Vector Calculus 2013/14

Tutorial Sheet 6: Surface integrals

$*$ denotes **harder** problems or parts of problems
♣ denotes hand–in questions

6.1 More revision of planar/double integration in plane-polar coordinates.

A non-conducting annular disc of outer radius b and inner radius a lies in the $x-y$ plane, with its centre at the origin. The disc carries charge/unit area

$$\sigma(x,y) \;=\; \lambda_0 \frac{x^2(b-y)}{(x^2+y^2)^2}$$

where λ_0 is a constant. Using plane polar coordinates, find the total charge

$$Q = \int_S \sigma(x,y)\,\mathrm{d}S$$

on the disc. What happens to Q as $a \to 0$, with $\sigma(x,y)$ unchanged? Does this make sense physically?

6.2 If $\underline{a} = 2x^2y\,\underline{e}_x + xz\,\underline{e}_y - y^3\,\underline{e}_z$, evaluate the surface integral $\int_S \underline{a}\cdot \mathrm{d}\underline{S}$ explicitly where S is the surface of the unit cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$.

Hint: Note that you need to integrate over all 6 faces of the cube.

6.3 Calculate the flux of the vector field

$$\underline{F}(\underline{r}) \;=\; \frac{\alpha\underline{r}}{(r^2+a^2)^{3/2}}$$

through the closed spherical surface $r = \sqrt{3}\,a$

6.4♣ A spherical surface S of radius a centred on the origin has a parametric representation

$$\underline{r} = a\sin\theta\cos\phi\,\underline{e}_1 + a\sin\theta\sin\phi\,\underline{e}_2 + a\cos\theta\,\underline{e}_3$$

where $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.

Write down an expression for the vector element of area on S, and hence evaluate the surface integrals

$$\int_S \underline{A}\cdot \mathrm{d}\underline{S} \quad \text{and} \quad \int_S \underline{A}\times \mathrm{d}\underline{S}$$

on the curved surface of the spherical *octant* $0 \le \theta \le \pi/2$ and $0 \le \phi \le \pi/2$, where the vector field $\underline{A}(\underline{r})$ is defined by

$$\underline{A}(\underline{r}) = \cos\theta\cos\phi\,\underline{e}_1 + \cos\theta\sin\phi\,\underline{e}_2 - \sin\theta\,\underline{e}_3.$$

6.5 The surface of a torus can be parametrised by two angles λ and μ

$$\begin{aligned} x &= (R + a\cos\mu)\cos\lambda \\ y &= (R + a\cos\mu)\sin\lambda \\ z &= a\sin\mu \end{aligned}$$

where $0 \le \lambda < 2\pi$, $0 \le \mu < 2\pi$, R is the distance from the centre of the tube to the centre of the torus, and a is the radius of the tube.

Calculate the surface area of the torus.

(PTO)

6.6♣ **Homework problem:** You should work through this problem on your own; you are not allowed to ask the tutors for help with this problem.

(i) An arbitrary point $\underline{r}$ on the curved surface S_C of a right circular cylinder

$$x^2 + y^2 = a^2, \quad 0 \le z \le c,$$

may be parameterised by the two real variables ϕ and z,

$$\underline{r} = a\cos\phi\,\underline{e}_1 + a\sin\phi\,\underline{e}_2 + z\,\underline{e}_3 \qquad \{0 \le \phi \le 2\pi,\ 0 \le z \le c\}\,.$$

Find an expression for the tangent vectors along (a) the lines of constant ϕ, (b) the lines of constant z. Deduce an expression for the vector element of area $\mathrm{d}\underline{S}$ on the curved surface, and evaluate

$$\int_{S_C} \underline{r} \cdot \mathrm{d}\underline{S}$$

over the curved surface of the cylinder.

(ii) Show that an arbitrary point $\underline{r}$ on the top (flat) surface S_T of the cylinder may be parameterised by the two real variables ρ and ϕ,

$$\underline{r} \;=\; \rho\cos\phi\,\underline{e}_1 + \rho\sin\phi\,\underline{e}_2 + c\,\underline{e}_3 \qquad \{0 \le \rho \le a\,,\ 0 \le \phi \le 2\pi\}\,.$$

Deduce an expression for the vector element of area $\mathrm{d}\underline{S}$ on the top surface of the cylinder, and evaluate

$$\int_{S_T} \underline{r} \cdot \mathrm{d}\underline{S}$$

over the top surface of the cylinder.

(iii) Repeat part (ii) for the bottom surface S_B of the cylinder.

Check that your vector elements of area point in the direction of the *outward* normals in each of parts (i), (ii) and (iii).

(iv) Verify that the sum of the surface integrals in parts (i), (ii) and (iii) is

$$\int_{S_C} \underline{r} \cdot \mathrm{d}\underline{S} + \int_{S_T} \underline{r} \cdot \mathrm{d}\underline{S} + \int_{S_B} \underline{r} \cdot \mathrm{d}\underline{S} \;=\; 3V$$

where V is the volume of the cylinder.