Vector Calculus 2013/14

Tutorial Sheet 7: Volume integrals, spherical polars, and the divergence theorem

♣ denotes hand–in questions

7.1 Show that the volume integral

$$\int_0^1 \mathrm{d}x \int_0^{x^2} \mathrm{d}y \int_{xy}^1 2x^2 z \, \mathrm{d}z \;=\; \frac{28}{165}$$

7.2 Consider the integrals

$$I_1 = \int_V (x + y + z) \, \mathrm{d}V \,, \qquad I_2 = \int_V z \, \mathrm{d}V \,,$$

where the volume V is the positive octant of the unit sphere:

$$x^2 + y^2 + z^2 \le 1, \qquad x \ge 0,\ y \ge 0,\ z \ge 0 \,.$$

(i) Explain why $I_1 = 3I_2$

(ii) Show that in Cartesian coordinates

$$I_2 \;=\; \int_0^1 \mathrm{d}x \int_0^{\sqrt{1-x^2}} \mathrm{d}y \int_0^{\sqrt{1-x^2-y^2}} z \, \mathrm{d}z \;=\; \pi/16 \,.$$

Using spherical polar coordinates (r, θ, ϕ):

(iii) Show that $I_2 = \pi/16$.

(iv) Evaluate the centre of mass vector for such an octant of uniform mass density.

7.3 Question (6.2) was: *If* $\underline{a} = 2x^2 y \, \underline{e}_x + xz \, \underline{e}_y - y^3 \, \underline{e}_z$, *evaluate the surface integral* $\int_S \underline{a} \cdot \mathrm{d}\underline{S}$ *explicitly where* S *is the surface of the unit cube bounded by* $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$.

Hint: Note that you need to integrate over all 6 *faces of the cube.*

Evaluate $\underline{\nabla} \cdot \underline{a}$, and hence evaluate the surface integral using the divergence theorem.

7.4 Question (6.3) was: *Calculate the flux of the vector field*

$$\underline{F}(\underline{r}) \;=\; \frac{\alpha \underline{r}}{(r^2 + a^2)^{3/2}}$$

through the closed spherical surface $r = \sqrt{3}\, a$

Evaluate $\underline{\nabla} \cdot \underline{F}$, and hence evaluate the surface integral using the divergence theorem.

Hints: Use a trigonometric substitution for the resulting integral. The answer is $3\sqrt{3}\pi\alpha/2$.

7.5♣ Use the divergence theorem to evaluate the flux $\int_S \underline{F} \cdot \mathrm{d}\underline{S}$ where the vector field $\underline{F}(\underline{r}) = xz \, \underline{e}_1 + 3x \, \underline{e}_2 - 2z \, \underline{e}_3$, and

(i) S is the closed cylinder bounded by the surface $x^2 + y^2 = 1$, and the planes $z = 0$ and $z = 3$;

(ii) S is the **open** curved cylindrical surface $x^2 + y^2 = 1$, $0 \le z \le 3$.

(PTO)

7.6 Spherical polar coordinates are defined by $\underline{r} = r\sin\theta\cos\phi\,\underline{e}_1 + r\sin\theta\sin\phi\,\underline{e}_2 + r\cos\theta\,\underline{e}_3$

(i) Show that $h_r \equiv \left|\frac{\partial \underline{r}}{\partial r}\right| = 1\,,\quad h_\theta \equiv \left|\frac{\partial \underline{r}}{\partial \theta}\right| = r\,,\quad h_\phi \equiv \left|\frac{\partial \underline{r}}{\partial \phi}\right| = r\sin\theta\,.$

The quantities h_r, h_θ and h_ϕ are called the *scale factors* for spherical polars.

(ii) Hence show that

$$\begin{aligned} \underline{e}_r &\equiv \frac{1}{h_r}\frac{\partial \underline{r}}{\partial r} = \sin\theta\cos\phi\,\underline{e}_1 + \sin\theta\sin\phi\,\underline{e}_2 + \cos\theta\,\underline{e}_3 \\ \underline{e}_\theta &\equiv \frac{1}{h_\theta}\frac{\partial \underline{r}}{\partial \theta} = \cos\theta\cos\phi\,\underline{e}_1 + \cos\theta\sin\phi\,\underline{e}_2 - \sin\theta\,\underline{e}_3 \\ \underline{e}_\phi &\equiv \frac{1}{h_\phi}\frac{\partial \underline{r}}{\partial \phi} = -\sin\phi\,\underline{e}_1 + \cos\phi\underline{e}_2 \end{aligned}$$

(iii) Show that $\underline{e}_r$, $\underline{e}_\theta$, and $\underline{e}_\phi$ form an *orthonormal* basis for spherical polars, *i.e.* that $\underline{e}_r \cdot \underline{e}_r = \underline{e}_\theta \cdot \underline{e}_\theta = \underline{e}_\phi \cdot \underline{e}_\phi = 1$, and $\underline{e}_r \cdot \underline{e}_\theta = \underline{e}_\theta \cdot \underline{e}_\phi = \underline{e}_\phi \cdot \underline{e}_r = 0$

(iv) Show that $\underline{e}_r \times \underline{e}_\theta = \underline{e}_\phi\,,\quad \underline{e}_\theta \times \underline{e}_\phi = \underline{e}_r\,,\quad \underline{e}_\phi \times \underline{e}_r = \underline{e}_\theta\,.$

(v) Hence show that the vector element of area on a sphere of radius r may be expressed as

$$\mathrm{d}\underline{S}_r = h_\theta\, h_\phi\, \underline{e}_r\, \mathrm{d}\theta\, \mathrm{d}\phi = r^2 \sin\theta\, \underline{e}_r\, \mathrm{d}\theta\, \mathrm{d}\phi$$

(vi) Find similar expressions for the vector elements of area on the cone $\theta = \theta_0$ where θ_0 is constant, and on the plane $\phi = \phi_0$, where ϕ_0 is constant. Illustrate your results with sketches.

(vii) Show that the volume element is

$$\mathrm{d}V = h_r\, h_\theta\, h_\phi\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi = r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi$$

7.7♣ **Homework problem:** You should work through this problem on your own; you are not allowed to ask the tutors for help with this problem.

Consider the surface integral

$$\int_{S_C} \underline{r} \cdot \mathrm{d}\underline{S}$$

where S_C is that part of the surface $z = a^2 - x^2 - y^2$ (a paraboloid) for which $z \geq 0$.

(i) Verify that the surface can be parametrised as $x = a\sin\theta\cos\phi$, $y = a\sin\theta\sin\phi$, $z = a^2\cos^2\theta$. What are the limits on the integrals over θ and ϕ?

(ii) Show that

$$\underline{r} \cdot \mathrm{d}\underline{S} = a^4 \left(2\sin^3\theta\cos\theta + \cos^3\theta\sin\theta\right) \mathrm{d}\theta\, \mathrm{d}\phi$$

(iii) Hence show that $\int_{S_C} \underline{r} \cdot \mathrm{d}\underline{S} = 3\pi a^4/2\,.$

(iv) Let S_B be the circular base of the paraboloid described above, which satisfies $x^2 + y^2 = a^2$ with $z = 0$. Show that $\int_{S_B} \underline{r} \cdot \mathrm{d}\underline{S} = 0$.

(v) Evaluate the integral

$$\int_V \underline{\nabla} \cdot \underline{r}\, \mathrm{d}V$$

where V is the volume bounded by the surface S_C and the plane $z = 0$. Compare your answer to the sum of the results for parts (iii) and (iv).