Vector Calculus 2013/14

Tutorial Sheet 10: The vector potential, orthogonal curvilinear coordinates

♣ denotes hand–in questions

$*$ denotes **harder** problems or parts of problems

Since this is the last week, you are *encouraged* to ask the tutors for help with the hand-in questions, the second of which is quite short.

10.1 Show that $\underline{A}(\underline{r}) = 1/2\ (\underline{B} \times \underline{r})$ is a vector potential for the *constant* magnetic field $\underline{B}(\underline{r}) = \underline{B}$. Verify that $\underline{A}$ satisfies the gauge condition $\underline{\nabla} \cdot \underline{A} = 0$.

10.2♣ Show that the following vector fields $\underline{B}(\underline{r})$ are solenoidal, where $\underline{c}$ and $\underline{d}$ are constant vectors and $n > -3$.

(i) $\underline{B} = 3\,(\underline{c} \cdot \underline{r})\,\underline{d} - (\underline{c} \cdot \underline{d})\,\underline{r}$ (ii) $\underline{B} = r^n(\underline{c} \times \underline{r})$

Use the integral expression for $\underline{A}(\underline{r})$ given in lectures to show that these fields have vector potentials

(i) $\underline{A} = (\underline{c} \cdot \underline{r})\,(\underline{d} \times \underline{r})$ (ii) $\underline{A} = \dfrac{r^n}{n+3}\Big((\underline{r} \cdot \underline{c})\,\underline{r} - r^2\underline{c}\Big)$

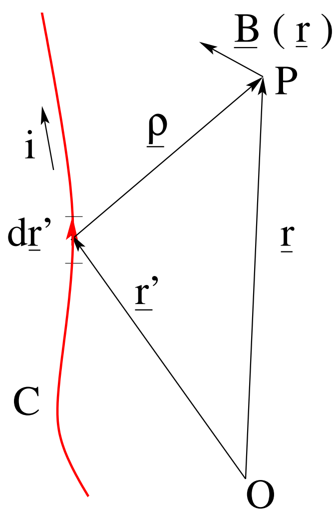
In each case, verify that the vector potentials you obtain satisfy $\underline{B} = \underline{\nabla} \times \underline{A}$.

10.3 Derivation of Maxwell's second equation from the Biot-Savart law.

The magnetic field $\underline{B}(\underline{r})$ at the point P, which lies a distance $\underline{\rho} = \underline{r} - \underline{r}'$ from the element $\mathrm{d}\underline{r}'$ of a thin wire carrying current i, is

$$\underline{B}(\underline{r}) \;=\; \frac{\mu_0}{4\pi}\int_C \frac{i\,\mathrm{d}\underline{r}' \times (\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3} \;=\; \frac{\mu_0}{4\pi}\int_C \frac{i\,\mathrm{d}\underline{r}' \times \underline{\rho}}{\rho^3}$$

where the line integral is along the curve C of the wire.



For a current density $\underline{j}(\underline{r}')$ in a volume V this becomes

$$\underline{B}(\underline{r}) \;=\; \frac{\mu_0}{4\pi}\int_V \frac{\underline{j}(\underline{r}') \times \underline{\rho}}{\rho^3}\,\mathrm{d}V'$$

Verify that

$$\underline{B}(\underline{r}) \;=\; -\frac{\mu_0}{4\pi}\int_V \underline{j}(\underline{r}') \times \underline{\nabla}_\rho\left(\frac{1}{\rho}\right)\,\mathrm{d}V'$$

where $\underline{\nabla}_\rho$ denotes the gradient with respect to the vector $\underline{\rho}$.

Use the identity $\underline{\nabla} \times (f\underline{a}) \;=\; (\underline{\nabla}f) \times \underline{a} \;+\; f\,(\underline{\nabla} \times \underline{a})$ to verify that

$$\underline{B}(\underline{r}) \;=\; \underline{\nabla}_r \times \underline{A}(\underline{r}) \qquad \text{where} \qquad \underline{A}(\underline{r}) \;=\; \frac{\mu_0}{4\pi}\int_V \frac{\underline{j}(\underline{r}')}{|\underline{r} - \underline{r}'|}\,\mathrm{d}V'$$

is called the *magnetic vector potential* for the magnetic field $\underline{B}$.

[*Hint:* $\underline{\nabla}_r = \underline{\nabla}_\rho$ because $\underline{r}'$ is constant when taking derivatives at the point $\underline{r}$.]

Hence show that $\underline{\nabla} \cdot \underline{B} = 0$, which is Maxwell's second equation of electromagnetism.

(PTO)

10.4 Show that the scale factors for circular cylindrical coordinates $(\rho,\, \phi,\, z)$ are

$$h_\rho = 1 \qquad h_\phi = \rho \qquad h_z = 1$$

Write down expressions for $\underline{\nabla} f$, $\underline{\nabla} \cdot \underline{a}$, $\underline{\nabla} \times \underline{a}$ and $\nabla^2 f$ in circular cylindrical coordinates. Evaluate the curl of each the following vector fields

(i) $\underline{a} = \underline{e}_\phi$ (ii) $\underline{a} = \rho\, \underline{e}_\phi$ (iii) $\underline{a} = \dfrac{1}{\rho}\, \underline{e}_\phi$

Note that each of the fields $\underline{a}$ has a non-zero circulation around the z axis, but this does not imply that $\underline{\nabla} \times \underline{a} \neq 0$.

10.5 Parabolic cylindrical coordinates (u, v, w) are defined by

$$x = uv\,, \qquad y = \frac{1}{2}(v^2 - u^2)\,, \qquad z = w$$

where (x, y, z) are Cartesian coordinates. Find the scale factors h_u, h_v and h_w. Hence find expressions for the curvilinear basis vectors $\{\underline{e}_u,\, \underline{e}_v,\, \underline{e}_w\}$ in terms of $\underline{e}_x$, $\underline{e}_y$ and $\underline{e}_z$, and show that they are mutually orthogonal.

The vector field $\underline{a}(\underline{r})$ has components $(a_u = v,\ a_v = -u,\ a_w = 0)$ in the basis $\{\underline{e}_u,\, \underline{e}_v,\, \underline{e}_w\}$,

$$\underline{a}(\underline{r}) = v\, \underline{e}_u - u\, \underline{e}_v$$

Show that $\underline{a}$ is solenoidal.

10.6 Do Question (7.6) on Sheet 7 (if you haven't already done it).

10.7♣ Physics applications of spherical polars – in supernatural units!

The electrostatic potential $\Phi(\underline{r})$ due to an electric dipole at the origin is given in spherical polars by

$$\Phi(\underline{r}) \;=\; \frac{\cos\theta}{r^2}\,,$$

where r is the length of the position vector $\underline{r}$. Show that, for $r \neq 0$,

$$\nabla^2 \Phi \;=\; 0\,.$$

The wavefunction $\psi(\underline{r})$ of a three-dimensional quantum harmonic oscillator in its ground state is given in spherical polars by

$$\psi(\underline{r}) \;=\; \exp\left(-r^2/2\right)\,.$$

Show that ψ satisfies Schrödinger's equation

$$-\nabla^2\psi + r^2\psi \;=\; 2E_0\,\psi\,,$$

and thus determine the value of the ground-state energy E_0 in supernatural units.

10.8* Your Easter treat! A vector field $\underline{A}$ has components in spherical polar coordinates

$$A_r = r^2\cos^2\theta\sin\theta\cos\phi\,, \qquad A_\theta = -r^2\cos\theta\sin^2\theta\cos\phi\,, \qquad A_\phi = 0\,.$$

Show that

(i) $\underline{\nabla} \cdot \underline{A} = r\sin\theta\cos\phi$

(ii) $\underline{B} = \underline{\nabla} \times \underline{A}$ has components

$$B_r = -r\sin\theta\cos\theta\sin\phi\,, \qquad B_\theta = -r\cos^2\theta\sin\phi\,, \qquad B_\phi = -r\cos\theta\cos\phi\,.$$

(iii) $\nabla^2 \underline{A} = 0$

Find the components of $\underline{A}$ in cartesian coordinates, and comment on the suitability of spherical polars for this problem.