Vector Calculus 2013/14

Tutorial Sheet 9: The continuity equation, Stokes' theorem, scalar potentials

♣ denotes hand–in questions
∗ denotes **harder** problems or parts of problems

9.1 In Question (4.3), you were asked to show that $\underline{\nabla} \cdot (\phi \underline{\nabla} \psi - \psi \underline{\nabla} \phi) \; = \; \phi \nabla^2 \psi - \psi \nabla^2 \phi$, where $\phi(\underline{r})$ and $\psi(\underline{r})$ are scalar fields. Make sure you can still show it!

If $\psi(\underline{r}, t)$ is a complex scalar field, use this result to show that

$$\underline{\nabla} \cdot (\psi^* \underline{\nabla} \psi - \psi \underline{\nabla} \psi^*) \; = \; \psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*$$

The Schrödinger equation (SE) for a particle of mass m in a potential $V(\underline{r})$ is

$$i\hbar \frac{\partial}{\partial t} \psi(\underline{r}, t) \; = \; \left\{ -\frac{\hbar^2}{2m} \, \nabla^2 + V(\underline{r}) \right\} \psi(\underline{r}, t)$$

Write down the complex conjugate of the Schrödinger equation. We shall denote the resulting equation by $(\mathrm{SE})^*$.

The equation obtained by multiplying the SE on the left by ψ^* is denoted by $\psi^* \,(\mathrm{SE})$.

By considering $\psi^* \,(\mathrm{SE}) - \psi \,(\mathrm{SE})^*$, derive the continuity equation

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{j} \; = \; 0$$

where $\qquad \rho = \psi^* \psi \qquad$ and $\qquad \underline{j} = -\dfrac{i\hbar}{2m} \left(\psi^* \underline{\nabla} \psi \; - \; \psi \underline{\nabla} \psi^* \right)$.

Integrate the continuity equation over a volume V, bounded by a closed surface S, and use the divergence theorem to deduce that $\underline{j}$ is the probability-current density.

9.2 Show that the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is πab by

(i) evaluating $\int \mathrm{d}S$ with the parametrisation

$$\underline{r} = a\lambda \cos\phi \, \underline{e}_x + b\lambda \sin\phi \, \underline{e}_y$$

where $0 \leq \lambda \leq 1$ and $0 \leq \phi \leq 2\pi$.

(ii) Using $S = \frac{1}{2} \oint_C (x \, \mathrm{d}y - y \, \mathrm{d}x)$ with the parametrisation

$$\underline{r} = a \cos\phi \, \underline{e}_x + b \sin\phi \, \underline{e}_y$$

where $0 \leq \phi \leq 2\pi$.

9.3 By applying Stokes' theorem to $\underline{a} \times \underline{c}$, where $\underline{a}(\underline{r})$ is a vector field and $\underline{c}$ is an arbitrary constant vector, show that

$$\int_S (\mathrm{d}\underline{S} \times \underline{\nabla}) \times \underline{a} = \oint_C \mathrm{d}\underline{r} \times \underline{a}$$

(PTO)

9.4* The vector field $\underline{a}(\underline{r})$ is given by

$$\underline{a} = (3x^2yz + y^3z + xe^{-x})\,\underline{e}_x + (3xy^2z + x^3z + ye^x)\,\underline{e}_y + (x^3y + y^3x + xy^2z^2)\,\underline{e}_z$$

(i) Calculate the integral

$$\oint_C \underline{a} \cdot \mathrm{d}\underline{r}$$

where C is the three-dimensional closed curve $OABCDEO$ defined by successive vertices $(0,0,0)$, $(1,0,0)$, $(1,0,1)$, $(1,1,1)$, $(1,1,0)$, $(0,1,0)$, $(0,0,0)$.

(ii) Check your answer using Stokes' theorem.
Hints: Split the surface into two planar surfaces. The result is $e/2 - 5/6$.

9.5 A particle moves in three dimensions under the influence of a force $\underline{F}(\underline{r}) = -k\underline{r}$ where k is a constant. Show that $\underline{F}$ is conservative.

Calculate the potential energy of the particle at the point $\underline{r}$ by evaluating the line integral

$$U(\underline{r}) = -\int_C \underline{F}(\underline{r}') \,.\, \mathrm{d}\underline{r}'$$

along the following paths C:

1. The straight line from the origin to the point $\underline{r}$.
2. Three contiguous straight lines parallel to the cartesian coordinate axes: $(0,0,0)$ to $(x_1,0,0)$, then $(x_1,0,0)$ to $(x_1,x_2,0)$, and finally $(x_1,x_2,0)$ to (x_1,x_2,x_3).

Verify that $\underline{F}(\underline{r}) = -\underline{\nabla}\, U(\underline{r})$.

9.6♣ Which of the following vector fields are conservative? (*Answer:* (i), (ii), (v), (viii))
Find a scalar potential for those that are.

(i)	$\underline{a} = \underline{c}$	(v)	$\underline{a} = r\underline{c} + (\underline{c}\cdot\underline{r})\underline{r}/r$
(ii)	$\underline{a} = (\underline{c}\cdot\underline{r})\underline{c}$	(vi)	$\underline{a} = r\underline{c} - (\underline{c}\cdot\underline{r})\underline{r}/r$
(iii)	$\underline{a} = (\underline{c}\cdot\underline{r})\underline{r}$	(vii)	$\underline{a} = \underline{c}f(r)$
(iv)	$\underline{a} = \underline{c}\times\underline{r}$	(viii)	$\underline{a} = \underline{r}f(r)$

In the above: $\underline{c}$ is a constant vector &$f(r)$ is an arbitrary function of $r = |\underline{r}|$. If $f(r) = r^n$ in (viii), for which value of n does the potential diverge at both $\underline{r}_0 = 0$ and ∞?

9.7♣ **Homework problem:** You should work through this problem on your own; you are not allowed to ask the tutors for help with this problem.

The vector field $\underline{a}(\underline{r})$ is defined as

$$\underline{a} = \left(xy^2 + 2x\right)\underline{e}_x + 3x^2y\,\underline{e}_y\,.$$

(i) By evaluating the line integral explicitly, show that

$$\oint_C \underline{a} \cdot \mathrm{d}\underline{r} = \frac{b^4}{2}\,,$$

where the closed curve C lies in the $x{-}y$ plane and consists of the section of the x axis between the points (0, 0, 0) and (b, 0, 0), the arc of the circle of radius b centred on the origin and hence passing through the points (b, 0, 0) and (0, b, 0), and the section of the y axis between the points (0, b, 0) and (0, 0, 0).

(ii) By performing an appropriate integral over the planar surface S enclosed by the curve C, use Stokes' theorem to check your result for the line integral in part (i).