Vector Calculus 2013/14

Tutorial Sheet 5: Line integrals

$*$ denotes **harder** problems or parts of problems
♣ denotes hand–in questions

5.1 Evaluate the line integral

$$\int_C (\underline{a} \times \underline{r}) \cdot \mathrm{d}\underline{r}$$

where the curve C is a semicircle of unit radius in the (x, y) plane: $\{x^2 + y^2 = 1; x \geq 0\}$, and $\underline{a}$ is a constant vector. Use a parametric representation for the curve C.

5.2 A particle moves under the influence of a force $\underline{F}(\underline{r})$.

(i) Calculate the work done by evaluating the line integral

$$\int_C \underline{F} \cdot \mathrm{d}\underline{r}$$

when $\underline{F}(\underline{r}) = (y, -x, 0)$, from the point $(a, 0, 0)$ to the point $(a, 0, 2\pi b)$ along

(a) a circular helix between the two points, parameterised by

$$\underline{r} = (a\cos\lambda,\ a\sin\lambda,\ b\lambda) \qquad \text{where} \quad 0 \leq \lambda \leq 2\pi$$

(b) a straight line between the two points, parameterised by

$$\underline{r} = (a,\ 0,\ b\lambda)\ .$$

(ii) Repeat for the force $\underline{F}(\underline{r}) = \underline{r}$.
[You should find the results to be the same for both paths in part(ii)]

5.3 For the vector field $\underline{F}(\underline{r}) = (2y + 3)\,\underline{e}_x + xz\,\underline{e}_y + (yz - x)\,\underline{e}_z$, evaluate the line integral $\int_C \underline{F} \cdot \mathrm{d}\underline{r}$ to find the work done by the force $\underline{F}(\underline{r})$ along the following paths C:

(i) $\{x = 2t^2,\ y = t,\ z = t^3\}$ from $t = 0$ to $t = 1$

(ii) the straight lines from $(0, 0, 0)$ to $(0, 0, 1)$, then to $(0, 1, 1)$, and then to $(2, 1, 1)$

(iii) the straight line from $(0, 0, 0)$ to $(2, 1, 1)$. (Use a parametric representation.)

5.4♣ Evaluate the line integrals

$$\int_C \underline{r} \times \mathrm{d}\underline{r} \quad \text{and} \quad \int_C \underline{r}\,\mathrm{d}s$$

from the point $(a, 0, 0)$ to the point $(a, 0, 2\pi b)$ on the circular helix

$$\underline{r} = (a\cos t, a\sin t, bt)$$

Hint: for the 2nd part, you will need to show that $\mathrm{d}s = \sqrt{a^2 + b^2}\,\mathrm{d}t$.

(PTO)

5.5 (i) If $\underline{F}(\underline{r}) = 2y\,\underline{e}_x - z\,\underline{e}_y + x\,\underline{e}_z$ show that

$$\int_C \underline{F} \times \mathrm{d}\underline{r} \;=\; (2 - \frac{\pi}{4})\,\underline{e}_x + (\pi - \frac{1}{2})\,\underline{e}_y$$

where the integral is evaluated along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t = 0$ to $t = \pi/2$.

(ii) If $\underline{a}(\underline{r}) = (3x + y)\underline{e}_x - x\underline{e}_y + (y - 2)\underline{e}_z$, and $\underline{b} = 2\underline{e}_x - 3\underline{e}_y + \underline{e}_z$ show that

$$\oint_C (\underline{a} \times \underline{b}) \times \mathrm{d}\underline{r} \;=\; 4\pi(7\underline{e}_x + 3\underline{e}_y)$$

where the integral is evaluated anticlockwise around a circle of radius 2, which lies in the x–y plane and is centred at the origin.

5.6♣ **Homework problem:** You should work through this problem on your own; you are not allowed to ask the tutors for help with this problem.

The magnetic field $\underline{B}(\underline{r})$ due to a long wire which lies along the x_3 axis and carries current I is

$$\underline{B}(\underline{r}) = \frac{K}{x_1^2 + x_2^2}\left[-x_2\,\underline{e}_1 + x_1\,\underline{e}_2\right]$$

where $K = \mu_o I/2\pi$ is a constant.

(i) Calculate the line integral

$$\int_C \underline{B} \cdot \mathrm{d}\underline{r}$$

from the point $(1, 0, 0)$ to $(0, 1, 1)$ along

1. C_1: $x_1 = \cos\lambda$, $x_2 = \sin\lambda$, $x_3 = \frac{2}{\pi}\lambda$, with $0 \le \lambda \le \frac{\pi}{2}$ [ans: $K\pi/2$]
2. C_2: $x_1 = \cos\lambda$, $x_2 = -\sin\lambda$, $x_3 = \frac{2}{3\pi}\lambda$, with $0 \le \lambda \le \frac{3\pi}{2}$ [ans: $-3K\pi/2$]
3. C_3: $x_2 = x_3 = 1 - x_1$, with $1 \ge x_1 \ge 0$ [ans: $K\pi/2$]

(ii) Show that in the plane polar basis $\underline{B} = \dfrac{K}{\rho}\,\underline{e}_\phi$