Vector Calculus 2013/14

Tutorial Sheet 2: Lines, planes, directional derivatives and gradients

$*$ denotes **harder** problems or parts of problems
♣ denotes hand–in questions

2.1 Write down an expression for the *unit* normal to the level surface $\phi(x, y, z) = c$ at the point P with position vector $\underline{r}_0 = (x_0, y_0, z_0)$. Hence write down an equation for the tangent plane to the level surface at $\underline{r}_0$.

Use your result to show that the equation of the tangent plane to the surface of the sphere $x^2 + y^2 + z^2 = a^2$ at the point $(0, 0, a)$ is $z = a$, and that the equation of the line normal to the surface at that point may be written as $x = y = 0$.

2.2♣ (i) The temperature of points in space is given by $T(\underline{r}) = x^2 + y^2 - z$. A mosquito located at $(1, 1, 2)$ desires to fly in such a direction that it *cools* as quickly as possible. In which direction should it set out?

Another mosquito is flying at a speed of $5\,\mathrm{ms}^{-1}$ in the direction of the vector $(4, 4, -2)$. What is its rate of increase of temperature (with respect to time) as it passes through the point $(1, 1, 2)$?

(ii) A masochistic hiker wants to find the steepest route up a mountain of altitude $h(x, y) = 100 - x^2 - 2y^2$. Assuming she starts at $(x, y) = (\sqrt{2}, 7)$ (*i.e.* $h = 0$) find the projection of the steepest path upon the x–y plane.

Hint: Show that the hiker should always move in a direction $\mathrm{d}\underline{r}$ such that $dy/dx = 2y/x$.

(iii) Find the equation of the normal line and tangent plane to the surface $z = 1+xy$ at the point $(1, 1, 2)$.

2.3 The *electrostatic potential* $\phi(\underline{r})$ due to a charge q at the origin is

$$\phi(\underline{r}) \;=\; \frac{q}{4\pi\epsilon_0 r}$$

Show that the electric field defined as $\underline{E}(\underline{r}) = -\underline{\nabla}\phi(\underline{r})$ is

$$\underline{E}(\underline{r}) \;=\; \frac{q}{4\pi\epsilon_0}\,\frac{\underline{r}}{r^3}$$

Show that for small $\underline{a}$, the potential $\phi(\underline{r} + \underline{a})$ may be approximated to first order in a by

$$\phi(\underline{r} + \underline{a}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{\underline{a}\cdot\underline{r}}{r^3}\right] + O(a^2)$$

By taking the gradient of the RHS of this expression, find $\underline{E}(\underline{r} + \underline{a})$ to first order in a.

2.4 Solve the following sets of equations for $\underline{r}$, and give a geometrical interpretation to your results.

(i) $\underline{a}\cdot\underline{r} = l$; $\underline{b}\times\underline{r} = \underline{c}$, where $\underline{a}\cdot\underline{b} \neq 0$;

(ii) $\underline{a}\cdot\underline{r} = l$; $\underline{b}\cdot\underline{r} = m$; $\underline{c}\cdot\underline{r} = n$, where $(\underline{a}, \underline{b}, \underline{c}) \neq 0$.

Hint for part (i): take the cross product of $\underline{a}$ with the second equation, and simplify the result.

Hint for part (ii): write $\underline{r}$ as a linear combination of $\underline{a}\times\underline{b}$, $\underline{b}\times\underline{c}$ and $\underline{c}\times\underline{a}$ and take scalar products with $\underline{a}, \underline{b}$, and $\underline{c}$.

(PTO)

2.5 Calculate the divergence and curl of the following vector fields:

(i) $\underline{a} = c\,\underline{e}_x$

(ii) $\underline{a} = cx\,\underline{e}_x$

(iii) $\underline{a} = cy\underline{e}_x$

(iv) $\underline{a} = -c\,\underline{r}$

(v) $\underline{a} = \underline{d} \times \underline{r}$

where $\underline{r} = x\underline{e}_x + y\underline{e}_y + z\underline{e}_z$ and c and $\underline{d}$ are constant.

Sketch the fields in the $\underline{e}_x - \underline{e}_y$ plane. (For (v) choose your basis so that $\underline{d} = d\underline{e}_z$.)

2.6♣ **Homework problem:** You should work through this problem on your own; you are not allowed to ask the tutors for help with this problem.

In this question, $\underline{r}$ is the position vector, $r = |\underline{r}|$ is the length of the position vector, $\underline{a}$ and $\underline{b}$ are constant vectors, and n is an integer.

(i) Show that $(\underline{a} \cdot \underline{\nabla})\,\underline{r} = \underline{a}$

(ii) Evaluate $\underline{\nabla}\,\{\underline{a} \cdot (\underline{b} \times \underline{r})\}$

Use the product and chain rules for the gradient to evaluate $\underline{\nabla}\phi$ for the following scalar fields

(iii) $\phi(\underline{r}) = |\underline{a} \times \underline{r}|^2$ [*Hint:* First show that $|\underline{a} \times \underline{r}|^2 = a^2 r^2 - (\underline{a} \cdot \underline{r})^2$]

(iv) $\phi(\underline{r}) = |\underline{r} - \underline{a}|^n$ [*Hint:* First show that $|\underline{r} - \underline{a}|^n = (r^2 - 2\underline{r} \cdot \underline{a} + a^2)^{n/2}$]

(v) $\phi(\underline{r}) = r^m\,|\underline{r} - \underline{a}|^n$

(vi) $\phi(\underline{r}) = (\underline{a} \cdot \underline{r})^m\,(\underline{b} \cdot \underline{r})^n\,\{\underline{a} \cdot (\underline{b} \times \underline{r})\}$

NB: You may have noticed that parts (iii) and (iv) were part of last week's homework question. The point is that this week you should use the product and chain rules for the gradient to evaluate them, which is much quicker! Do *not* write out the fields in explicit Cartesian coordinates.