Quantum Theory 2015/16

Tutorial Sheet 1

1.1 Consider a collision experiment in which (in the centre-of-mass frame) two incoming particles A and B of equal mass scatter elastically to two final state particles α and β emerging at right angles to the orginal axis of the collision. Show that the probability of this happening when the two particles A and B are identical spinless bosons is twice what it would be if they were not identical. What is the probability for identical 'spinless fermions'?

[Hint: Read Feynman and Hibbs, Chapter 1]

1.2 Consider a two slit experiment in which the slits are a distance d apart, and the detector is a distance L from the slits, $L \gg d$. Show that if the particles leave the slits in an arbitrary direction with momentum p, the probability of observing a particle a distance y, with $y \ll L$, from the centreline of the experiment vanishes whenever $y = n\pi\hbar L/dp$, where n is an odd integer.

[Hint: the amplitude for a particle of momentum p and energy E to travel a distance x in time t is proportional to $e^{ipx/\hbar - iEt/\hbar}$.]

1.3 Show that if $\hat{H}$ is a nonhermitian operator, $\hat{H}^\dagger \hat{H}$ is hermitian with nonnegative real eigenvalues, so the expectation value $\langle \hat{H}^\dagger \hat{H} \rangle \geq 0$.

Show further that if $\hat{A}$ and $\hat{B}$ are two noncommuting hermitian operators, then

$$\langle \hat{A}^2 \rangle \langle \hat{B}^2 \rangle \geq -\tfrac{1}{4} \langle [\hat{A}, \hat{B}] \rangle^2.$$

[Hint: take $\hat{H} = \hat{A} + i\gamma\hat{B}$, and minimise $\langle \hat{H}^\dagger \hat{H} \rangle$ by varying the real variable γ.]

Deduce that if $\Delta\alpha \equiv \sqrt{\langle \hat{\alpha}^2 \rangle - \langle \hat{\alpha} \rangle^2}$, etc, then

$$\Delta\alpha\Delta\beta \geq \tfrac{1}{2} |\langle [\hat{\alpha}, \hat{\beta}] \rangle|.$$

[Hint: apply the previous result to $\hat{A} = \hat{\alpha} - \langle \hat{\alpha} \rangle$, etc.]

Consider the Heisenberg Uncertainty Relation as a special case.

1.4 If $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$, evaluate matrix elements of the double commutator $[[\hat{x}, \hat{H}], \hat{x}]$ to obtain the sum rule

$$\sum_n (E_n - E_m) |\langle n | \hat{x} | m \rangle|^2 = \hbar^2/2m,$$

where $|n\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue E_n.

1.5 If $\hat{A}$ and $\hat{B}$ are two operators which both commute with their commutator show that $[\hat{A}, \hat{B}^n] = n[\hat{A}, \hat{B}]\hat{B}^{n-1}$, and thus that $[\hat{A}, e^{\lambda\hat{B}}] = \lambda[\hat{A}, \hat{B}]e^{\lambda\hat{B}}$.

Deduce that if $\hat{F}(\lambda) = e^{\lambda\hat{A}} e^{\lambda\hat{B}} e^{-\lambda(\hat{A}+\hat{B})}$, then

$$\frac{d\hat{F}}{d\lambda} = \lambda[\hat{A}, \hat{B}]\hat{F},$$

and thus that

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A}, \hat{B}]}.$$