Quantum Theory 2015/16

Gaussian Integrals

In this course there are a lot of integrals, but most of the ones we do explicitly will be Gaussian. Gaussian integrals should be familiar from other MP courses, in particular Honours Complex Variables, Complex Analysis and Methods of Mathematical Physics, so here we merely compile some results, leaving the derivations as a revision and/or tutorial exercise.

The basic Gaussian integral is

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}. \tag{1}$$

where a is real and positive. To derive this, square it, and use polar co-ordinates. The integral also converges for complex a, provided only that the real part of a is strictly positive.

Gaussian integrals with positive powers of x^2 may be obtained by differentiation with respect to a:

$$\int_{-\infty}^{\infty} dx\, x^2 e^{-ax^2} = \frac{1}{2a}\sqrt{\frac{\pi}{a}}, \qquad \int_{-\infty}^{\infty} dx\, x^4 e^{-ax^2} = \frac{3}{2a}\frac{1}{2a}\sqrt{\frac{\pi}{a}}, \tag{2}$$

and so on. Integrals over odd powers vanish by symmetry.

An exponential factor may be taken care of by completing the square:

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2+bx} = \int_{-\infty}^{\infty} dx\, e^{-a(x-b/2a)^2+b^2/4a} = \sqrt{\frac{\pi}{a}} e^{b^2/4a}, \tag{3}$$

where we perform the second integral by a simple change of variables $y = x - b/2a$. This shift works even if b is complex: in particular if $b = ik$,

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2+ikx} = \int_{-\infty}^{\infty} dx\, e^{-a(x-ik/2a)^2-k^2/4a} = \sqrt{\frac{\pi}{a}} e^{-k^2/4a}, \tag{4}$$

where now the change of variables $z = x - ik/2a$ shifts the contour into the complex plane. The Fourier transform of a Gaussian is thus a Gaussian with reciprocal width.

Integrals with factors of powers of x may be again obtained by differentiation with respect to a and/or b: for example

$$\int_{-\infty}^{\infty} dx\, x e^{-ax^2+bx} = \frac{b}{2a}\sqrt{\frac{\pi}{a}} e^{b^2/4a}, \tag{5}$$

$$\int_{-\infty}^{\infty} dx\, x^2 e^{-ax^2+bx} = \left(1 + \frac{b^2}{2a}\right)\frac{1}{2a}\sqrt{\frac{\pi}{a}} e^{b^2/4a}, \tag{6}$$

or by completing the square and using previous results.

We will also need Gaussian integrals with an i in the exponent: the simplest is the Fresnel integral

$$\int_{-\infty}^{\infty} dx\, e^{iax^2} = e^{i\pi/4}\int_{-\infty}^{\infty} dy\, e^{-ay^2} = e^{i\pi/4}\sqrt{\frac{\pi}{a}} \equiv \sqrt{\frac{i\pi}{a}}, \tag{7}$$

since the change of variables $x = e^{i\pi/4}y$ rotates the contour through $\pi/4$. It follows that such integrals can be obtained from the previous ones simply by the replacement $a \to -ia$: for example (cf. eq.(5))

$$\int_{-\infty}^{\infty} dx\, e^{iax^2+ikx} = \sqrt{\frac{i\pi}{a}} e^{-ik^2/4a}, \tag{8}$$

and so on. It is important to note that these integrals are only convergent when the imaginary part of a is strictly positive: elsewhere they are defined by analytic continuation. It is usual to take the cut in the square root down the negative real axis of the a-plane.