Vector Calculus 2013/14

Tutorial Sheet 1: Vector products, scalar fields, level surfaces, gradient

$*$ denotes **harder** problems or parts of problems
♣ denotes hand–in questions

1.1 For vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$, state whether the following are true or false:

(i) $(\underline{a} \cdot \underline{b})\, \underline{a} = (\underline{a} \cdot \underline{a})\, \underline{b}$

(ii) If $\underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c}$ then $\underline{a} = \underline{b}$

(iii) $(\underline{a} \times \underline{b}) \times (\underline{a} \times \underline{b}) = 0$

(iv) $\underline{a} \times (\underline{a} \times \underline{b}) = 0$

Explain your answers. For those that are false, give a counterexample.

1.2 Prove the identity $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c}$, and use it to deduce the identities

(i) $(\underline{a} \times \underline{b}) \times \underline{c} = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{b} \cdot \underline{c})\underline{a}$

(ii) $\underline{a} \times (\underline{b} \times \underline{c}) + \underline{b} \times (\underline{c} \times \underline{a}) + \underline{c} \times (\underline{a} \times \underline{b}) = 0$

(iii) $(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c})$

(iv) $(\underline{a} \times \underline{b}) \times (\underline{c} \times \underline{a}) = -(\underline{a}, \underline{b}, \underline{c})\, \underline{a}$

(v) $(\underline{a} \times \underline{b},\ \underline{b} \times \underline{c},\ \underline{c} \times \underline{a}) = (\underline{a}, \underline{b}, \underline{c})^2$

In the above equations, $(\underline{a}, \underline{b}, \underline{c})$ denotes the scalar triple product $\underline{a} \cdot (\underline{b} \times \underline{c})$. Remember the cyclic symmetry properties of $(\underline{a}, \underline{b}, \underline{c})$.

1.3 If $\phi(x, y, z) = 3x^2y - y^3z^2$, find $\underline{\nabla}\phi$ and its value at the point $(1, -2, 1)$

1.4 Calculate $\underline{\nabla}\phi$ for each of the scalar fields

(i) $\phi(x, y, z) = (xyz)^3$

(ii) $\phi(x, y, z) = xy + yz + zx$

(iii) $\phi(x, y, z) = x\, e^{yz}$

1.5♣ Describe the level surfaces (equipotentials), and calculate the gradient $\underline{\nabla}\phi$ for the following scalar fields.

[*Hints:* Parts (iii) & (iv) are in 3D (not just 2D). For part (v) try considering $\cos\phi$.]

(i) $\phi(\underline{r}) = x_1 + 2x_2 - 3x_3$

(ii) $\phi(\underline{r}) = 2x_1^2 + x_2^2 + 3x_3^2$

(iii) $\phi(\underline{r}) = x_1^2 + x_2^2$

(iv) $\phi(\underline{r}) = (x_1^2 + x_2^2)^{-1}$

(v) $\phi(\underline{r}) = \cos^{-1}(x_1^2 + x_2^2 - x_3)$

(PTO)

1.6 Calculate $\underline{\nabla}\phi$ and describe the level surfaces for each of following scalar fields, where in each case $\underline{r}$ is the position vector, $r = |\underline{r}|$ is the length of the position vector, and $\underline{a}$ is a constant vector (*i.e.* it doesn't depend on $\underline{r}$.)

(i) $\phi(\underline{r}) = \underline{a} \cdot \underline{r}$

(ii) $\phi(\underline{r}) = (\underline{a} \cdot \underline{r})^2$

(iii) $\phi(\underline{r}) = r$

(iv) $\phi(\underline{r}) = |\underline{r} - \underline{a}|^2$

1.7 Consider the complex scalar field

$$\phi = \frac{\exp(ikr)}{4\pi r}$$

where r is the length of the position vector (in 3 dimensions).

Describe the level surfaces. Hence explain why the field is known as a spherical wave. What is its wavelength?

1.8♣ **Homework problem:** You should work through this problem on your own; you are not allowed to ask the tutors for help with this problem.

In this question, $\underline{r}$ is the position vector, $r = |\underline{r}|$ is the length of the position vector, $\underline{a}$ and $\underline{b}$ are constant vectors, and n is an integer.

Let $\phi(\underline{r}) = r^n$.

(i) Show that $r^n = (x_1^2 + x_2^2 + x_3^2)^{n/2}$ and hence evaluate $\dfrac{\partial \phi}{\partial x_1}$, $\dfrac{\partial \phi}{\partial x_2}$ and $\dfrac{\partial \phi}{\partial x_3}$.

(ii) Hence show that $\underline{\nabla} r^n = n r^{n-2} \underline{r}$

For each of following scalar fields, evaluate $\dfrac{\partial \phi}{\partial x_1}$, $\dfrac{\partial \phi}{\partial x_2}$ and $\dfrac{\partial \phi}{\partial x_3}$, and hence obtain $\underline{\nabla}\phi$.

(iii) $\phi(\underline{r}) = |\underline{a} \times \underline{r}|^2$ [*Hint:* First show that $|\underline{a} \times \underline{r}|^2 = a^2 r^2 - (\underline{a} \cdot \underline{r})^2$]

(iv) $\phi(\underline{r}) = (\underline{a} \cdot \underline{r})\, r^2$

(v) $\phi(\underline{r}) = (\underline{a} \cdot \underline{r})\,(\underline{b} \cdot \underline{r})$

Show that

(vi) $\underline{\nabla}\left(|\underline{r} - \underline{a}|^n\right) = n|\underline{r} - \underline{a}|^{n-2}\,(\underline{r} - \underline{a})$