Vector Calculus 2013/14

Vector Calculus – a résumé

These notes summarise many of the results of vector calculus derived/used in pre-Honours *Vector Calculus*. They are far from complete, *e.g.* basic line, surface and volume integrals aren't defined, for example.

Both index notation (with the i^{th} Cartesian coordinate denoted by x_i and the i^{th} basis vector by $\underline{e}_i$), and 'long-hand' notation (x, y, z and $\underline{e}_x$, $\underline{e}_y$, $\underline{e}_z$) are used.

Vectors

The vector $\underline{a}$ has components a_i (or a_x, a_y, a_z) :

$$\underline{a} \;=\; \sum_{i=1}^{3} a_i\,\underline{e}_i \qquad \Big(= a_x\,\underline{e}_x + a_y\,\underline{e}_y + a_z\,\underline{e}_z\Big)$$

where $\{\underline{e}_i\}$ or $\{\underline{e}_x, \underline{e}_y, \underline{e}_z\}$ are a set of orthonormal basis vectors satisfying:

$$\underline{e}_i \cdot \underline{e}_j \;=\; \delta_{ij} \qquad \Big(\underline{e}_x \cdot \underline{e}_x = 1,\; \underline{e}_x \cdot \underline{e}_y = 0,\; etc\Big)\,.$$

The position vector $\underline{r}$ has components x_i (or x, y, z):

$$\underline{r} \;=\; \sum_i x_i\,\underline{e}_i \qquad \Big(= x\,\underline{e}_x + y\,\underline{e}_y + z\,\underline{e}_z\Big)\;.$$

Scalar and vector fields

$$\begin{aligned} \phi(\underline{r}) &= \text{scalar function of } \underline{r} \;=\; \text{a scalar field} \\ \underline{a}(\underline{r}) &= \text{vector function of } \underline{r} \;=\; \text{a vector field} \;=\; \sum_i a_i(\underline{r})\,\underline{e}_i\;. \end{aligned}$$

The *scalar product* of $\underline{a}$ and $\underline{b}$ is $\underline{a} \cdot \underline{b} = \sum_i a_i b_i$ $\;(= a_x b_x + a_y b_y + a_z b_z)$

The *vector product* of $\underline{a}$ and $\underline{b}$ is denoted $\underline{a} \times \underline{b}$, and defined by

$$\underline{a} \times \underline{b} \equiv \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The *vector operator* $\underline{\nabla}$ ('del'), defined as

$$\underline{\nabla} \;=\; \sum_i \underline{e}_i \frac{\partial}{\partial x_i} \;=\; \underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y} + \underline{e}_z \frac{\partial}{\partial z}$$

can operate on scalar and vector fields to give, in terms of Cartesian coordinates:

$$\begin{aligned}
\operatorname{grad}\phi &= \underline{\nabla}\,\phi &&= \sum_i \underline{e}_i \frac{\partial \phi}{\partial x_i} &&= \underline{e}_x \frac{\partial \phi}{\partial x} + \underline{e}_y \frac{\partial \phi}{\partial y} + \underline{e}_z \frac{\partial \phi}{\partial z} \\
\operatorname{div}\underline{a} &= \underline{\nabla}\cdot\underline{a} &&= \sum_i \frac{\partial a_i}{\partial x_i} &&= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \\
\operatorname{curl}\underline{a} &= \underline{\nabla}\times\underline{a} && &&= \underline{e}_x \left\{ \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right\} + \cdots \; .
\end{aligned}$$

The *chain rule*: If $F(\phi(\underline{r}))$ is a scalar field, then $\underline{\nabla}\, F\,(\phi) \;=\; \dfrac{dF(\phi)}{d\phi}\, \underline{\nabla}\,\phi(\underline{r})$

If ϕ and $\underline{F}$ are built from two or more other fields (*e.g.* $\phi = \underline{a}\cdot\underline{b}$, $\underline{F} = \phi\,\underline{a}$, $\underline{F} = \underline{a}\times\underline{b}$) where ϕ and ψ are scalar fields, and $\underline{a}$, $\underline{b}$ and $\underline{F}$ are vector fields, then the following identities are found to be useful:

$$\begin{aligned}
\underline{\nabla}\,(\phi\psi) &= \phi(\underline{\nabla}\,\psi) + (\underline{\nabla}\,\phi)\psi & \qquad \underline{\nabla}\cdot(\underline{a}\times\underline{b}) &= \underline{b}\cdot(\underline{\nabla}\times\underline{a}) - \underline{a}\cdot(\underline{\nabla}\times\underline{b}) \\
\underline{\nabla}\cdot(\phi\underline{a}) &= (\underline{\nabla}\,\phi)\cdot\underline{a} + \phi(\underline{\nabla}\cdot\underline{a}) & \qquad \underline{\nabla}\times(\phi\underline{a}) &= \phi(\underline{\nabla}\times\underline{a}) + (\underline{\nabla}\,\phi)\times\underline{a}
\end{aligned}$$

$$\begin{aligned}
\underline{\nabla}\,(\underline{a}\cdot\underline{b}) &= (\underline{a}\cdot\underline{\nabla}\,)\underline{b} + (\underline{b}\cdot\underline{\nabla}\,)\underline{a} + \underline{a}\times(\underline{\nabla}\times\underline{b}) + \underline{b}\times(\underline{\nabla}\times\underline{a}) \\
\underline{\nabla}\times(\underline{a}\times\underline{b}) &= (\underline{\nabla}\cdot\underline{b})\,\underline{a} + (\underline{b}\cdot\underline{\nabla}\,)\,\underline{a} - (\underline{\nabla}\cdot\underline{a})\,\underline{b} - (\underline{a}\cdot\underline{\nabla}\,)\,\underline{b}.
\end{aligned}$$

Second-order operators can be formed from $\underline{\nabla}$, such as:

$$\begin{aligned}
\underline{\nabla}\cdot(\underline{\nabla}\,\phi) &= \sum_i \frac{\partial^2 \phi}{\partial x_i^2} \;\equiv\; \nabla^2\phi && \text{[Laplacian acting on a } \textit{scalar}\text{]} \\
\underline{\nabla}\times(\underline{\nabla}\times\underline{a}) &= \underline{\nabla}\,(\underline{\nabla}\cdot\underline{a}) \;-\; \nabla^2\underline{a} && \text{[Laplacian acting on a } \textit{vector}\text{]}
\end{aligned}$$

Also $\qquad \underline{\nabla}\cdot(\underline{\nabla}\times\underline{a}) \;\equiv\; 0 \qquad$ and $\qquad \underline{\nabla}\times(\underline{\nabla}\,\phi) \;\equiv\; 0$

For the *position vector* $\underline{r}$ we have the special results:

$$\underline{\nabla}\cdot\underline{r} \;=\; 3\,, \qquad \underline{\nabla}\times\underline{r} \;=\; 0\,, \qquad \underline{\nabla}\times(\underline{c}\times\underline{r}) \;=\; 2\underline{c}$$

$$(\underline{c}\cdot\underline{\nabla}\,)\,\underline{r} \;=\; \underline{\nabla}\,(\underline{c}\cdot\underline{r}) \;=\; \underline{c} \qquad \text{where } \underline{c} \text{ is a } \textit{constant} \text{ vector}$$

$$\underline{\nabla}\; f(r) \;=\; f'(r)\,\underline{r}/r \quad \text{where } r = |\underline{r}|\;.$$

$$\textit{e.g.} \qquad \underline{\nabla}\; r^n \;=\; n\,r^{n-2}\,\underline{r} \qquad (\text{because } f(r) = r^n \Rightarrow f'(r) = nr^{n-1})$$

Integral Vector Calculus

The line, surface and volume integrals of these fields are related by the *divergence theorem* (sometimes called Gauss' theorem) and *Stokes' theorem*:

$$\int_V (\underline{\nabla} \cdot \underline{a})\, \mathrm{d}V = \int_S \underline{a} \cdot \mathrm{d}\underline{S} \quad \text{with } S \text{ a } \textit{closed} \text{ surface bounding volume } V.$$

$$\int_S (\underline{\nabla} \times \underline{a}) \cdot \mathrm{d}\underline{S} = \oint_C \underline{a} \cdot \mathrm{d}\underline{r} \quad \text{with } S \text{ an } \textit{open} \text{ surface, bounded by a } \textit{closed} \text{ curve } C.$$

Scalar and vector potentials

Scalar potential:

$$\underline{\nabla} \times \underline{a} = 0 \quad \Leftrightarrow \quad \exists\, \phi \text{ such that } \underline{a} = \underline{\nabla}\, \phi$$

Vector potential:

$$\underline{\nabla} \cdot \underline{B} = 0 \quad \Leftrightarrow \quad \exists\, \underline{A} \text{ such that } \underline{B} = \underline{\nabla} \times \underline{A}$$

Orthogonal curvilinear coordinates

If the symmetry of the problem suggests a coordinate system other than Cartesian, we need to generalise the relations between div, grad, curl and ∇^2 and the partial derivatives as given in the first part of this résumé.

For *orthogonal curvilinear coordinates* $\{u_i\}$, the unit vectors $\{\underline{e}_i\}$ and scale factors $\{h_i\}$, for all $i = 1, 2, 3$, are defined as follows:

$$\frac{\partial \underline{r}}{\partial u_i} = \underline{e}_i\, h_i$$

The basic operations are given in terms of h_i, $\underline{e}_i$ and $\dfrac{\partial}{\partial u_i}$ are as follows (with $\underline{a} = a_i\, \underline{e}_i$)

$$\underline{\nabla}\,\phi = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial \phi}{\partial u_i}\,\underline{e}_i$$

$$\begin{aligned}\underline{\nabla}\cdot\underline{a} &= \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(h_2h_3a_1) + \frac{\partial}{\partial u_2}(h_3h_1a_2) + \frac{\partial}{\partial u_3}(h_1h_2a_3)\right\} \\ &= \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(h_2h_3a_1) + \text{cyclic combinations}\right\}\end{aligned}$$

$$\underline{\nabla}\times\underline{a} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\,\underline{e}_1 & h_2\,\underline{e}_2 & h_3\,\underline{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1a_1 & h_2a_2 & h_3a_3 \end{vmatrix}$$

$$\begin{aligned}\nabla^2\phi &= \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}\left\{\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right\} + \frac{\partial}{\partial u_2}\left\{\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial u_2}\right\} + \frac{\partial}{\partial u_3}\left\{\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right\}\right\}. \\ &= \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}\left\{\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right\} + \text{cyclic combinations}\right\}.\end{aligned}$$

The most commonly used systems are:

1. Circular cylindrical coordinates (ρ, ϕ, z):

$$x = \rho\cos\phi\,, \quad y = \rho\sin\phi\,, \quad z = z\,, \quad \text{with} \quad h_\rho = 1\,, \quad h_\phi = \rho\,, \quad h_z = 1.$$

2. Spherical polar coordinates (r, θ, ϕ):

$$x = r\sin\theta\cos\phi\,, \quad y = r\sin\theta\sin\phi\,, \quad z = r\cos\theta \quad \text{with} \quad h_r = 1\,, \quad h_\theta = r\,, \quad h_\phi = r\sin\theta.$$

The elements of length, area and volume may be written in terms of the scale factors.