Quantum Theory 2015/16

Tutorial Sheet 3

3.1 Check all the Gaussian integrals on the handout. Consider in particular the convergence of the integrals when a is complex, and discuss how they may be defined throughout the complex plane by analytic continuation.

3.2 Show by substitution that the free particle amplitude derived in lectures satisfies the nonlinear consistency condition

$$\langle x_b, t_b | x_a, t_a \rangle = \int_{-\infty}^{\infty} dx \langle x_b, t_b | x, t \rangle \, \langle x, t | x_a, t_a \rangle,$$

for all $t_a < t < t_b$.

3.3 Show that the Gaussian integral

$$\int_{-\infty}^{\infty} d\eta_n \, e^{i\left((\eta_{n+1}-\eta_n)^2/a\right) + i\eta_n^2/(na)} = \sqrt{i\pi a}\sqrt{\frac{n}{n+1}}\, e^{i\eta_{n+1}^2/((n+1)a)}.$$

Use this result to evaluate the normalization factor

$$F_0(T) = \int_0^0 \mathcal{D}\eta e^{iS[\eta]/\hbar} \equiv \lim_{N\to\infty} (\nu(\varepsilon))^{N+1} \Big(\prod_{n=1}^{N} \int_{-\infty}^{\infty} d\eta_n \Big)\, e^{i(m/2\hbar\varepsilon)\sum_{n=1}^{N+1}(\eta_n - \eta_{n-1})^2},$$

where $\eta_0 = \eta_{N+1} = 0$, $\varepsilon = T/(N+1)$, and $\nu(\varepsilon) = \sqrt{m/2\pi i\hbar\varepsilon}$, by repeated Gaussian integration.

3.4 Show that if $\psi(x,t) \equiv \langle x, t | \psi \rangle$, then

$$\psi(x,t) = \int dx' G_0(x - x', t - t')\, \psi(x', t'), \tag{1}$$

where $G_0(x - x', t - t') = \langle x, t | x', t' \rangle$ is the free particle amplitude, and $t \geq t'$.

Suppose a free particle has definite momentum p at time $t = 0$ (so the wave function is proportional to $e^{ipx/\hbar}$). Show using the result derived in lectures for the free particle amplitude that at some later time the particle has the same definite momentum, but that the wave function also varies in time as $e^{-ip^2t/2m\hbar}$, and thus that the particle has definite energy $E = p^2/2m$.

3.5 (a) Show by explicit computation that $\psi(x,t) \equiv \langle x, t | \psi \rangle$ (as given by equation (1) in question (3.4)) satisfies the time dependent free particle Schrödinger equation, ie that

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t).$$

(b) Compute the momentum space free particle amplitude $\langle p, t | p', t' \rangle$ from the position space amplitude. Show thus that the momentum space wave function $\widetilde{\psi}(p,t) \equiv \langle p, t | \psi \rangle$ satisfies the momentum space free particle Schrödinger equation, *i.e.* that

$$i\hbar\frac{\partial}{\partial t}\widetilde{\psi}(p,t) = \frac{p^2}{2m}\widetilde{\psi}(p,t).$$

3.6 The wavefunction $\psi(x,t) = \langle x,t|\psi\rangle$ for a free particle moving in one dimension is given at time $t=0$ by the *Gaussian wave packet*

$$\psi(x,0) = C\ \exp(ip_0\,x/\hbar)\ \exp(-x^2/4\sigma_0^2).$$

(i) Show that the normalisation constant is $C = \left(2\pi\sigma^2\right)^{-1/4}$, and that the uncertainty Δx at $t=0$ is $\Delta x_0 = \sigma_0$.

(ii) Show that the normalised momentum space wavefunction at time $t=0$ is

$$\widetilde{\psi}(p,0) = \widetilde{C}\ \exp(-\sigma_0^2\,(p-p_0)^2/\hbar^2)$$

and determine the normalisation constant $\widetilde{C}$. Show that the expectation value of $\hat{p}$ is p_0 and hence that the momentum uncertainty Δp at $t=0$ is $\Delta p_0 = \hbar/2\sigma_0$, and thus that the uncertainty relation is saturated, i.e.

$$\Delta x_0\ \Delta p_0\ =\ \tfrac{1}{2}\hbar$$

(iii) Determine the evolved wave functions $\psi(x,t)$ and $\widetilde{\psi}(p,t)$ using the free particle amplitudes in position and momentum space respectively. Hence show that the expectation value of the position at time t is $\langle\hat{x}\rangle_t = p_0\,t/m$, while the uncertainty about the mean value of x at time t is

$$\Delta x_t = \Delta x_0\ \sqrt{1 + \frac{\hbar^2\,t^2}{4\,\Delta x_0^4\,m^2}},$$

and deduce similar expressions for the expectation value of the momentum and its uncertainty. Does the uncertainty relation remain saturated?