Vector Calculus 2013/14

Tutorial Sheet 8: The continuity equation, divergence and Stokes' theorems

♣ denotes hand–in questions

8.1 Maxwell's first and fourth equations for the electric field $\underline{E}(\underline{r},t)$ and the magnetic field $\underline{B}(\underline{r},t)$ are

$$\underline{\nabla}\cdot\underline{E} = \frac{\rho}{\epsilon_0} \qquad \frac{1}{\mu_0}\,\underline{\nabla}\times\underline{B} = \underline{j} + \epsilon_0\frac{\partial\underline{E}}{\partial t}$$

where $\rho(\underline{r},t)$ and $\underline{j}(\underline{r},t)$ are the electric charge density and the electric current density respectively, and ϵ_0 and μ_0 are constants.

By taking the divergence of the fourth equation, show that ρ and $\underline{j}$ satisfy a continuity equation, and hence deduce that electric charge is conserved.

8.2 If the temperature at any point $\underline{r}$ of a solid at time t is $T(\underline{r},t)$, and if κ, ρ and c are respectively the thermal conductivity, density and specific heat of the solid (assumed constant), show, by considering the net total heat flux from a volume V bounded by a closed surface S, that

$$\frac{\partial T}{\partial t} = \lambda\nabla^2 T$$

where $\lambda = \kappa/\rho c$.

Hints: The heat content of a small volume $\mathrm{d}V$, at temperature T, is $c\,T\rho\,\mathrm{d}V$, and the heat flux density is given by Fourier's law $\underline{J} = -\kappa\,\underline{\nabla}T$, where $|\underline{J}|$ is the magnitude of the heat flow per unit area per unit time across an element of surface normal to $\underline{J}$.

8.3 If $f(\underline{r})$ is a scalar field defined in a volume V, bounded by a closed surface S, show that

$$\int_V (\underline{\nabla}f)\ \mathrm{d}V \ = \ \int_S f\ \mathrm{d}\underline{S}$$

Hint: apply the divergence theorem to the vector field $\underline{A} = f(\underline{r})\,\underline{c}$, where $\underline{c}$ is an arbitrary *constant* vector.

8.4 (i) Evaluate the line integral

$$\oint_C (\underline{\omega}\times\underline{r})\cdot\mathrm{d}\underline{r}$$

where the curve C is the unit circle in the (x,y) plane, and $\underline{\omega}$ is a constant angular-velocity vector. Use a parametric representation for the curve C.

(ii) Check your answer using Stokes' theorem.

8.5♣ Evaluate $\displaystyle\int_S (\underline{\nabla}\times\underline{F})\cdot\mathrm{d}\underline{S}$ where S is the *open* hemispherical surface $x^2+y^2+z^2=a^2$, with $z\geq 0$, and

$$\underline{F}(\underline{r}) = (1-ay)\,\underline{e}_1 + 2y^2\,\underline{e}_2 + (x^2+1)\,\underline{e}_3$$

(i) By direct evaluation. Take the vector element of area $\mathrm{d}\underline{S}$ to point away from the origin.

(ii) By using the divergence theorem applied to the vector field $\underline{\nabla}\times\underline{F}$. (Recall that the divergence theorem applies to a *closed* surface.)

(iii) By using Stokes' theorem applied to the vector field $\underline{F}$.

8.6 In the study of triclinic crystals it is useful to define general linear coordinates (u, v, w), which are related to Cartesian coordinates (x, y, z) by a 3×3 matrix M, such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

where the elements of the matrix M are constants.

(i) Show that the infinitesimal element of volume in general linear coordinates is

$$\mathrm{d}V = |\det M| \, \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}w$$

(ii) If M is an orthogonal matrix (*i.e.* $M^T M = I$, where M^T denotes the transpose of M, and I is the unit matrix), show that

$$\mathrm{d}V = \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}w$$

What is the geometrical interpretation of this result?

8.7♣ **Homework problem:** You should work through this problem on your own; you are not allowed to ask the tutors for help with this problem.

A closed surface S encloses a volume V. Use the divergence theorem to establish that

$$V = \frac{1}{3} \int_S \underline{r} \cdot \mathrm{d}\underline{S}$$

where $\underline{r}$ is the position vector.

A right circular cone has height h and a base of radius a. Choose the origin to be at the apex of the cone, and the positive z axis to be the axis of symmetry of the cone, *i.e.* the cone is "standing" on its apex.

(i) Using cylindrical coordinates (ρ, ϕ, z), show that a point on the curved surface, S_C, of the cone can be parameterised as

$$\underline{r} = z \left(\frac{a}{h} \cos\phi \, \underline{e}_x + \frac{a}{h} \sin\phi \, \underline{e}_y + \underline{e}_z \right)$$

Hence show that, on S_C

$$\mathrm{d}\underline{S} = z \left(\frac{a}{h} \cos\phi \, \underline{e}_x + \frac{a}{h} \sin\phi \, \underline{e}_y - \frac{a^2}{h^2} \underline{e}_z \right) \mathrm{d}\phi \, \mathrm{d}z$$

(ii) Evaluate $\underline{r} \cdot \mathrm{d}\underline{S}$ and draw a diagram to explain your answer. Evaluate $\int_{S_C} \underline{r} \cdot \mathrm{d}\underline{S}$.

(iii) Obtain an expression for $\mathrm{d}\underline{S}$ on the base S_B of the cone. Evaluate $\int_{S_B} \underline{r} \cdot \mathrm{d}\underline{S}$, and hence show that $V = \pi a^2 h/3$.

(iv) Generalise your calculation to the case of a cone of height h whose (non-circular) base is a plane surface of area A.

Hints for part (iv): (a) Take the origin at the apex of the cone and the z axis perpendicular to the base, (b) the angle between the position vector and the unit normal to the curved surface is $\pi/2$. This part takes some thought, but not much calculation.