\, \underline{e}_y\right) d\phi \end{aligned}$$

Hence

$$\oint_C \underline{B} \times d\underline{r} = \int_0^{2\pi} \left(a^2\cos^2\phi + a^2\sin^2\phi\right) \underline{e}_z\, d\phi = \underline{e}_z\, a^2 \int_0^{2\pi} d\phi = 2\pi a^2\, \underline{e}_z$$

17 Surface Integrals



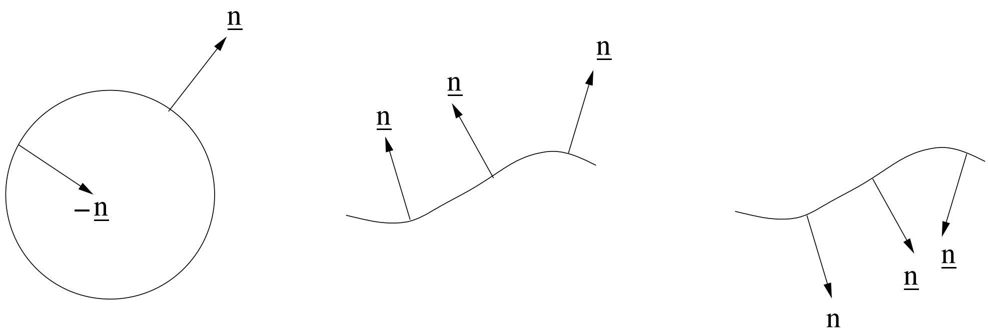
Let S be a two-sided surface in ordinary three-dimensional space as shown. If an infinitesimal element of surface with (scalar) area dS has unit normal $\underline{n}$, then the infinitesimal *vector element of area* is *defined* by

$$\boxed{d\underline{S} \; = \; \underline{n}\, dS}$$

Example: If S lies in the (x, y) plane, then $d\underline{S} = \underline{e}_z \, dx\, dy$ in Cartesian coordinates.

Physical interpretation: $\hat{\underline{a}} \cdot d\underline{S}$ gives the projected (scalar) element of area onto the plane with unit normal $\hat{\underline{a}}$.

For *closed* surfaces (*e.g.* a sphere) we *choose* $\underline{n}$ to be the *outward* normal. For *open* surfaces, the sense of $\underline{n}$ is arbitrary – except that it is chosen in the same sense for all elements of the surface.



If $\underline{a}(\underline{r})$ is a vector field defined on S, we define the (normal) *surface integral*

$$\int_S \underline{a} \cdot d\underline{S} \;=\; \int_S \underline{a} \cdot \underline{n}\, dS \;=\; \lim_{\substack{m \to \infty \\ \delta S \to 0}} \sum_{i=1}^{m} \left(\underline{a}\left(\underline{r}^{(i)}\right) \cdot \underline{n}^{(i)} \right) \, \delta S^{(i)}$$

where we have formed the (Riemann) sum by dividing the surface S into m small areas, the i^{th} area having vector area $\delta \underline{S}^{(i)}$. Clearly, the quantity $\underline{a}(\underline{r}^{(i)}) \cdot \underline{n}^{(i)}$ is the component of $\underline{a}$ *normal* to the surface at the point $\underline{r}^{(i)}$

- We use the notation $\int_S \underline{a} \cdot d\underline{S}$ for both *open* and *closed* surfaces. Sometimes the integral over a *closed* surface is denoted by $\oint_S \underline{a} \cdot d\underline{S}$ (*not* used here).
- Note that the integral over S is a *double integral* in each case. Hence surface integrals are sometimes denoted by $\iint_S \underline{a} \cdot d\underline{S}$.

Example: Let S be the surface of a unit cube ($S =$ sum over *all six faces*).

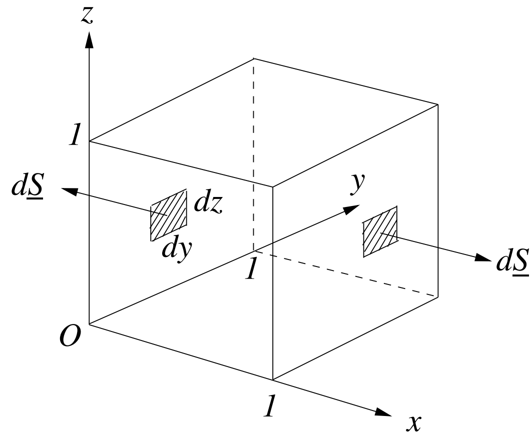
On the *front* face, parallel to the (y,z) plane, at $x = 1$,

$$d\underline{S} \;=\; \underline{n}\, dS \;=\; \underline{e}_x\, dy\, dz$$

On the *back* face at $x = 0$ in the (y,z) plane,

$$d\underline{S} \;=\; \underline{n}\, dS \;=\; -\,\underline{e}_x\, dy\, dz$$

In each case, the unit normal $\underline{n}$ is an *outward* normal because S is a *closed* surface.



If $\underline{a}(\underline{r})$ is a vector field, then the integral $\int_S \underline{a} \cdot d\underline{S}$ over the front face shown is

$$\int_{z=0}^{z=1} \int_{y=0}^{y=1} \underline{a} \cdot \underline{e}_x\, dy\, dz \;=\; \int_{z=0}^{z=1} \int_{y=0}^{y=1} a_x \bigg|_{x=1} dy\, dz$$

The integral over y and z is an ordinary double integral over a square of side 1. The integral over the back face is

$$-\int_{z=0}^{z=1} \int_{y=0}^{y=1} \underline{a} \cdot \underline{e}_x\, dy\, dz \;=\; -\int_{z=0}^{z=1} \int_{y=0}^{y=1} a_x \bigg|_{x=0} dy\, dz$$

The total integral is the sum of contributions from all 6 faces.

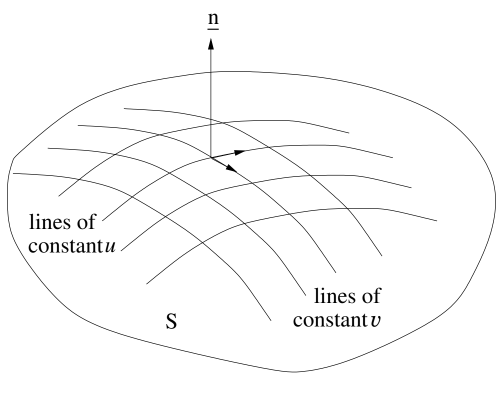
17.1 Parametric form of the surface integral

Suppose the points on a surface S are defined by *two* real parameters u and v

$$\underline{r} \;=\; \underline{r}(u,v) \;=\; (x(u,v),\, y(u,v),\, z(u,v))$$

- the lines $\underline{r}(u,v)$ for fixed u, variable v, and
- the lines $\underline{r}(u,v)$ for fixed v, variable u

are *parametric lines* and form a *grid* on the surface S as shown.



If we change u and v by du and dv respectively, then $\underline{r}$ changes by $d\underline{r}$ with

$$d\underline{r} \;=\; \frac{\partial \underline{r}}{\partial u}\, du + \frac{\partial \underline{r}}{\partial v}\, dv$$

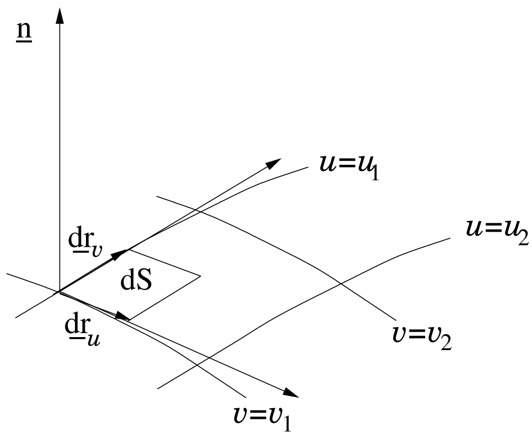
Along the curves $v = constant$, we have $dv = 0$, and so $d\underline{r}$ is simply

$$d\underline{r}_u \;=\; \frac{\partial \underline{r}}{\partial u}\, du$$

where $\dfrac{\partial \underline{r}}{\partial u}$ is a vector which is tangent to the surface, and tangent to the lines $v = constant$.
Similarly, for $u = constant$, we have

$$d\underline{r}_v \;=\; \frac{\partial \underline{r}}{\partial v}\, dv$$

so $\dfrac{\partial \underline{r}}{\partial v}$ is tangent to lines $u = constant$.



We can therefore construct a *unit vector* $\underline{n}$, *normal* to the surface at $\underline{r}$

$$\underline{n} \;=\; \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right) \Big/ \left|\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right|$$

The vector element of area, $d\underline{S}$, has magnitude equal to the area of the infinitesimal parallelogram shown, and points in the direction of $\underline{n}$, therefore we can write

$$d\underline{S} \;=\; d\underline{r}_u \times d\underline{r}_v \;=\; \left(\frac{\partial \underline{r}}{\partial u}\, du\right) \times \left(\frac{\partial \underline{r}}{\partial v}\, dv\right) \;=\; \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right) du\, dv$$

$$\boxed{d\underline{S} = \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right) du\, dv}$$

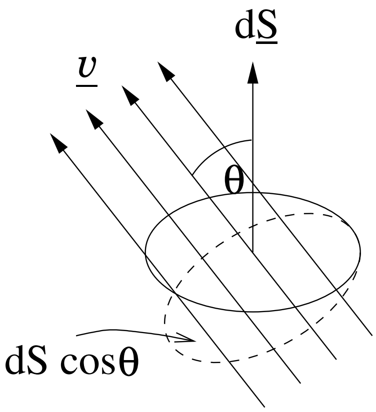
Finally, our integral is parameterised as

$$\boxed{\int_S \underline{a} \cdot d\underline{S} \;=\; \int_v \int_u \underline{a} \cdot \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right) du\, dv}$$

We use two integral signs when writing surface integrals in terms of *explicit* parameters u and v. The limits for the integrals over u and v must be chosen appropriately for the surface.

18 More on Surface and Volume Integrals

18.1 The Concept of Flux



Let $\underline{v}(\underline{r})$ be the velocity at a point $\underline{r}$ in a moving fluid. In a small region, where $\underline{v}$ is approximately constant, the *volume* of fluid crossing the element of vector area $d\underline{S} = \underline{n}\, dS$ in time dt is

$$\left(|\underline{v}|\, dt\right)\ (dS\, \cos\theta) \quad = \quad \left(\underline{v} \cdot d\underline{S}\right) dt$$

because the area *normal* to the direction of flow is $\hat{\underline{v}} \cdot d\underline{S} = dS\, \cos\theta$.

Therefore

$$\underline{v} \cdot d\underline{S} \quad = \quad \textit{volume per unit time} \text{ of fluid crossing } d\underline{S}$$

$$\text{hence} \quad \int_S \underline{v} \cdot d\underline{S} \quad = \quad \textit{volume per unit time} \text{ of fluid crossing a finite surface } S$$

More generally, for a vector field $\underline{a}(\underline{r})$,

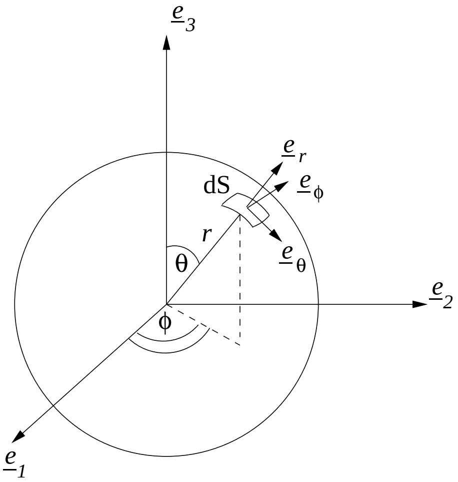
The surface integral $\int_S \underline{a} \cdot d\underline{S}$ is called the *flux* of $\underline{a}$ through the surface S

The concept of flux is useful in many different contexts *e.g.* flux of molecules in a gas; electromagnetic flux, *etc.*

Example: Let S be the surface of sphere $x_1^2 + x_2^2 + x_3^2 = R^2$. Find $\underline{n}$ and $d\underline{S}$, and evaluate the total flux of the vector field $\underline{a}(\underline{r}) = \hat{\underline{r}}/r^2$ out of the sphere.

An arbitrary point $\underline{r}$ on S may be parameterised in spherical polar co-ordinates θ and ϕ as

$$\begin{aligned}
\underline{r} &= R\sin\theta\cos\phi\, \underline{e}_1 + R\sin\theta\sin\phi\, \underline{e}_2 + R\cos\theta\, \underline{e}_3 \qquad \{0 \le \theta < \pi,\ 0 \le \phi < 2\pi\} \\
\text{so} \quad \frac{\partial \underline{r}}{\partial \theta} &= R\cos\theta\cos\phi\, \underline{e}_1 + R\cos\theta\sin\phi\, \underline{e}_2 - R\sin\theta\, \underline{e}_3 \\
\text{and} \quad \frac{\partial \underline{r}}{\partial \phi} &= -R\sin\theta\sin\phi\, \underline{e}_1 + R\sin\theta\cos\phi\, \underline{e}_2 + 0\, \underline{e}_3
\end{aligned}$$



Therefore

$$
\begin{aligned}
\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} &= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ R\,\cos\theta\cos\phi & R\,\cos\theta\sin\phi & -R\,\sin\theta \\ -R\sin\theta\sin\phi & R\sin\theta\cos\phi & 0 \end{vmatrix} \\
&= R^2\,\sin^2\theta\cos\phi\,\underline{e}_1 + R^2\,\sin^2\theta\sin\phi\,\underline{e}_2 + R^2\,\sin\theta\cos\theta\,\left(\cos^2\phi + \sin^2\phi\right)\underline{e}_3 \\
&= R^2\sin\theta\,\left(\sin\theta\cos\phi\,\underline{e}_1 + \sin\theta\sin\phi\,\underline{e}_2 + \cos\theta\,\underline{e}_3\right) \\
&= R^2\sin\theta\,\hat{\underline{r}}
\end{aligned}
$$

Hence $\quad \underline{n} = \hat{\underline{r}}$

and $\quad d\underline{S} = \frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi}\,d\theta\,d\phi = R^2\sin\theta d\theta\,d\phi\,\hat{\underline{r}}$

This provides an algebraic derivation of the result we obtained geometrically in Section (16.1). On the surface S, we have $r = R$, and so the vector field $\underline{a}(\underline{r}) = \hat{\underline{r}}/R^2$. The flux of $\underline{a}$ through the closed surface S is then

$$\int_S \underline{a}\cdot d\underline{S} = \int_0^{\pi}\sin\theta\,d\theta\,\int_0^{2\pi}d\phi = 4\pi$$

In this simple example, the result of the integral is just the surface area of a unit sphere.

Spherical basis: The normalised vectors (shown in the figure)

$$\underline{e}_r = \hat{\underline{r}}\,, \qquad \underline{e}_\theta = \frac{\partial \underline{r}}{\partial \theta}\bigg/\left|\frac{\partial \underline{r}}{\partial \theta}\right|\,, \qquad \underline{e}_\phi = \frac{\partial \underline{r}}{\partial \phi}\bigg/\left|\frac{\partial \underline{r}}{\partial \phi}\right|\,,$$

which we may write explicitly as

$$
\begin{aligned}
\underline{e}_r &= \sin\theta\cos\phi\,\underline{e}_1 + \sin\theta\sin\phi\,\underline{e}_2 + \cos\theta\,\underline{e}_3 \\
\underline{e}_\theta &= \cos\theta\cos\phi\,\underline{e}_1 + \cos\theta\sin\phi\,\underline{e}_2 - \sin\theta\,\underline{e}_3 \\
\underline{e}_\phi &= -\sin\phi\,\underline{e}_1 + \cos\phi\underline{e}_2
\end{aligned}
$$

form an *orthonormal* basis, *i.e.* $\underline{e}_r\cdot\underline{e}_\theta = \underline{e}_\theta\cdot\underline{e}_\phi = \underline{e}_\phi\cdot\underline{e}_r = 0$.

Since $\underline{e}_r$, $\underline{e}_\theta$ and $\underline{e}_\phi$ depend on position $\underline{r}$, they form a non-Cartesian or *curvilinear* basis.

18.2 Other Surface Integrals

If $f(\underline{r})$ is a scalar field, we may define a scalar surface integral

$$\int_S f\, dS$$

For example, the *surface area* of the surface S is

$$\int_S dS \;=\; \int_S \left|d\underline{S}\right| \;=\; \int_v \int_u \left| \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right| \, du\, dv$$

We may also define vector surface integrals

$$\int_S f\, d\underline{S} \qquad \int_S \underline{a}\, dS \qquad \int_S \underline{a} \times d\underline{S}$$

Each of these is a double integral, and is evaluated in a similar fashion to the scalar integrals, the result being a vector in each case.

Example: The vector area of the (open) hemisphere, $x_1^2 + x_2^2 + x_3^2 = R^2$, $(x_3 \geq 0)$, of radius R, is, using spherical polars,

$$\underline{S} \;=\; \int_S d\underline{S} \;=\; \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} R^2 \sin\theta\, \underline{e}_r\, d\theta\, d\phi\ .$$

Using $\underline{e}_r = \sin\theta\cos\phi\,\underline{e}_1 + \sin\theta\sin\phi\,\underline{e}_2 + \cos\theta\,\underline{e}_3$ we obtain

$$\begin{aligned}
\underline{S} \;&=\; \underline{e}_1 R^2 \int_0^{\pi/2} \sin^2\theta\, d\theta \int_0^{2\pi} \cos\phi\, d\phi \;+\; \underline{e}_2 R^2 \int_0^{\pi/2} \sin^2\theta\, d\theta \int_0^{2\pi} \sin\phi\, d\phi \;+ \\
&\quad\; \underline{e}_3 R^2 \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^{2\pi} d\phi \\
&=\; 0 + 0 + \pi R^2\, \underline{e}_3
\end{aligned}$$

The vector surface integral over the full sphere is zero because the contributions from the upper and lower hemispheres cancel. Similarly, the vector area of a *closed* hemisphere is zero because the vector area of the bottom face is $-\pi R^2\,\underline{e}_3$.

In fact, for *any* closed surface,

$$\int_S d\underline{S} = 0$$

For a proof, it is simplest to use the divergence theorem applied to an arbitrary constant vector $\underline{c}$ – see later.)

18.3 Parametric form of Volume Integrals

We have already met and revised volume integrals in Section (16.2). They are conceptually simpler than line and surface integrals because the element of volume dV is a scalar quantity.

In this section we discuss the *parametric* form of volume integrals. Suppose we can write $\underline{r}$ in terms of three real parameters u, v and w, so that $\underline{r} = \underline{r}(u, v, w)$. If we make a small change in each of these parameters, then $\underline{r}$ changes by

$$d\underline{r} \;=\; \frac{\partial \underline{r}}{\partial u}\, du + \frac{\partial \underline{r}}{\partial v}\, dv + \frac{\partial \underline{r}}{\partial w}\, dw$$

Along the curves $\{v = \text{constant},\ w = \text{constant}\}$, we have $dv = dw = 0$, so $d\underline{r}$ is simply

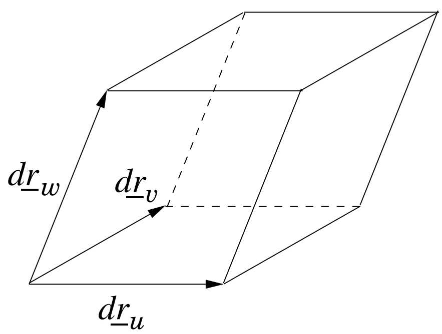
$$d\underline{r}_u \;=\; \frac{\partial \underline{r}}{\partial u}\, du$$

with $d\underline{r}_v$ and $d\underline{r}_w$ having analogous definitions.

The vectors $d\underline{r}_u$, $d\underline{r}_v$ and $d\underline{r}_w$ form the sides of an infinitesimal parallelepiped of volume

$$dV \;=\; \left| d\underline{r}_u \cdot d\underline{r}_v \times d\underline{r}_w \right|$$

$$\boxed{dV \;=\; \left| \frac{\partial \underline{r}}{\partial u} \cdot \frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial w} \right| du\, dv\, dw}$$



Example: Consider a circular cylinder of radius a, height c. We can parameterise $\underline{r}$ using cylindrical coordinates. Within the cylinder, we have

$$\underline{r} \;=\; \rho \cos\phi\, \underline{e}_1 + \rho \sin\phi\, \underline{e}_2 + z\underline{e}_3 \quad \{0 \le \rho \le a,\ 0 \le \phi \le 2\pi,\ 0 \le z \le c\}$$

Thus

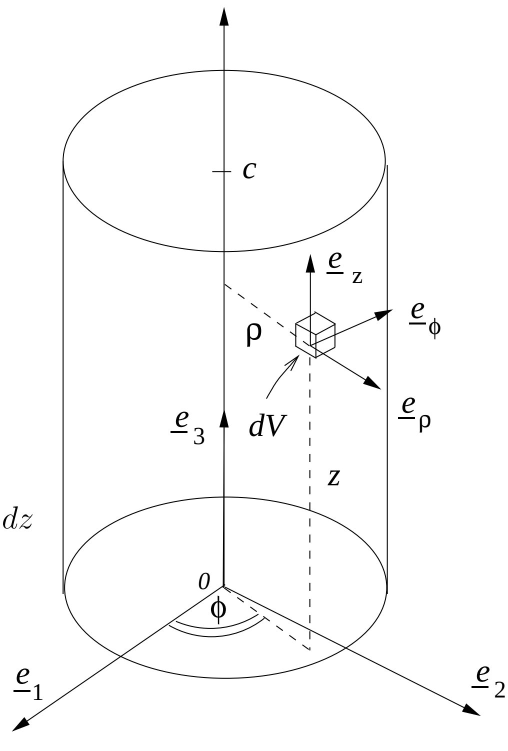
$$\frac{\partial \underline{r}}{\partial \rho} \;=\; \cos\phi\, \underline{e}_1 + \sin\phi\, \underline{e}_2$$

$$\frac{\partial \underline{r}}{\partial \phi} \;=\; -\rho\, \sin\phi\, \underline{e}_1 + \rho\, \cos\phi\, \underline{e}_2$$

$$\frac{\partial \underline{r}}{\partial z} \;=\; \underline{e}_3$$

And so

$$dV \;=\; \left| \frac{\partial \underline{r}}{\partial \rho} \cdot \frac{\partial \underline{r}}{\partial \phi} \times \frac{\partial \underline{r}}{\partial z} \right| d\rho\, d\phi\, dz \;=\; \rho\, d\rho\, d\phi\, dz$$



The *volume* of the cylinder is just

$$\int_V dV \;=\; \int_{z=0}^{z=c}\int_{\phi=0}^{\phi=2\pi}\int_{\rho=0}^{\rho=a} \rho\, d\rho\, d\phi\, dz \;=\; \pi\, a^2 c.$$

Cylindrical basis: the normalised vectors

$$\underline{e}_\rho \;=\; \frac{\partial \underline{r}}{\partial \rho} \Big/ \left|\frac{\partial \underline{r}}{\partial \rho}\right| \quad ; \quad \underline{e}_\phi \;=\; \frac{\partial \underline{r}}{\partial \phi} \Big/ \left|\frac{\partial \underline{r}}{\partial \phi}\right| \quad ; \quad \underline{e}_z \;=\; \frac{\partial \underline{r}}{\partial z} \Big/ \left|\frac{\partial \underline{r}}{\partial z}\right|$$

(shown on the figure) form an orthonormal curvilinear non-Cartesian basis.

Exercise: For spherical polars: $\underline{r} = r\sin\theta\cos\phi\,\underline{e}_1 + r\sin\theta\sin\phi\,\underline{e}_2 + r\cos\theta\,\underline{e}_3$ show that

$$dV \;=\; \left|\frac{\partial \underline{r}}{\partial r}\cdot\frac{\partial \underline{r}}{\partial \theta}\times\frac{\partial \underline{r}}{\partial \phi}\right| dr\, d\theta\, d\phi \;=\; r^2\sin\theta\, dr\, d\theta\, d\phi$$

19 The Divergence Theorem

19.1 Integral definition of divergence

Let $\underline{a}$ be a vector field in the region R, and let P be a point in R, then the divergence of $\underline{a}$ at P may be *defined* by

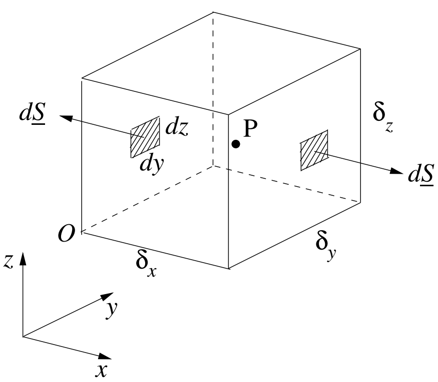
$$\text{div}\,\underline{a} \;=\; \lim_{\delta \mathrm{V}\to 0} \frac{1}{\delta \mathrm{V}} \int_{\delta \mathrm{S}} \underline{a}\cdot \mathrm{d}\underline{S}$$

where δS is a *closed* surface in R which encloses the volume δV. The limit must be taken so that the point P is within δV. (We assume the limit is independent of the shape of δV.)

This definition of $\text{div}\,\underline{a}$ is *basis independent.*

We now prove that our original definition of *div* is recovered in Cartesian co-ordinates

Let P be a point with Cartesian coordinates $(x_0,\, y_0,\, z_0)$ situated at the *centre* of a small rectangular block of size $\delta_x \times \delta_y \times \delta_z$, so that its volume is $\delta V = \delta_x\,\delta_y\,\delta_z$.



- On the *front* face of the block, orthogonal to the x axis at $x = x_0 + \delta_x/2$, we have *outward* normal $\underline{n} = \underline{e}_x$ and so $d\underline{S} = \underline{e}_x\, dy\, dz$
- On the *back* face of the block, orthogonal to the x axis at $x = x_0 - \delta_x/2$, we have *outward* normal $\underline{n} = -\underline{e}_x$ and so $d\underline{S} = -\underline{e}_x\, dy\, dz$

Hence $\underline{a}\cdot d\underline{S} = \pm\, a_x\, dy\, dz$ on these two faces. Let us denote the union of the two surfaces orthogonal to the $\underline{e}_1$ axis by S_x.

The contribution of these two surfaces to the integral $\int_S \underline{a} \cdot d\underline{S}$ is given by

$$\begin{aligned}
\int_{S_x} \underline{a} \cdot d\underline{S} &= \int_z\int_y \left\{ a_x(x_0 + \delta_x/2, y, z) - a_x(x_0 - \delta_x/2, y, z) \right\} dy\, dz \\
&= \int_z\int_y \left\{ \left[a_x(x_0, y, z) + \frac{\delta_x}{2}\frac{\partial a_x}{\partial x}(x_0, y, z) + O(\delta_x^2) \right] \right. \\
&\qquad \left. - \left[a_x(x_0, y, z) - \frac{\delta_x}{2}\frac{\partial a_x}{\partial x}(x_0, y, z) + O(\delta_x^2) \right] \right\} dy\, dz \\
&= \int_z\int_y \delta_x \frac{\partial a_x}{\partial x}(x_0, y, z)\, dy\, dz
\end{aligned}$$

where we have dropped terms of $O(\delta_x^2)$ in the Taylor expansion of a_x about (x_0, y, z).

So

$$\frac{1}{\delta V}\int_{S_x} \underline{a} \cdot d\underline{S} = \frac{1}{\delta_y\,\delta_z}\int_z\int_y \frac{\partial a_x}{\partial x}(x_0, y, z)\, dy\, dz$$

As we take the limit $\delta_x, \delta_y, \delta_z \to 0$ the integral tends to $\left.\frac{\partial a_x}{\partial x}\right|_{(x_0,y_0,z_0)} \delta_y\,\delta_z$ and we obtain

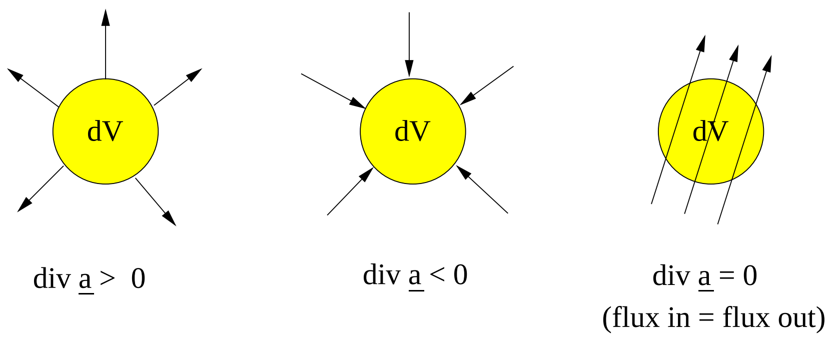
$$\lim_{\delta V \to 0} \frac{1}{\delta V}\int_{S_x} \underline{a} \cdot d\underline{S} = \frac{\partial a_x}{\partial x}(x_0, y_0, z_0)$$

With similar contributions from the other 4 faces, we find

$$\text{div}\,\underline{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = \underline{\nabla} \cdot \underline{a}$$

in agreement with our original definition in Cartesian co-ordinates. Thus we can continue to use the notation $\underline{\nabla} \cdot \underline{a}$ for *both* forms.

Note that the integral definition gives an intuitive understanding of the divergence in terms of net flux leaving a small volume around a point $\underline{r}$. *In pictures:* for a small volume dV



19.2 The Divergence Theorem (Gauss' Theorem)

Let $\underline{a}$ be a vector field in a volume V, and S be the closed surface bounding V, then

$$\boxed{\int_V \underline{\nabla} \cdot \underline{a}\, dV \;=\; \int_S \underline{a} \cdot d\underline{S}}$$

Proof: We derive the divergence theorem by making use of the integral definition of $\underline{\nabla} \cdot \underline{a}$

$$\underline{\nabla} \cdot \underline{a} \;=\; \lim_{\delta V \to 0} \frac{1}{\delta V} \int_{\delta S} \underline{a} \cdot d\underline{S}.$$

Since this *definition* of $\underline{\nabla} \cdot \underline{a}$ is valid for volumes of arbitrary shape, we can build a smooth surface S from a large number, N, of blocks of volume $\Delta V^{(i)}$ and surface $\Delta S^{(i)}$. We have

$$\underline{\nabla} \cdot \underline{a}(\underline{r}^{(i)}) \;=\; \frac{1}{\Delta V^{(i)}} \int_{\Delta S^{(i)}} \underline{a} \cdot d\underline{S} + O(\epsilon^{(i)})$$

where $\epsilon^{(i)} \to 0$ as $\Delta V^{(i)} \to 0$. Now multiply both sides by $\Delta V^{(i)}$ and sum over all i

$$\sum_{i=1}^{N} \underline{\nabla} \cdot \underline{a}(\underline{r}^{(i)})\, \Delta V^{(i)} \;=\; \sum_{i=1}^{N} \int_{\Delta S^{(i)}} \underline{a} \cdot d\underline{S} \;+\; \sum_{i=1}^{N} \epsilon^{(i)}\, \Delta V^{(i)}$$

On the RHS the contributions from surface elements *interior* to S cancel. This is because where two blocks touch, the outward normals are in *opposite* directions, implying that the contributions to the respective integrals cancel.

Taking the limit $N \to \infty$ we have, as claimed,

$$\int_V \underline{\nabla} \cdot \underline{a}\, dV \;=\; \int_S \underline{a} \cdot d\underline{S}\,.$$

19.3 Examples of the use of the Divergence Theorem

Volume of a body: This is simply given by

$$V = \int_V dV$$

Recalling that $\underline{\nabla} \cdot \underline{r} = 3$ we can write

$$V = \frac{1}{3} \int_V \underline{\nabla} \cdot \underline{r}\, dV$$

On applying the divergence theorem, this becomes

$$V = \frac{1}{3} \int_S \underline{r} \cdot d\underline{S}$$

Example: Consider the hemisphere $x^2 + y^2 + z^2 \le R^2$ centered on $\underline{e}_3$ with its bottom face at $z = 0$. Recalling that the divergence theorem holds for a *closed* surface, the above equation for the volume of the hemisphere tells us

$$V = \frac{1}{3}\left[\int_{\text{hemisphere}} \underline{r} \cdot d\underline{S} + \int_{\text{bottom}} \underline{r} \cdot d\underline{S}\right] .$$

On the bottom face $d\underline{S} = -\underline{e}_z\, dS$ so that $\underline{r} \cdot d\underline{S} = -z\, dS = 0$ since $z = 0$. Hence the only contribution comes from the (open) surface of the hemisphere and we see that

$$V = \frac{1}{3}\int_{\text{hemisphere}} \underline{r} \cdot d\underline{S} .$$

We can evaluate this by using spherical polars for the surface integral. For a hemisphere of radius R we showed previously that

$$d\underline{S} = R^2 \sin\theta\, d\theta\, d\phi\, \underline{e}_r .$$

On the hemisphere, $\underline{r} \cdot d\underline{S} = R\,\underline{e}_r \cdot d\underline{S} = R^3 \sin\theta\, d\theta\, d\phi$ so that

$$\int_S \underline{r} \cdot d\underline{S} = R^3 \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 2\pi R^3$$

giving the anticipated result

$$V = \frac{2\pi R^3}{3}$$

19.4 The Continuity Equation

Consider a fluid with density field $\rho(\underline{r})$ and velocity field $\underline{v}(\underline{r})$. We have seen previously that the volume flux (volume per unit time) flowing across a surface is given by $\int_S \underline{v} \cdot d\underline{S}$. The corresponding mass flux (mass per unit time) is given by

$$\int_S (\rho\underline{v}) \cdot d\underline{S} \equiv \int_S \underline{J} \cdot d\underline{S}$$

where $\underline{J} = \rho\underline{v}$ is called the *mass current density* .

Now consider a volume V bounded by the *closed* surface S containing no sources or sinks of fluid. Conservation of mass means that the outward mass flux through the surface S must be equal to the rate of *decrease* of mass contained in the volume V.

$$\int_S \underline{J} \cdot d\underline{S} = -\frac{\partial M}{\partial t} .$$

The mass in V may be written as $M = \int_V \rho\, dV$. Therefore we have

$$\frac{\partial}{\partial t}\int_V \rho\, dV + \int_S \underline{J} \cdot d\underline{S} = 0 .$$

We now use the divergence theorem to rewrite the second term as a volume integral and we obtain

$$\int_V \left[\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J}\right] dV = 0$$

Since this holds for arbitrary V, we must have that

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0}$$

This equation, known as the *continuity equation*, appears in many different contexts because it holds for any *conserved* quantity. Here we considered mass density ρ and mass current density $\underline{J}$ of a fluid; but equally it could have been number density of molecules in a gas and current density of molecules; electric charge density and electric current density vector; thermal energy density and heat current density vector; or even more abstract conserved quantities such as probability density and probability current density in quantum mechanics.

To understand better the divergence of a vector field consider the divergence of the current in the continuity equation:

$$\text{if} \quad \underline{\nabla} \cdot \underline{J}(\underline{r}) > 0 \quad \text{then} \quad \frac{\partial \rho}{\partial t} < 0 \quad \text{and the mass density at } \underline{r} \text{ decreases}$$

$$\text{if} \quad \underline{\nabla} \cdot \underline{J}(\underline{r}) < 0 \quad \text{then} \quad \frac{\partial \rho}{\partial t} > 0 \quad \text{and the mass density at } \underline{r} \text{ increases}$$

19.5 Sources and Sinks

Static case: Consider *time independent* behaviour where $\frac{\partial \rho}{\partial t} = 0$. The continuity equation tells us that for the density to be constant in time we must have $\underline{\nabla} \cdot \underline{J} = 0$ so that flux into a point equals flux out.

However if we have a *source* or a *sink* of the field, the divergence is not zero at that point. In general the quantity

$$\frac{1}{V} \int_S \underline{a} \cdot d\underline{S}$$

tells us whether there are sources or sinks of the vector field $\underline{a}$ within V. If V contains

- a *source*, then $\displaystyle\int_S \underline{a} \cdot d\underline{S} \;=\; \int_V \underline{\nabla} \cdot \underline{a}\, dV \;>\; 0$
- a *sink*, then $\displaystyle\int_S \underline{a} \cdot d\underline{S} \;=\; \int_V \underline{\nabla} \cdot \underline{a}\, dV \;<\; 0$

If S contains neither sources nor sinks, then $\displaystyle\int_S \underline{a} \cdot d\underline{S} = 0$.

Electrostatics: As an example consider *electrostatics*. You will have learned that electric field lines can only start and stop at charges. A positive charge is a source of electric field (*i.e.* it creates a positive flux) and a negative charge is a sink (*i.e.* it absorbs flux, or, equivalently, creates a negative flux).

The electric field at $\underline{r}$ due to a charge q at the origin is

$$\underline{E} \;=\; \frac{q}{4\pi\epsilon_0} \frac{\hat{\underline{r}}}{r^2}$$

Then, for $r \neq 0$,

$$\underline{\nabla} \cdot \underline{E} \;=\; \frac{q}{4\pi\epsilon_0}\, \underline{\nabla} \cdot \left(\frac{\underline{r}}{r^3}\right) \;=\; \frac{q}{4\pi\epsilon_0} \left(\underline{\nabla}\left(\frac{1}{r^3}\right) \cdot \underline{r} + \frac{\underline{\nabla} \cdot \underline{r}}{r^3}\right) \;=\; \frac{q}{4\pi\epsilon_0} \left(-\frac{3\underline{r}}{r^5} \cdot \underline{r} + \frac{3}{r^3}\right) \;=\; 0$$

We also have

$$\int_{\text{sphere}} \underline{E} \cdot d\underline{S} \;=\; \frac{q}{4\pi\epsilon_0} \int_S \frac{\underline{r} \cdot d\underline{S}}{r^3} \;=\; \frac{q}{4\pi\epsilon_0}\, 4\pi \;=\; \frac{q}{\epsilon_0}$$

(where the surface integral has been evaluated in section 18.1).

Now $\underline{\nabla} \cdot \underline{E} = 0$, $\forall \underline{r} \neq 0$, so for *any* surface S enclosing a volume V which includes the origin, we have

$$\int_S \underline{E} \cdot d\underline{S} \;=\; \int_V \underline{\nabla} \cdot \underline{E}\, dV \;=\; \int_{\text{sphere}} \underline{\nabla} \cdot \underline{E}\, dV \;=\; \int_{\text{sphere}} \underline{E} \cdot d\underline{S} \;=\; \frac{q}{\epsilon_0}$$

We can replace the single charge q by a collection of charges $\sum_i q_i$ or a *charge density* $\rho(\underline{r})$. Hence

$$\int_V \underline{\nabla} \cdot \underline{E}\, dV \;=\; \int_S \underline{E} \cdot d\underline{S} \;=\; \frac{1}{\epsilon_0} \int_V \rho(\underline{r})\, dV$$

The second equality is *Gauss' Law* of electrostatics.

Since this must hold for arbitrary V, we find

$$\boxed{\underline{\nabla} \cdot \underline{E} \;=\; \frac{\rho(\underline{r})}{\epsilon_0}}$$

which holds for all $\underline{r}$. This is *Maxwell's first equation* of electromagnetism.

19.6 Corollaries of the divergence theorem

We may deduce several immediate consequences of the divergence theorem

$$\int_V \underline{\nabla} \cdot \underline{a}\, dV \;=\; \int_S \underline{a} \cdot d\underline{S}$$

(i) Let $\underline{a} = \underline{c}$ where $\underline{c}$ is a constant vector, then $\int_S \underline{c} \cdot d\underline{S} = 0$. Since this holds for arbitrary $\underline{c}$, we must have

$$\int_S d\underline{S} = 0$$

for any *closed* surface S as claimed previously.

(ii) Apply the divergence theorem to $\underline{a} \times \underline{c}$ with $\underline{c} = \text{constant}$. Then

$$\underline{\nabla} \cdot (\underline{a} \times \underline{c}) = \partial_i\,(\epsilon_{ijk}\, a_j\, c_k) = c_k\, \epsilon_{kij}\, \partial_i\, a_j = \underline{c} \cdot (\underline{\nabla} \times \underline{a})$$

and therefore

$$\underline{c} \cdot \int_V (\underline{\nabla} \times \underline{a})\, dV \;=\; \int_V \underline{\nabla} \cdot (\underline{a} \times \underline{c})\, dV \;=\; \int_S d\underline{S} \cdot (\underline{a} \times \underline{c}) \;=\; \underline{c} \cdot \int_S d\underline{S} \times \underline{a}$$

This holds for all constant vectors $\underline{c}$, hence

$$\int_V \underline{\nabla} \times \underline{a}\, dV \;=\; \int_S d\underline{S} \times \underline{a}$$

(iii) In suffix notation, the divergence theorem becomes

$$\int_V \partial_i \, a_i \, dV \;=\; \int_S a_i \, dS_i$$

For a second-rank tensor T, we regard one index (j in this case) as a 'spectator' index, so

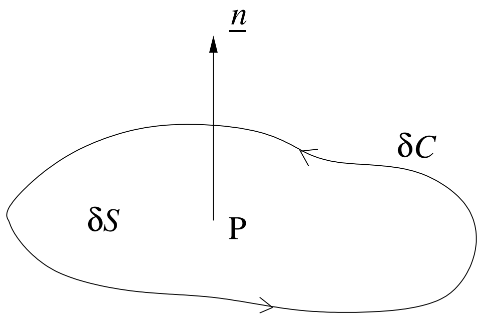
$$\int_V \partial_i \, T_{ij} \, dV \;=\; \int_S T_{ij} \, dS_i$$

This is the *generalised divergence theorem.* In particular with $T_{ij} = -\epsilon_{ijk} a_k$ we recover the result in (ii) above.

20 Line Integral Definition of Curl, Stokes' Theorem

20.1 Line Integral Definition of Curl

Let δS be a small planar surface containing the point P, bounded by a *closed* curve δC, with unit normal $\underline{n}$ and (scalar) area δS. Let $\underline{a}$ be a vector field defined on δS.



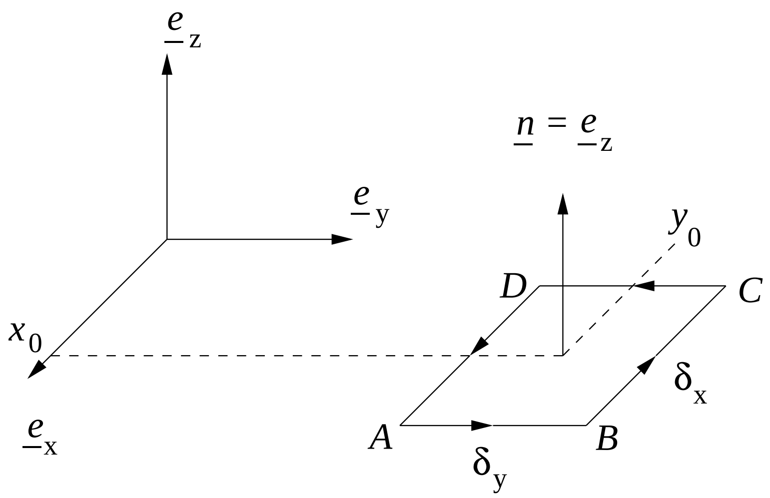
The component of $\underline{\nabla} \times \underline{a}$ parallel to $\underline{n}$ is defined to be

$$\underline{n} \cdot \left(\underline{\nabla} \times \underline{a} \right) \;=\; \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_{\delta C} \underline{a} \cdot d\underline{r}$$

NB: The integral around δC is taken in the right-hand sense with respect to the normal $\underline{n}$ to the surface – as in the figure above.

This definition of curl is *independent of the choice of basis.* The usual Cartesian form for curl can be recovered from this general definition by considering small rectangles parallel to the $x{-}y$, $y{-}z$, and $z{-}x$ planes respectively.

Let P be a point with Cartesian coordinates (x_0, y_0, z_0) situated at the *centre* of a small rectangle $\delta C = ABCD$ of size $\delta_x \times \delta_y$, area $\delta S = \delta_x \, \delta_y$, parallel to the $x{-}y$ plane.



The line integral around δC is given by the sum of four terms

$$\oint_{\delta C} \underline{a} \cdot d\underline{r} \;=\; \int_A^B \underline{a} \cdot d\underline{r} \;+\; \int_B^C \underline{a} \cdot d\underline{r} \;+\; \int_C^D \underline{a} \cdot d\underline{r} \;+\; \int_D^A \underline{a} \cdot d\underline{r}$$

Since $\underline{r} = x\underline{e}_x + y\underline{e}_y + z\underline{e}_z$ we have $d\underline{r} = \underline{e}_x\, dx$ along $D \to A$ and $C \to B$, and $d\underline{r} = \underline{e}_y\, dy$ along $A \to B$ and $D \to C$. Therefore

$$\oint_{\delta C} \underline{a} \cdot d\underline{r} \;=\; \int_A^B a_y\, dy \;-\; \int_C^B a_x\, dx \;-\; \int_D^C a_y\, dy \;+\; \int_D^A a_x\, dx$$

For small δ_x & δ_y, we can Taylor expand the integrands,

$$\begin{aligned}
\int_D^A a_x\, dx \;&=\; \int_D^A a_x(x, y_0 - \delta_y/2, z_0)\, dx \\
&=\; \int_{x_0-\delta_x/2}^{x_0+\delta_x/2} \left[a_x(x, y_0, z_0) \;-\; \frac{\delta_y}{2}\, \frac{\partial a_x}{\partial y}(x, y_0, z_0) \;+\; O(\delta_y^2) \right] dx \\
\int_C^B a_x\, dx \;&=\; \int_C^B a_x(x, y_0 + \delta_y/2, z_0)\, dx \\
&=\; \int_{x_0-\delta_x/2}^{x_0+\delta_y/2} \left[a_x(x, y_0, z_0) \;+\; \frac{\delta_y}{2}\, \frac{\partial a_x}{\partial y}(x, y_0, z_0) \;+\; O(\delta_y^2) \right] dx
\end{aligned}$$

so

$$\begin{aligned}
\frac{1}{\delta S}\left[\int_D^A \underline{a} \cdot d\underline{r} \;+\; \int_B^C \underline{a} \cdot d\underline{r} \right] \;&=\; \frac{1}{\delta_x\, \delta_y}\left[\int_D^A a_x\, dx \;-\; \int_C^B a_x\, dx \right] \\
&=\; \frac{1}{\delta_x \delta_y} \int_{x_0-\delta_x/2}^{x_0+\delta_x/2} \left[-\delta_y \frac{\partial a_x}{\partial y}(x, y_0, z_0) \;+\; O(\delta_y^2) \right] dx \\
&\to\; -\frac{\partial a_x}{\partial y}(x_0, y_0, z_0) \quad \text{as} \quad \delta_x,\, \delta_y \to 0
\end{aligned}$$

A similar analysis of the line integrals along $A \to B$ and $C \to D$ gives

$$\frac{1}{\delta S}\left[\int_A^B \underline{a} \cdot d\underline{r} \;+\; \int_C^D \underline{a} \cdot d\underline{r} \right] \;\to\; \frac{\partial a_y}{\partial x}(x_0, y_0, z_0) \quad \text{as} \quad \delta_x,\, \delta_y \to 0$$

Adding the results gives, for our line integral definition of curl,

$$\underline{e}_z \cdot (\underline{\nabla} \times \underline{a}) \;=\; (\underline{\nabla} \times \underline{a})_z \;=\; \left[\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right]_{(x_0,\,y_0,\,z_0)}$$

in agreement with our original definition in Cartesian coordinates.

The other components of $\underline{\nabla} \times \underline{a}$ can be obtained from similar rectangles parallel to the $y-z$ and $x-z$ planes, respectively.

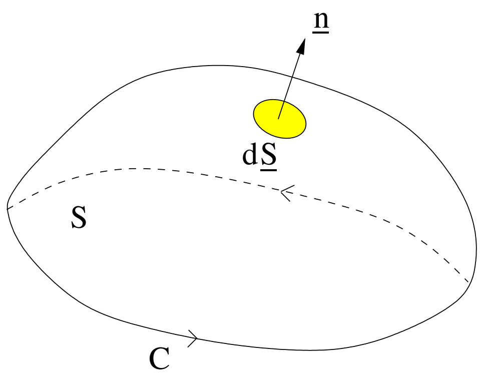
It can be shown that $\underline{\nabla} \times \underline{a}$, when defined in this way, is independent of the shape of the infinitesimal area δS.

20.2 Stokes' Theorem

Let S be an *open* surface, bounded by a simple *closed* curve C, and let $\underline{a}$ be a vector field defined on S, then

$$\oint_C \underline{a} \cdot d\underline{r} \;=\; \int_S (\underline{\nabla} \times \underline{a}) \cdot d\underline{S}$$

where C is traversed in a right-hand sense about $d\underline{S}$. As usual, $d\underline{S} = \underline{n}\, dS$ where $\underline{n}$ is the unit normal to S.



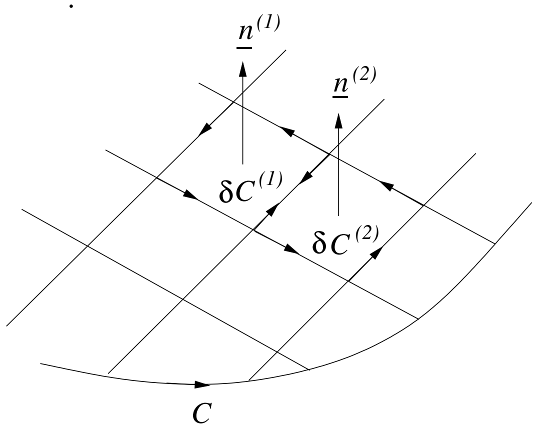
Proof: Divide the surface area S into N adjacent small surfaces as indicated in the diagram. Let $\delta \underline{S}^{(i)} = \delta S^{(i)}\, \underline{n}^{(i)}$ be the vector element of area at $\underline{r}^{(i)}$. Using the integral definition of curl,

$$\underline{n} \cdot (\underline{\nabla} \times \underline{a}) \;=\; \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_{\delta C} \underline{a} \cdot d\underline{r}$$

we multiply by $\delta S^{(i)}$ and sum over all i to get

$$\sum_{i=1}^{N} \left(\underline{\nabla} \times \underline{a}\left(\underline{r}^{(i)}\right)\right) \cdot \underline{n}^{(i)}\, \delta S^{(i)} \;=\; \sum_{i=1}^{N} \oint_{\delta C^{(i)}} \underline{a} \cdot d\underline{r} \;+\; \sum_{i=1}^{N} \epsilon^{(i)}\, \delta S^{(i)}$$

where $\delta C^{(i)}$ is the curve enclosing the area $\delta S^{(i)}$, and the quantity $\epsilon^{(i)} \to 0$ as $\delta S^{(i)} \to 0$.



Since each small closed curve $\delta C^{(i)}$ is traversed in the same sense, then, from the diagram, all contributions to $\displaystyle\sum_{i=1}^{N} \oint_{\delta C^{(i)}} \underline{a} \cdot d\underline{r}$ *cancel*, except on those curves where part of $\delta C^{(i)}$ lies on the curve C. For example, the line integrals along the common sections of the two small closed curves $\delta C^{(1)}$ and $\delta C^{(2)}$ in the figure *cancel exactly*. Therefore

$$\sum_{i=1}^{N} \oint_{\delta C^{(i)}} \underline{a} \cdot d\underline{r} \;=\; \oint_C \underline{a} \cdot d\underline{r}$$

Hence

$$\oint_C \underline{a} \cdot d\underline{r} \;=\; \int_S \left(\underline{\nabla} \times \underline{a}\right) \cdot d\underline{S} \;=\; \int_S \underline{n} \cdot \left(\underline{\nabla} \times \underline{a}\right)\; dS$$

20.3 Examples of the use of Stokes' Theorem

Hemisphere: Given the vector field $\underline{a} = 4y\underline{e}_x + x\underline{e}_y + 2z\underline{e}_z$, verify Stokes' theorem for the (open) hemispherical surface $x^2 + y^2 + z^2 = R^2$ with $z > 0$.

On the hemisphere, we have $\underline{\nabla} \times \underline{a} = -3\underline{e}_z$, and we have shown previously that $d\underline{S} = R^2 \sin\theta \, d\theta \, d\phi \, \underline{e}_r$.

Direct integration then gives

$$\begin{aligned}
\int_{\text{hemisphere}} \underline{\nabla} \times \underline{a} \cdot d\underline{S} \;&=\; \int_{\text{hemisphere}} R^2 \sin\theta \, d\theta \, d\phi \, \underline{e}_r \cdot (-3\underline{e}_z) \\
&=\; -6\pi R^2 \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \;=\; -\,3\pi R^2
\end{aligned}$$

The closed curve C bounding the hemisphere is a circle of radius R in the $x{-}y$ plane. Parameterising this by $x = R\cos\phi$, $y = R\sin\phi$, gives $dx = -R\sin\phi \, d\phi$, $\; dy = R\cos\phi \, d\phi$, and so

$$\begin{aligned}
\oint_C \underline{a} \cdot d\underline{r} \;&=\; \oint_C (4y dx + x dy) \\
&=\; \int_0^{2\pi} \left(-4R^2 \sin^2\phi + R^2 \cos^2\phi\right) \, d\phi \;=\; -\,3\pi R^2
\end{aligned}$$

Planar Areas: Consider a planar surface parallel to the $x{-}y$ plane, and let the vector field $\underline{a}$ be

$$\underline{a} = \frac{1}{2}\left[-y\,\underline{e}_x + x\,\underline{e}_y\right]$$

We find $\underline{\nabla} \times \underline{a} = \underline{e}_z$, and the vector element of area normal to the $x{-}y$ plane is $d\underline{S} = dS\,\underline{e}_z$. Hence

$$\int_S \underline{\nabla} \times \underline{a} \cdot d\underline{S} \;=\; \int_S \underline{e}_z \cdot d\underline{S} = \int_S dS \;=\; S$$

We can then use Stokes' theorem to find the area of the surface

$$S \;=\; \oint_C \underline{a} \cdot d\underline{r} \;=\; \frac{1}{2} \oint_C (-y\underline{e}_x + x\underline{e}_y) \cdot (dx\,\underline{e}_x + dy\,\underline{e}_y)$$

which gives

$$\boxed{S = \frac{1}{2} \oint_C (x\,dy - y\,dx)}$$

where C is the closed curve bounding the surface.

Example: Find the area inside the curve

$$x^{2/3} + y^{2/3} = 1\,.$$

Use the parameterisation $x = \cos^3\phi$, $y = \sin^3\phi$, for $0 \le \phi \le 2\pi$, so that

$$\frac{dx}{d\phi} = -3\cos^2\phi\,\sin\phi\,, \qquad \frac{dy}{d\phi} = 3\sin^2\phi\,\cos\phi$$

which gives

$$\begin{aligned} S &= \frac{1}{2}\oint_C \left(x\,\frac{dy}{d\phi} - y\,\frac{dx}{d\phi} \right) d\phi \\ &= \frac{1}{2}\int_0^{2\pi} \left(3\cos^4\phi\,\sin^2\phi + 3\sin^4\phi\,\cos^2\phi\right) d\phi \\ &= \frac{3}{2}\int_0^{2\pi} \sin^2\phi\,\cos^2\phi\,d\phi \;=\; \frac{3}{8}\int_0^{2\pi} \sin^2 2\phi\,d\phi \;=\; \frac{3\pi}{8} \end{aligned}$$

20.4 Corollaries of Stokes' Theorem

We may deduce several immediate consequences of Stokes' theorem,

$$\int_S \underline{\nabla} \times \underline{a} \cdot d\underline{S} \;=\; \oint_C \underline{a} \cdot d\underline{r}$$

where C is the boundary (traversed in the anticlockwise direction) of the open surface S.

(i) If $\underline{a} = \underline{c}$, where $\underline{c}$ is a constant vector, then $\underline{\nabla} \times \underline{a} = 0$. Therefore $\underline{c} \cdot \oint_C d\underline{r} = 0$, and because $\underline{c}$ is arbitrary, we have

$$\oint_C d\underline{r} \;=\; 0$$

(ii) Applying Stokes' theorem to $\underline{a} = \phi\,\underline{c}$ where $\underline{c}$ is a constant vector, we have

$$\left(\underline{\nabla} \times (\phi\,\underline{c})\right) \cdot d\underline{S} \;=\; \epsilon_{ijk}\,\partial_j(\phi\,c_k)\,dS_i \;=\; c_k\,\epsilon_{ijk}\,dS_i\,\partial_j\phi \;=\; \underline{c} \cdot (d\underline{S} \times \underline{\nabla}\phi)$$

which gives

$$\int_S \left(\underline{\nabla} \times (\phi\,\underline{c})\right) \cdot d\underline{S} \;=\; \underline{c} \cdot \int_S d\underline{S} \times \underline{\nabla}\phi \;=\; \underline{c} \cdot \oint_C \phi\,d\underline{r}$$

This holds for all constant vectors $\underline{c}$, so

$$\int_S d\underline{S} \times \underline{\nabla}\phi \;=\; \oint_C \phi\,d\underline{r}$$

(iii) In terms of indices, Stokes' theorem is

$$\epsilon_{ijk} \int_S \frac{\partial a_k}{\partial x_j} \, dS_i \;=\; \oint_C a_k \, dx_k$$

For a second-rank tensor T, we again regard one index as a 'spectator' index, so

$$\epsilon_{ijk} \int_S \frac{\partial T_{kl}}{\partial x_j} \, dS_i \;=\; \oint_C T_{kl} \, dx_k$$

This is the *generalised Stokes' theorem.* In particular, with $T_{kl} = \phi \, \delta_{kl}$ we recover the result in (ii) above.

21 The Scalar Potential

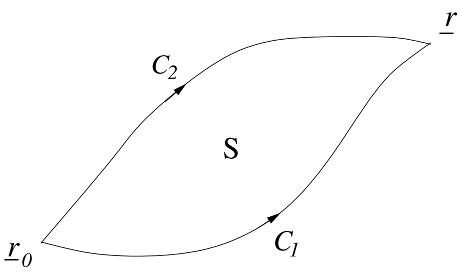
21.1 Defining the scalar potential

A vector field $\underline{a}(\underline{r})$ is irrotational or *conservative* if its curl vanishes, *i.e.*

$$\underline{\nabla} \times \underline{a} = 0$$

Path independence of line integrals for conservative fields

Let $\underline{\nabla} \times \underline{a} = 0$ and consider two (arbitrary) paths C_1 and C_2 from point $\underline{r}_0$ to point $\underline{r}$, say. Applying Stokes' theorem to the open surface S bounded by these two paths gives



$$0 \;=\; \int_S (\underline{\nabla} \times \underline{a}) \cdot d\underline{S} \;=\; \int_{C_1} \underline{a}(\underline{r}') \cdot d\underline{r}' \;-\; \int_{C_2} \underline{a}(\underline{r}') \cdot d\underline{r}'$$

where the $-$ve sign occurs in the second integral because *both* paths are from $\underline{r}_0$ to $\underline{r}$. (We use $\underline{r}'$ as integration variable to distinguish it from the *limits* of integration $\underline{r}_0$ and $\underline{r}$.)

Hence

$$\int_{C_1} \underline{a}(\underline{r}') \cdot d\underline{r}' = \int_{C_2} \underline{a}(\underline{r}') \cdot d\underline{r}'$$

This is true for *any* S, and therefore for *any* paths C_1 and C_2 between $\underline{r}_0$ and $\underline{r}$.

Since $\displaystyle\int \underline{a}(\underline{r}') \cdot d\underline{r}'$ is *path independent* it can be a function *only* of the *end points* of the path.

Clearly, the converse is also true: if the line integral between two points is path independent, then the line integral around any closed curve (connecting the two points) is zero, and hence $\underline{\nabla} \times \underline{a} = 0$. We just reverse the steps of the argument above.

Therefore

$$\boxed{\underline{\nabla} \times \underline{a} = 0 \quad \Leftrightarrow \quad \text{path independence of } \int_{\underline{r}_0}^{\underline{r}} \underline{a}(\underline{r}') \cdot d\underline{r}'}$$

21.2 A Theorem for Conservative Vector Fields

Since the line integral of a conservative vector field between two fixed points $\underline{r}_0$ and $\underline{r}$ is *path independent*, there must exist a function $\phi(\underline{r})$ such that

$$\phi(\underline{r}) - \phi(\underline{r}_0) = \int_{\underline{r}_0}^{\underline{r}} \underline{a}(\underline{r}') \cdot d\underline{r}' . \qquad (35)$$

The field $\phi(\underline{r})$ is called the *scalar potential* of the vector field $\underline{a}(\underline{r})$.

It is useful to invert this equation (and to give a more conventional result) by considering two neighbouring points $\underline{r}$ and $\underline{r} + d\underline{r}$, for which

$$\begin{aligned}
d\phi &= \phi(\underline{r} + d\underline{r}) - \phi(\underline{r}) \\
&= \big[\phi(\underline{r} + d\underline{r}) - \phi(\underline{r}_0)\big] - \big[\phi(\underline{r}) - \phi(\underline{r}_0)\big] \\
&= \int_{\underline{r}_0}^{\underline{r}+d\underline{r}} \underline{a}(\underline{r}') \cdot d\underline{r}' \; - \int_{\underline{r}_0}^{\underline{r}} \underline{a}(\underline{r}') \cdot d\underline{r}' \qquad \text{(using equation (35))} \\
&= \int_{\underline{r}}^{\underline{r}+d\underline{r}} \underline{a}(\underline{r}') \cdot d\underline{r}' \qquad \text{(using path independence)} \\
&= \underline{a}(\underline{r}) \cdot d\underline{r} \; + \; O(|d\underline{r}|^2)
\end{aligned}$$

But $d\phi = \underline{\nabla}\phi \cdot d\underline{r}$ (by definition), and since $d\underline{r}$ is *arbitrary*, we must have

$$\underline{a}(\underline{r}) = \underline{\nabla}\phi(\underline{r})$$

The converse is trivial to prove: if $\underline{a} = \underline{\nabla}\phi$, then $\underline{\nabla} \times \underline{a} = \underline{\nabla} \times (\underline{\nabla}\phi) \equiv 0$.

Therefore

$$\boxed{\underline{\nabla} \times \underline{a} = 0 \quad \Leftrightarrow \quad \underline{a} = \underline{\nabla}\phi}$$

To determine whether a vector field is conservative, one simply checks whether $\underline{\nabla} \times \underline{a} = 0$.

NB: The scalar potential $\phi(\underline{r})$ is only determined up to a *constant.* If $\psi = \phi + \textit{constant}$ then $\underline{\nabla}\,\psi = \underline{\nabla}\,\phi$, so ψ is an equally good potential. The freedom in the constant corresponds to the freedom in choosing $\underline{r}_0$ to calculate the potential. So $\phi(\underline{r}_0)$ in equation (35) is just an irrelevant constant. Equivalently, the absolute value of a scalar potential has no meaning, only *potential differences* are significant.

21.3 Finding Scalar Potentials

Method (1): Integration along a straight line

We have shown that the scalar potential $\phi(\underline{r})$ for a *conservative* vector field $\underline{a}(\underline{r})$ can be constructed from a line integral which is *independent* of the path of integration between the endpoints. Therefore, a convenient way of evaluating such integrals is to integrate along a *straight line*. Depending on the convergence of the integral, we have two (obvious) choices:

(i) $\underline{r}_0 = 0$: If $\phi(\underline{r} = 0)$ is *finite*, we can parameterise the straight line by $\underline{r}' = \lambda\,\underline{r}$, with $0 \le \lambda \le 1$, so $d\underline{r}' = d\lambda\,\underline{r}$, and hence

$$\phi(\underline{r}) \;=\; \int_0^{\underline{r}} \underline{a}(\underline{r}')\cdot d\underline{r}' \;=\; \int_{\lambda=0}^{\lambda=1} \underline{a}(\lambda\underline{r})\cdot\underline{r}\,d\lambda\,,$$

(ii) $|\underline{r}| = \infty$: If $\phi(\underline{r} \to \infty)$ is *finite*, we parameterise the straight line by $\underline{r}' = \lambda\,\underline{r}$, with $1 \le \lambda < \infty$, so $d\underline{r}' = d\lambda\,\underline{r}$, and hence

$$\phi(\underline{r}) \;=\; \int_\infty^{\underline{r}} \underline{a}(\underline{r}')\cdot d\underline{r}' \;=\; \int_{\lambda=\infty}^{\lambda=1} \underline{a}(\lambda\underline{r})\cdot\underline{r}\,d\lambda\,,$$

Example 1: Let $\underline{a} = (2xy + z^3)\underline{e}_x + x^2\underline{e}_y + 3xz^2\underline{e}_z$.

We first check that $\underline{\nabla} \times \underline{a} = 0$, so the field is conservative (exercise). Then

$$\begin{aligned}
\phi(\underline{r}) &= \int_0^1 \underline{a}(\lambda\underline{r})\cdot\underline{r}\,d\lambda \\
&= \int_0^1 \left[\left(2\lambda^2 xy + \lambda^3 z^3\right)x + \lambda^2 x^2 y + \lambda^3 3xz^3\right] d\lambda \\
&= \frac{2}{3}x^2 y + \frac{1}{4}xz^3 + \frac{1}{3}x^2 y + \frac{3}{4}xz^3 \\
&= x^2 y + xz^3
\end{aligned}$$

NB: Always check that your potential $\phi(\underline{r})$ satisfies $\underline{a}(\underline{r}) = \underline{\nabla}\,\phi(\underline{r})$.

Example 2: Let $\underline{a}(\underline{r}) = 2\,(\underline{c}\cdot\underline{r})\,\underline{r} + r^2\,\underline{c}$ where $\underline{a}$ is a constant vector.

Check that $\underline{a}$ is conservative:

$$\begin{aligned}
\underline{\nabla} \times \underline{a} &= 2\left[\underline{\nabla}\left(\underline{c}\cdot\underline{r}\right)\times\underline{r} + \left(\underline{c}\cdot\underline{r}\right)\underline{\nabla}\times\underline{r}\right] + \left(\underline{\nabla}\,r^2\right)\times\underline{c} \\
&= 2\left[\underline{c}\times\underline{r} + 0\right] + 2\,\underline{r}\times\underline{c} \;=\; 0
\end{aligned}$$

or, using indices,

$$\begin{aligned}
(\underline{\nabla}\times\underline{a})_i &= \epsilon_{ijk}\,\partial_j\left(2\,(c_l\,x_l)\,x_k + x_l\,x_l\,c_k\right) \\
&= \epsilon_{ijk}\left(2\,c_l\,\delta_{jl}\,x_k + 2\,\underline{c}\cdot\underline{r}\,\delta_{jk} + 2x_l\,\delta_{lj}\,c_k\right) \\
&= 2\epsilon_{ijk}\left(c_j\,x_k + x_j\,c_k\right) \;=\; 0
\end{aligned}$$

Thus

$$
\begin{aligned}
\phi(\underline{r}) &= \int_0^{\underline{r}} \underline{a}(\underline{r}') \cdot d\underline{r}' \;=\; \int_0^1 \underline{a}\left(\lambda\,\underline{r}\right) \cdot \left(d\lambda\,\underline{r}\right) \\
&= \int_0^1 \left(2\ \left(\underline{c} \cdot \lambda\,\underline{r}\right) \lambda\,\underline{r} \;+\; \lambda^2 r^2\,\underline{c}\right) \cdot \underline{r}\,d\lambda \\
&= \left(2\left(\underline{c} \cdot \underline{r}\right) \underline{r} \cdot \underline{r} \;+\; r^2\left(\underline{c} \cdot \underline{r}\right)\right) \int_0^1 \lambda^2\,d\lambda \\
&= r^2\left(\underline{c} \cdot \underline{r}\right)
\end{aligned}
$$

This is a fairly elegant method and is generally applicable.

Method (2): Direct integration "by inspection" (guessing)

Sometimes the result can be spotted directly.

Example 1 (revisited):

$$
\begin{aligned}
\underline{a} &= (2xy + z^3, x^2, 3xz^2) \\
&= \left(\frac{\partial}{\partial x}\left(x^2y + xz^3\right),\ \frac{\partial}{\partial y}\left(x^2y + xz^3\right),\ \frac{\partial}{\partial z}\left(x^2y + xz^3\right)\right) \\
&= \underline{\nabla}\,\left(x^2y + xz^3\right)
\end{aligned}
$$

Similarly, if $\underline{a}(\underline{r}) = (\underline{c} \cdot \underline{r})\,\underline{c}$ where $\underline{c}$ is a constant vector, then

$$
\underline{a}(\underline{r}) \;=\; (\underline{c} \cdot \underline{r})\,\underline{c} \;=\; (\underline{c} \cdot \underline{r})\,\underline{\nabla}\,(\underline{c} \cdot \underline{r}) \;=\; \underline{\nabla}\left(\frac{1}{2}\,(\underline{c} \cdot \underline{r})^2 \;+\; \text{const}\right)
$$

This can be tricky to spot though.

Method (3): Direct integration

Since $\underline{a} = \underline{\nabla}\phi$, we have

$$
\frac{\partial \phi}{\partial x} = a_x(x, y, z) \qquad \frac{\partial \phi}{\partial y} = a_y(x, y, z) \qquad \frac{\partial \phi}{\partial z} = a_z(x, y, z)
$$

We can integrate these equations separately to give

$$
\begin{aligned}
\phi(x, y, z) &= \int^x a_x(x', y, z)\,dx' + f(y, z) \\
\phi(x, y, z) &= \int^y a_y(x, y', z)\,dy' + g(x, z) \\
\phi(x, y, z) &= \int^z a_z(x, y, z')\,dz' + h(x, y)
\end{aligned}
$$

and then determine the "constants" of integration $f(y, z)$, $g(x, z)$ and $h(x, y)$ by consistency.

Example 1 (revisited): Let $\underline{a} = (2xy + z^3)\underline{e}_x + x^2\underline{e}_y + 3xz^2\underline{e}_z$. Then

$$\begin{aligned}\phi &= x^2y + xz^3 + f(y,z)\\ \phi &= x^2y + g(x,z)\\ \phi &= xz^3 + h(x,y)\end{aligned}$$

These agree if we choose $f(y,z) = 0$, $g(x,z) = xz^3$ and $h(x,y) = x^2y^3$.

This is a straightforward method but it can get very messy.

21.4 Conservative Forces: Conservation of Energy

We now show how the name *conservative field* arises in Physics. Let the vector field $\underline{F}(\underline{r})$ be the total force acting on a particle of mass m at position $\underline{r}$. We will show that for a conservative force, where we can write

$$\underline{F} = -\underline{\nabla}\, V\,,$$

the total energy is *constant* in time. The *force* is *minus* the gradient of the (scalar) potential. The minus sign is conventional.

Proof: Let $\underline{r}(t)$ be the position vector of a particle at time t. Denote the first and second derivatives of $\underline{r}$ with respect to time by $\dot{\underline{r}}$ (velocity) and $\ddot{\underline{r}}$ (acceleration) respectively.

The particle moves under the influence of Newton's Second Law:

$$m\ddot{\underline{r}} = \underline{F}(\underline{r})$$

Consider a small displacement along the path of the particle: $\underline{r} \to \underline{r} + d\underline{r}$ taking time dt. Then

$$m\,\ddot{\underline{r}}\cdot d\underline{r} \;=\; \underline{F}(\underline{r})\cdot d\underline{r} \;=\; -\,\underline{\nabla}\, V(\underline{r})\cdot d\underline{r}\,.$$

Integrating this expression along a path from $\underline{r}_A$ at time $t = t_A$, to $\underline{r}_B$ at time $t = t_B$ yields

$$m\int_{\underline{r}_A}^{\underline{r}_B} \ddot{\underline{r}}\cdot d\underline{r} \;=\; -\int_{\underline{r}_A}^{\underline{r}_B} \underline{\nabla}\, V(\underline{r})\cdot d\underline{r}.$$

We can simplify the left-hand side of this equation as follows,

$$m\int_{\underline{r}_A}^{\underline{r}_B} \ddot{\underline{r}}\cdot d\underline{r} \;=\; m\int_{t_A}^{t_B} \ddot{\underline{r}}\cdot\dot{\underline{r}}\,dt \;=\; m\int_{t_A}^{t_B} \frac{1}{2}\frac{d}{dt}\left(\dot{\underline{r}}\cdot\dot{\underline{r}}\right)dt \;=\; \frac{1}{2}m\left(v_B^2 - v_A^2\right),$$

where v_A and v_B are the magnitudes of the velocities at points A and B respectively.

The right-hand side gives

$$-\int_{\underline{r}_A}^{\underline{r}_B} \underline{\nabla}\, V(\underline{r})\cdot d\underline{r} \;=\; -\int_{\underline{r}_A}^{\underline{r}_B} dV = V_A - V_B$$

where V_A and V_B are the values of the potential V at $\underline{r}_A$ and $\underline{r}_B$, respectively. Therefore

$$\frac{1}{2}mv_A^2 + V_A \;=\; \frac{1}{2}mv_B^2 + V_B$$

and the *total energy* $E \equiv \dfrac{1}{2}mv^2 + V$ is *conserved* — it's *constant in time.*

(Choosing $\underline{F} = +\underline{\nabla}\, V$ would lead to $E \equiv \dfrac{1}{2}mv^2 - V$, a less desirable convention.)

21.5 Physical Examples of Conservative Forces

Newtonian gravity and the *electrostatic force* are both conservative. *Frictional forces* are not conservative: energy is dissipated and work is done in traversing a closed path. In general, time-dependent forces are not conservative.

The foundation of Newtonian Gravity is *Newton's Law of Gravitation.* The force $\underline{F}$ on a particle of mass m_1 at $\underline{r}$ due to a particle of mass m situated at the origin is given (in SI units) by

$$\underline{F}(\underline{r}) \;=\; -\,G\,m\,m_1\,\frac{\underline{r}}{r^3}$$

where $G = 6.672\,59(85) \times 10^{-11}\,Nm^2kg^2$ is Newton's Gravitational Constant.

The *gravitational field* $\underline{G}(\underline{r})$ due to the mass at the origin is *defined* by

$$\underline{F}(\underline{r}) \;\equiv\; m_1\,\underline{G}(\underline{r}) \qquad \text{or} \qquad \underline{G}(\underline{r}) \;=\; -\,G\,m\,\frac{\underline{r}}{r^3} \tag{36}$$

where the test mass m_1 is so small that its gravitational field can be ignored. The gravitational field is conservative because

$$\underline{\nabla}\times\left(\frac{\underline{r}}{r^3}\right) \;=\; \underline{\nabla}\left(\frac{1}{r^3}\right)\times\underline{r} \;+\; \frac{1}{r^3}\left(\underline{\nabla}\times\underline{r}\right) \;=\; \left(-\frac{3\underline{r}}{r^5}\right)\times\underline{r} \;+\; 0 \;=\; 0$$

or, using indices,

$$\left(\underline{\nabla}\times\left(\underline{r}/r^3\right)\right)_i \;=\; \epsilon_{ijk}\partial_j(x_k/r^3) \;=\; \epsilon_{ijk}\left(\delta_{jk}/r^3 - 3x_jx_k/r^5\right) \;=\; 0$$

The *gravitational potential* defined by

$$\underline{G} = -\underline{\nabla}\,\phi$$

can be obtained from equation (36) by spotting the direct integration, giving

$$\phi = -\frac{Gm}{r}$$

Alternatively, we may evaluate it by a line integral. Choosing $\underline{r}_0 = \infty$ gives

$$\begin{aligned}\phi(\underline{r}) \;&=\; -\int_\infty^{\underline{r}} \underline{G}(\underline{r}')\cdot d\underline{r}' \;=\; -\int_\infty^1 \underline{G}(\lambda\underline{r})\cdot d\lambda\underline{r} \\ &=\; (-)^2\int_\infty^1 \frac{Gm\,(\underline{r}\cdot\underline{r})}{r^3}\,\frac{d\lambda}{\lambda^2} \;=\; -\,\frac{Gm}{r}\end{aligned}$$

Note: In this example, the vector field $\underline{G}$ is singular at the origin $\underline{r} = 0$. This implies that we have to exclude the origin, so it's not possible to obtain the scalar potential at $\underline{r}$ by integration along a path from the origin. Instead we integrate from infinity, which in turn means that the gravitational potential at infinity is zero.

Note: Since $\underline{F} = m_1\,\underline{G} = -\underline{\nabla}\,(m_1\,\phi)$, the *potential energy* of the mass m_1 is $V = m_1\phi$. The distinction (a convention) between potential and potential energy is a common source of confusion.

Electrostatics: Coulomb's Law states that the force $\underline{F}(\underline{r})$ on a particle of charge q_1 situated at $\underline{r}$ in the electric field $\underline{E}(\underline{r})$ due to a particle of charge q situated at the origin is given (in SI units) by

$$\underline{F} \;=\; q_1\,\underline{E} \;=\; \frac{q_1\,q}{4\pi\epsilon_0}\,\frac{\underline{r}}{r^3}\;,$$

where $\epsilon_0 \;=\; 10^7/(4\pi c^2) \;=\; 8.854\,187\,817\cdots \times 10^{-12}\,C^2N^{-1}m^{-2}$ is called the *permittivity of free space*. Again the test charge q_1 is taken as small, so as not to disturb the electric field.

The *electrostatic potential* may be obtained by integrating $E = -\underline{\nabla}\,\phi$ from infinity to $\underline{r}$,

$$\phi \;=\; \frac{q}{4\pi\epsilon_0 r}$$

The *potential energy* of a charge q_1 in the electric field is $V = q_1\phi$.

Note that electrostatics and gravitation are very similar mathematically, the only real difference being that the gravitational force between two masses is always attractive, whereas like charges repel.