Vector Calculus 2013/14

Tutorial Sheet 4: Physics applications of *div*, *grad*, *curl* and the *Laplacian*

$*$ denotes **harder** problems or parts of problems
♣ denotes hand–in questions

4.1 For vector fields $\underline{a}(\underline{r})$ and $\underline{b}(\underline{r})$, show by writing out the components of $(\underline{a} \times \underline{b})$, that

$$\frac{\partial}{\partial x_i}(\underline{a} \times \underline{b}) \;=\; \frac{\partial \underline{a}}{\partial x_i} \times \underline{b} + \underline{a} \times \frac{\partial \underline{b}}{\partial x_i}$$

for $i = 1, 2, 3$.

4.2 By writing out the components, or using the following expressions

$$\underline{\nabla} \;=\; \sum_{i=1}^{3} \underline{e}_i \frac{\partial}{\partial x_i}\,, \qquad \underline{\nabla} \cdot \underline{a} = \sum_{i=1}^{3} \underline{e}_i \cdot \frac{\partial \underline{a}}{\partial x_i}\,, \qquad \underline{\nabla} \times \underline{a} = \sum_{i=1}^{3} \underline{e}_i \times \frac{\partial \underline{a}}{\partial x_i}\,,$$

establish the following identities for vector fields $\underline{a}(\underline{r})$ and $\underline{b}(\underline{r})$:

(i) $\underline{\nabla} \times (\underline{\nabla} \times \underline{a}) \;=\; \underline{\nabla}(\underline{\nabla} \cdot \underline{a}) - \nabla^2 \underline{a}$

(ii)* $\underline{\nabla}(\underline{a} \cdot \underline{b}) \;=\; (\underline{a} \cdot \underline{\nabla})\underline{b} + (\underline{b} \cdot \underline{\nabla})\underline{a} + \underline{a} \times (\underline{\nabla} \times \underline{b}) + \underline{b} \times (\underline{\nabla} \times \underline{a})$

Hint: In part (ii), start by considering $\underline{a} \times (\underline{\nabla} \times \underline{b})$.

4.3 If $\phi(\underline{r})$ and $\psi(\underline{r})$ are scalar fields, and $\underline{a}(\underline{r})$ is a vector field, show that

(i) $\underline{\nabla} \cdot (\phi \underline{\nabla} \psi - \psi \underline{\nabla} \phi) \;=\; \phi \nabla^2 \psi - \psi \nabla^2 \phi$

(ii) $(\underline{a} \cdot \underline{\nabla})\,\underline{a} \;=\; \frac{1}{2}\underline{\nabla} a^2 \;-\; \underline{a} \times (\underline{\nabla} \times \underline{a})$, where, $a^2 \equiv \underline{a} \cdot \underline{a}$ as usual.

(iii) $\underline{\nabla} \cdot \{(\underline{a} \times \underline{r})\,/r^3\} = \underline{r} \cdot (\underline{\nabla} \times \underline{a})/r^3$

(iv) $\underline{\nabla} \times (\underline{r} \times (\underline{\nabla} \times \underline{a})) \;=\; -\,2\,\underline{\nabla} \times \underline{a} - (\underline{r} \cdot \underline{\nabla})(\underline{\nabla} \times \underline{a})$

Hint: You may use the product rules and any of the results derived in this course thus far, including the previous questions on this tutorial sheet.

4.4♣ The electric field at $\underline{r}$ due to a particle of charge q at the origin is

$$\underline{E}(\underline{r}) \;=\; \frac{q}{4\pi\epsilon_0}\frac{\underline{r}}{r^3}$$

(i) Show that, for $\underline{r} \neq 0$, $\underline{\nabla} \cdot \underline{E} = 0$ and $\underline{\nabla} \times \underline{E} = 0$

(ii) If the particle has mass m, deduce that the gravitational field $\underline{G} = (-Gm/r^3)\,\underline{r}$ due to the particle satisfies $\underline{\nabla} \cdot \underline{G} = 0$ and $\underline{\nabla} \times \underline{G} = 0$, when $\underline{r} \neq 0$

Let $\underline{E}(\underline{r})$ be an electric field which satisfies $\underline{\nabla} \times \underline{E} = 0$, but is otherwise arbitrary.

(iii) If $\underline{a}$ is a constant vector, show that $\underline{\nabla}(\underline{a} \cdot \underline{E}) = (\underline{a} \cdot \underline{\nabla})\,\underline{E}$

Hint: You may use the product rules and any of the results derived in this course thus far, including the questions above.

(PTO)

4.5♣ The velocity of an element of fluid at position $\underline{r}(t)$ at time t is described by a vector field $\underline{u}(\underline{r}(t), t)$. Use the chain rule to show that the *total derivative* of this velocity field with respect to time is

$$\frac{d\underline{u}}{dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\,\underline{u}$$

where $\partial \underline{u}/\partial t$ is the partial derivative with respect to t at fixed $\underline{r}$. Hence show that

$$\frac{d\underline{u}}{dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{\nabla} \times \underline{u}) \times \underline{u} + \frac{1}{2}\underline{\nabla} u^2$$

Hint: For the last bit, you will need a result derived in one of the previous questions.

4.6 The condition for hydrostatic equilibrium in a fluid or gas of density (mass per unit volume) $\rho(\underline{r})$ is

$$\rho\,\underline{F} = \underline{\nabla}\,p$$

where $\underline{F}(\underline{r})$ is the external force per unit mass at the point with position vector $\underline{r}$, and $p(\underline{r})$ is the pressure at $\underline{r}$.

(i) Show that $\underline{F}$ satisfies $\underline{F} \cdot (\underline{\nabla} \times \underline{F}) = 0$.

(ii) The density of the Earth's atmosphere close to its surface may be taken to be constant. Taking $\underline{F}$ to be the gravitational force per unit mass, obtain a differential equation for the variation of pressure with height z above the Earth's surface. Hence show that

$$p(z) = p_0 - \rho g z\,,$$

and interpret the constant p_0.

4.7 Revision of planar/double integration in Cartesian and plane-polar coordinates.

Do *not* use plane polar coordinates in parts (i) or (ii) of the question.

(i) Use Cartesian coordinates to evaluate the integral

$$\int_S xy\,\mathrm{d}S$$

where S is the first quadrant of the unit circle $x^2 + y^2 < 1$, $x \geq 0$, $y \geq 0$, and $\mathrm{d}S$ is the (infinitesimal) element of area in the $x{-}y$ plane.

(ii) Use Cartesian coordinates to evaluate the area $S = \int_S \mathrm{d}S$.

(iii) Repeat parts (i) and (ii) using plane polar coordinates, $x = \rho\cos\phi$, $y = \rho\sin\phi$.

4.8♣ **Homework problem:** You should work through this problem on your own; you are not allowed to ask the tutors for help with this problem.

Maxwell's equations for the electric field $\underline{E}(\underline{r})$ and the magnetic field $\underline{B}(\underline{r})$ in a charge- and current-free region are:

$$\underline{\nabla} \cdot \underline{E} = 0 \qquad \underline{\nabla} \cdot \underline{B} = 0 \qquad \underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \frac{1}{\mu_0}\underline{\nabla} \times \underline{B} = \epsilon_0 \frac{\partial \underline{E}}{\partial t}$$

(i) By considering $\underline{\nabla} \times (\underline{\nabla} \times \underline{E})$, use Maxwell's equations and an appropriate vector identity to show that

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\underline{E} = 0$$

Hence obtain an expression for the constant c.

(ii) Find a similar equation for the magnetic field $\underline{B}$.