Quantum Theory 2015/16

Tutorial Sheet 4

4.1 Write down the quantum transition amplitudes $\langle x, t|x', t'\rangle$ for each of the classical systems described in questions 2.2a, 2.2c, 2.3 and 2.4 of example sheet 2. In each case you should take care to fix up the normalisation correctly.

4.2 Find the wave function $\psi(x,t)$ for a harmonic oscillator $L = \frac{1}{2}m(\dot{x}^2 - \omega^2 x^2)$ if at $t = 0$,

$$\psi(x, 0) = \sqrt{\frac{2\pi\hbar}{m\omega}} \exp\left(-\frac{m\omega}{2\hbar}(x-a)^2\right),$$

and deduce the time dependence of the expectation value $\langle x\rangle$.

4.3 For an infinitesimal time step $t \to t + \epsilon$, the transition amplitude

$$\langle x, t+\epsilon|y, t\rangle = A \exp\left(\frac{im\eta^2}{2\hbar\epsilon} - \frac{i\epsilon}{\hbar}V(x,t)\right),$$

where $\eta = y - x$ and the normalization factor $A = (m/2\pi i\hbar\epsilon)^{\frac{1}{2}}$. By using this amplitude to evolve a wave function $\psi(y,t)$ forwards in time to $\psi(x, t+\epsilon)$, and then expanding to first order in ϵ and second order in $\eta = O(\epsilon^{1/2})$ (so that the kinetic energy term remains of order unity) and doing several Gaussian integrals, show that $\psi(x,t)$ satisfies Schrödinger's equation.

4.4 Consider a particle of mass m and charge e moving in the x-y plane under the influence of a constant magnetic field B normal to the plane. Initially the particle is localised at the origin, with Gaussian wave function

$$\langle x, y; 0|\psi\rangle = N(a)e^{-(x^2+y^2)/4a^2},$$

where $N(a)$ is a normalisation factor fixed in the usual way. Show that at a later time t the probability density

$$|\langle x, y; t|\psi\rangle|^2 = (N(a(t)))^2 \, e^{-(x^2+y^2)/2a(t)^2},$$

where

$$a(t)^2 = a^2 \cos^2 \frac{\omega t}{2} + \frac{\hbar^2}{m^2\omega^2 a^2} \sin^2 \frac{\omega t}{2},$$

where $\omega = eB/mc$.

[Hint: Use the transition amplitude computed in questions 2.4 and 4.1.]

4.5 Take the transition amplitude $\langle x, t|x', t'\rangle$ for the harmonic oscillator and expand it in powers of $e^{-i\omega T}$, where $T = t - t'$, to deduce the stationary state wave functions $u_n(x) \equiv \langle x|n\rangle$, $n = 0, 1, 2, \ldots$.

[You should find that (up to a constant phase factor) $u_0 = (m\omega/\pi\hbar)^{1/4} \exp(-m\omega x^2/2\hbar)$, $u_1 = (2m\omega/\hbar)^{1/2} x u_0$, $u_2 = 2^{-1/2}((2m\omega/\hbar)x^2 - 1)u_0$, etc.]

4.6 The Hamiltonian of a general linear oscillator in one dimension is $\hat{H} = \hat{p}^2/2m + \frac{1}{2}\, m\,\omega^2\,\hat{q}^2$.

(i) Show that in the Heisenberg picture $\dot{\hat{q}} = \hat{p}/m$ and $\dot{\hat{p}} = -m\,\omega^2\,\hat{q}$, and hence that $\ddot{\hat{q}} + \omega^2\,\hat{q} = 0$.

(ii) By taking matrix elements of $\dot{\hat{q}}$ in the energy eigenbasis show that $q_{nk}(t) = q_{nk}(0)e^{i\omega_{nk}t}$ where $\omega_{nk} = (E_n - E_k)/\hbar$ and thence that $q_{nk} = 0$ if $\omega_{nk} \neq \pm\omega$.

(iii) By taking matrix elements of the commutator $[\hat{q}(t), \hat{p}(t)] = i\hbar$ obtain the recurrence relation

$$q_{n,\,n+1}\, q_{n+1,\,n} = q_{n-1,\,n}\, q_{n,\,n-1} + \hbar/2m\omega$$

and hence show that

$$q_{n,\,n+1}\, q_{n+1,\,n} = (n+1)\hbar/2m\omega, \qquad p_{n,\,n+1}\, p_{n+1,\,n} = \tfrac{1}{2} m\hbar\omega\,(n+1).$$

(iv) Use these results to deduce the energy eigenvalues $E_n = (n + \frac{1}{2})\,\hbar\omega$.

[This was the method originally used by Born, Heisenberg & Jordan to solve the SHO.]