Vector Calculus 2013/14

Tutorial Sheet 3: *Div*, *grad*, *curl* and the *Laplacian*

$*$ denotes **harder** problems or parts of problems
♣ denotes hand–in questions

3.1 Given a vector field $\underline{a} = xyz(x-y)\,\underline{e}_x + xyz(x+y)\,\underline{e}_y + xyz^2\,\underline{e}_z$ show that

$$\underline{\nabla} \cdot \underline{a} = 6xyz + z(x^2 - y^2)$$

and that

$$\underline{\nabla} \times \underline{a} = [x(z^2 - y^2 - xy)]\,\underline{e}_x + [y(x^2 - z^2 - xy)]\,\underline{e}_y + [z(4xy + y^2 - x^2)]\,\underline{e}_z$$

3.2 If $f(\underline{r})$ and $g(\underline{r})$ are scalar fields, and $\underline{a}(\underline{r})$ is a vector field, verify the following identities by working in Cartesian coordinates.

[In parts (i) and (iii) you should verify that the identity holds for all three Cartesian components.]

(i) $\underline{\nabla}(fg) = f\underline{\nabla}g + g\underline{\nabla}f$

(ii) $\underline{\nabla} \cdot (f\underline{a}) = (\underline{\nabla}f) \cdot \underline{a} + f(\underline{\nabla} \cdot \underline{a})$

(iii) $\underline{\nabla} \times (f\underline{a}) = f(\underline{\nabla} \times \underline{a}) + (\underline{\nabla}f) \times \underline{a}$

Repeat part (ii) using the identity $\underline{\nabla} \cdot \underline{b} = \sum_{i=1}^{3} \underline{e}_i \cdot \dfrac{\partial \underline{b}}{\partial x_i}$ with $\underline{b}(\underline{r}) \equiv f(\underline{r})\,\underline{a}(\underline{r})$

3.3♣ If $g(\underline{r})$ and $h(\underline{r})$ are scalar fields, use the product rule for scalar fields to show that

(i) $\nabla^2(gh) = g\nabla^2 h + 2(\underline{\nabla}g) \cdot (\underline{\nabla}h) + (\nabla^2 g)h$

If $\underline{r}$ is the position vector and $f(r)$ is a scalar field which depends only on the magnitude r of the position vector $\underline{r}$, use the product and chain rules to show that

(ii) $\underline{\nabla} \cdot (r^n \underline{r}) = (n+3)r^n$

(iii) $\underline{\nabla} \cdot \{f(r)\underline{r}\} = 3f(r) + rf'(r)$

(iv) $\underline{\nabla} \times (r^n \underline{r}) = 0$

(v) $\underline{\nabla} \times (f(r)\underline{r}) = 0$

(vi) $\nabla^2 f(r) = f''(r) + 2f'(r)/r$, and, in particular, $\nabla^2(1/r) = 0 \quad (r \neq 0)$

The quantity $f'(r)$ denotes the derivative of $f(r)$ with respect to the scalar quantity r.

3.4 If $\underline{a}$ and $\underline{b}$ are vector fields, establish the following identities using Cartesian coordinates

(i) $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{a}) = 0$

(ii) $\underline{\nabla} \cdot (\underline{a} \times \underline{b}) = \underline{b} \cdot (\underline{\nabla} \times \underline{a}) - \underline{a} \cdot (\underline{\nabla} \times \underline{b})$

(iii) $\underline{\nabla} \times (\underline{a} \times \underline{b}) \;=\; \underline{a}\,(\underline{\nabla} \cdot \underline{b}) - \underline{b}\,(\underline{\nabla} \cdot \underline{a}) + (\underline{b} \cdot \underline{\nabla})\,\underline{a} - (\underline{a} \cdot \underline{\nabla})\,\underline{b}$

In parts (ii) and (iii), you should evaluate the LHS and RHS, and show they are equal. This requires quite a bit of algebra, but is conceptually simpler than the proofs given in lectures!

3.5 Evaluate the following expressions, in which $\underline{a}$ and $\underline{b}$ are constant vectors, $\underline{r}$ is the position vector, $r = |\underline{r}|$ is the length of the position vector, and $\hat{\underline{r}} = \underline{r}/r$ is a unit vector in the direction of $\underline{r}$

(i)	$\underline{\nabla} \cdot \hat{\underline{r}}$	(iv)	$\underline{\nabla} \times \{\underline{a} \times (\underline{r} \times \underline{b})\}$
(ii)	$\underline{\nabla} \times \hat{\underline{r}}$	(v)	$\underline{\nabla} \times \{(\underline{a} \times \underline{r})/r^3\}$
(iii)	$\underline{\nabla} \cdot \{(\underline{a} \cdot \underline{r})\underline{b}\}$	(vi)	$\nabla^2 r^n$

3.6♣ **Homework problem:** You should work through this problem on your own; you are not allowed to ask the tutors for help with this problem.

Evaluate the following expressions, in which $\underline{a}$ and $\underline{b}$ are *constant* vectors, $\underline{r}$ is the position vector, and $|\underline{r}|$ is the length of the position vector.

(i)	$\underline{\nabla} \times \{(\underline{a} \cdot \underline{r})\underline{b}\}$	(iv)	$\underline{\nabla}\{(\underline{a} \cdot \underline{r})/r^3\}$
(ii)	$\underline{\nabla} \cdot \{\underline{a} \times (\underline{r} \times \underline{b})\}$	(v)	$\underline{\nabla} \times \{(\underline{a} \cdot \underline{r})(\underline{a} \times \underline{r})\}$
(iii)	$\underline{\nabla} \cdot \{(\underline{a} \times \underline{r})/r^3\}$	(vi)	$\nabla^2 \{(\underline{a} \cdot \underline{r})r^n\}$

Your answers should be written in terms of vectors, not components of vectors.