Vector Calculus – 2013/14

[PHYS08043, Dynamics and Vector Calculus]

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Abstract

In this course, we shall study vector calculus, which is the branch of mathematics that deals with differentiation and integration of scalar and vector fields. We shall encounter many examples of vector calculus in physics.

Timetable

- Monday 11:10-12:00 Lecture (JCMB Lecture Theatre A)
- Thursday 14:10-16:00 Tutorial Workshop (JCMB Teaching Studio 3217)
- Monday 11:10-12:00 Lecture (JCMB Lecture Theatre A)
- Thursday 14:10-16:00 Tutorial Workshop (JCMB Teaching Studio 3217)

Students should attend both lectures and one tutorial workshop each week. Tutorials start in Week 2.

Genealogy

For historians of pre-Honours courses . . .

This course was known as *Mathematics for Physics 4: Fields* until 2012-13, when it became *Vector Calculus*.

There will be some evolution from last year's instance of the course, but I'm not planning any *major* structural changes. There should be some new material on index notation and the concept of solid angle, a number of new tutorial problems, and there will most likely be some further tweaks to be decided *en route*.

Contents

Synopsis

We will cover all the topics below, but not necessarily in that order.

- Introduction, scalar and vector fields in gravitation and electrostatics. Revision of vector algebra and products.
- Fields, potentials, grad, div and curl and their physical interpretation, the Laplacian, vector identities involving grad, div, curl and the Laplacian. Physical examples.
- Lines and surfaces. Line integrals, vector integration, physical applications.
- Surface and volume integrals, divergence and Stokes' theorems, Green's theorem and identities, scalar and vector potentials; applications in electromagnetism and fluids.
- Curvilinear coordinates, line, surface, and volume elements; grad, div, curl and the Laplacian in curvilinear coordinates. More examples.

Syllabus

The Contents section of this document is the course syllabus! There will be corrections to your printed version as this year's course evolves. The online version is always up to date.

Books

The course will not use any particular textbook. The first six listed below are standard texts; Spiegel contains many examples and problems. All are available from Amazon.co.uk

- KF Riley and MP Hobson,
Essential Mathematical Methods for the Physical Sciences (CUP)
(also useful for Junior Honours)
or Foundation Mathematics for the Physical Sciences (CUP) (not so good for JH)
- KF Riley, MP Hobson and SJ Bence,
Mathematical Methods for Physics and Engineering, (CUP).
(This is an older, but more comprehensive version of the books above.)
- DE Bourne and PC Kendall,
Vector Analysis and Cartesian Tensors, (Chapman and Hall).
- PC Matthews, Vector Calculus, (Springer). (Also useful for JH SoCM)
- ML Boas, Mathematical Methods in the Physical Sciences, (Wiley).
- GB Arfken and HJ Weber, Mathematical Methods for Physicists, (Academic Press).
- MR Spiegel, Vector Analysis, (Schaum, McGraw-Hill).
- Any mathematical methods book you're comfortable with.

1 Fields and why we need them in Physics

1.1 Vectors and scalars

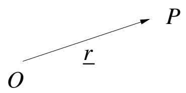
We start by recalling two basic definitions (simple-minded Physics versions) in order to establish our notation (which is similar to that used in *Linear Algebra and Several Variable Calculus.*)

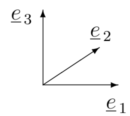
Scalar: a quantity specified by a single number;

Vector: a quantity specified by a number (magnitude) and a *direction* (two numbers in three dimensions, *e.g.* two angles);

Examples: mass is a scalar, velocity is a vector.

Example: A *position vector* is a vector bound to some origin and gives the position of some point P, say, relative to that origin. It is often denoted by $\underline{r}$ (or $\underline{x}$ or $\overrightarrow{OP}$).





Define an orthonormal[1] right-handed Cartesian basis of unit vectors $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$, such that

$$\underline{e}_1 \cdot \underline{e}_1 = \underline{e}_2 \cdot \underline{e}_2 = \underline{e}_3 \cdot \underline{e}_3 = 1 \quad \text{and} \quad \underline{e}_1 \cdot \underline{e}_2 = \underline{e}_1 \cdot \underline{e}_3 = \underline{e}_2 \cdot \underline{e}_3 = 0 \,. \tag{1}$$

In such a basis, we may write the position vector $\underline{r}$ in terms of its Cartesian *components* (x_1, x_2, x_3) as follows

$$\underline{r} = x_1\,\underline{e}_1 + x_2\,\underline{e}_2 + x_3\,\underline{e}_3$$

The *length* or *magnitude* of the vector $\underline{r}$ is a scalar and is denoted by $r \equiv |\underline{r}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Any vector $\underline{a}$ may written in this notation as[2]

$$\underline{a} = a_1\,\underline{e}_1 + a_2\,\underline{e}_2 + a_3\,\underline{e}_3 = \sum_{i=1}^{3} a_i\,\underline{e}_i \,.$$

1.2 Fields

In physics we have quantities that vary in some region of space, *e.g.* the temperature $T(\underline{r})$ of the ocean depends on position $\underline{r}$. To study this variation we require the concept of a *field*. In these lectures we shall develop the calculus of *scalar fields* and *vector fields*.

[1]The word *orthonormal* means mutually *ortho*gonal (perpendicular) and *normal*ised to have unit length.

[2]You may be more comfortable with the 'xyz' notation in which the Cartesian components of a vector $\underline{a}$ are written as (a_x, a_y, a_z), and a vector is written in terms of orthonormal basis vectors $\{\underline{i}, \underline{j}, \underline{k}\}$ as

$$\underline{a} = a_x\,\underline{i} + a_y\,\underline{j} + a_z\,\underline{k} \qquad \text{or perhaps as} \qquad \underline{a} = a_x\,\underline{e}_x + a_y\,\underline{e}_y + a_z\,\underline{e}_z \,.$$

We will often use the '(a_x, a_y, a_z)' notation for components of a vector, but we won't use the '$\{\underline{i}, \underline{j}, \underline{k}\}$' notation for basis vectors. There are good reasons for our conventions: the '123' notation is succinct; it's easier to generalise to an arbitrary number of dimensions; and it avoids possible confusion between the index i and the unit vector $\underline{i}$.

If to each point $\underline{r}$ in some region of space there corresponds a *scalar* $\phi(x_1, x_2, x_3)$, then $\phi(\underline{r})$ is a *scalar field*: ϕ is a function of the three Cartesian position coordinates (x_1, x_2, x_3).

Examples: the temperature distribution in a body $T(\underline{r})$, pressure in the atmosphere $P(\underline{r})$, electric charge (or mass) density $\rho(\underline{r})$, electrostatic potential $\phi(\underline{r})$, the Higgs field $h(\underline{r})$.

Similarly a *vector field* assigns a vector $\underline{V}(\underline{r})$ to each point $\underline{r}$ of some region.

Examples: velocity in a fluid $\underline{v}(\underline{r})$, electric current density $\underline{j}(\underline{r})$, electric field $\underline{E}(\underline{r})$, magnetic field $\underline{B}(\underline{r})$ (actually a pseudo-vector field).

A vector field in 2-d can be represented graphically, at a carefully selected set of points $\underline{r}$, by an arrow whose length and direction is proportional to $\underline{V}(\underline{r})$, *e.g.* wind velocity on a weather forecast chart.

1.3 Examples: Gravitation and Electrostatics

Let us revisit two familiar examples of fundamental fields in Nature.

Gravitation: The foundation of Newtonian Gravity is *Newton's Law of Gravitation*, which Newton deduced from observations of the motion of the planets by Tycho Brahe, and their analysis by Kepler.

The force $\underline{F}$ *on* a particle[3] of mass m_1 at the point with position vector $\underline{r}$ *due to* a particle of mass m situated at the origin is given (in SI units) by

$$\underline{F}(\underline{r}) \;=\; -\,G m m_1 \frac{\underline{r}}{r^3}$$

where $G = 6.672\,59(85) \times 10^{-11}\, Nm^2kg^2$ is Newton's Gravitational Constant. The magnitude of $\underline{F}(\underline{r})$ is proportional to the length of the vector $\underline{r}/r^3$, which is just $|\underline{r}|/r^3 = r/r^3 = 1/r^2$, so we have the well-known *inverse square law.*

The *gravitational field* $\underline{G}(\underline{r})$ at $\underline{r}$ due to the mass m at the origin is *defined* by

$$\underline{F}(\underline{r}) \;\equiv\; m_1\,\underline{G}(\underline{r}) \qquad \Rightarrow \qquad \underline{G}(\underline{r}) \;=\; -\,G\,m\,\frac{\underline{r}}{r^3} \tag{2}$$

where the test mass m_1 is so small that its own gravitational field can be ignored. $\underline{G}(\underline{r})$ is a *vector field.*

At this point, we shall simply state that the *gravitational potential* due to the field $\underline{G}(\underline{r})$ is

$$\phi(\underline{r}) = -\frac{Gm}{r} \tag{3}$$

and the *potential energy* of the mass m_1 in the field is

$$V(\underline{r}) = m_1 \phi(\underline{r})\,.$$

Gravitational potential and potential energy are *scalar fields.* The distinction (a convention) between potential and potential energy is a common source of confusion. We shall return to these potentials later in the course, so don't worry if you don't know how to obtain them.

[3]In this course, a "particle" will refer to an idealised point particle, *i.e.* a particle of negligible size.

Electrostatics: Coulomb's Law was also deduced experimentally; it states that the force $\underline{F}(\underline{r})$ on a particle of charge q_1 situated at $\underline{r}$ due to a particle of charge q situated at the origin is given (in SI units) by

$$\underline{F} \;=\; \frac{q_1\, q}{4\pi\epsilon_0}\,\frac{\underline{r}}{r^3}\,,$$

where $\epsilon_0 \equiv 10^7/(4\pi c^2) = 8.854\,187\,817\ldots \times 10^{-12}\, C^2 N^{-1} m^{-2}$ is called the *permittivity of free space*. The electric field $\underline{E}(\underline{r})$ at $\underline{r}$ due to the charge q at the origin is *defined* by

$$\underline{F}(\underline{r}) \;\equiv\; q_1\, \underline{E}(\underline{r}) \qquad \Rightarrow \qquad \underline{E}(\underline{r}) \;=\; \frac{q}{4\pi\epsilon_0}\,\frac{\underline{r}}{r^3} \tag{4}$$

Again the test charge q_1 is taken as small, so as not to disturb the electric field from q.

The *electrostatic potential* $\phi(\underline{r})$ is then (see later)

$$\phi(\underline{r}) \;=\; \frac{q}{4\pi\epsilon_0 r} \tag{5}$$

and the *potential energy* of a charge q_1 in the electric field is $V = q_1\phi$.

Note that electrostatics and gravitation are very similar mathematically, the only real difference being that the gravitational force between two masses is always attractive, whereas like charges repel.

1.4 The need for vector calculus

At this point, we may ask several questions:

(i) How are equations (2) through (5) changed when the mass m or the charge q moves away from the origin?

(ii) How do the vector fields in equations (2) and (4) change when the mass m_1 or the charge q_1 moves a small distance from position $\underline{r}$ to position $\underline{r} + \delta\underline{r}$ — where $\delta\underline{r}$ is very small but its direction is arbitrary? In the language of mathematics, how do we define derivatives of a vector field with respect to the position vector?

(iii) Similarly, how do the potentials change when $\underline{r} \to \underline{r} + \delta\underline{r}$?

(iv) What happens when there are more than two masses or charges, or when the masses and charges have a finite size? For example, when the masses are lumpy asteroids or the charged objects are irregular lumps of metal.

You should be able to answer the first question already (exercise).

In order to address the second and third questions, we need to develop the sub-branch of mathematics known as differential vector calculus, to which we shall soon turn our attention. The answer to the fourth question requires integral vector calculus, which will come later.

In what follows, we will assume some familiarity with several variable calculus at the level of *Linear Algebra and Several Variable Calculus* (or the specialist Mathematics course *Several Variable Calculus and Differential Equations*), but these notes should be largely self-contained. We will also assume a working knowledge of vectors and bases, matrices and determinants. We shall develop the mathematics of scalar and vector fields required for third- and fourth-year courses on electromagnetism, quantum mechanics, etc, and for courses on meteorology, fluid mechanics, etc, from other schools.

1.5 Revision of vector algebra

In this section we collect together many of the results on vector algebra that will be assumed in these lectures.

A **vector** is *represented* in some orthonormal basis $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ by its *components*, an *ordered* set of 3 numbers with certain laws of addition. For example

$$\begin{aligned} \underline{a} &\quad \text{is represented by} \quad (a_1,\, a_2,\, a_3) \\ \underline{a} + \underline{b} &\quad \text{is represented by} \quad (a_1 + b_1,\, a_2 + b_2\,, a_3 + b_3)\,. \end{aligned}$$

Note that *in this basis*, the basis vectors themselves are represented by

$$\underline{e}_1 = (1,0,0) \qquad \underline{e}_2 = (0,1,0) \qquad \underline{e}_3 = (0,0,1)\,.$$

The various 'products' of vectors are defined in an orthonormal basis as follows:

The scalar product is denoted by $\underline{a}\cdot\underline{b}$ and is the single number defined as

$$\begin{aligned} \underline{a}\cdot\underline{b} &\equiv |\underline{a}|\,|\underline{b}|\cos\theta_{ab} = a_1\,b_1 + a_2\,b_2 + a_3\,b_3 = \sum_{i=1}^{3} a_i\,b_i \\ \sqrt{\underline{a}\cdot\underline{a}} &= \sqrt{a_1^2 + a_2^2 + a_3^2} = |\underline{a}| \quad \text{defines the } \textit{length} \text{ or } \textit{magnitude},\ a, \text{ of the vector } \underline{a}. \end{aligned}$$

where θ_{ab} is the angle between $\underline{a}$ and $\underline{b}$.

The vector product or **cross product** is denoted by $\underline{a}\times\underline{b}$ and is defined in a right-handed orthonormal basis as the vector[4]

$$\underline{a}\times\underline{b} \equiv |\underline{a}|\,|\underline{b}|\sin\theta_{ab}\,\underline{n} \;=\; \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where the vertical lines signify the *determinant* of the matrix, and $\underline{n}$ is a unit vector orthogonal to $\underline{a}$ and $\underline{b}$. This gives

$$\underline{a}\times\underline{b} = -\underline{b}\times\underline{a} \quad \text{and hence} \quad \underline{a}\times\underline{a} = 0\,.$$

The basis vectors satisfy the cyclic properties

$$\underline{e}_1\times\underline{e}_2 = \underline{e}_3 \qquad \underline{e}_2\times\underline{e}_3 = \underline{e}_1 \qquad \underline{e}_3\times\underline{e}_1 = \underline{e}_2$$

The scalar triple product is the single number

$$(\underline{a},\underline{b},\underline{c}) \;\equiv\; \underline{a}\cdot(\underline{b}\times\underline{c}) \;=\; \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The properties of the determinant imply that $(\underline{a},\underline{b},\underline{c}) = (\underline{b},\underline{c},\underline{a}) = (\underline{c},\underline{a},\underline{b})$ – cylic permutation, and $(\underline{a},\underline{b},\underline{c}) = -(\underline{b},\underline{a},\underline{c})$, etc – non-cyclic permutation.

If $\underline{a}$, $\underline{b}$ and $\underline{c}$ are three concurrent edges of a parallelepiped, its volume is $(\underline{a},\underline{b},\underline{c})$.

[4] To be more precise, the result of the vector product an axial- or pseudo-vector, but this subtle difference is not needed for this course. See Junior Honours *Symmetries of Classical Mechanics*.

The vector triple product is the vector $\underline{a} \times (\underline{b} \times \underline{c})$ and it may be shown that

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c}\,.$$

You **must** know this result – memorise it! It was derived in *Linear Algebra and Several Variable Calculus* by writing out the components in full. It is sometimes known as the 'bac-cab rule', but note that you must write the vectors in front of the scalar products on the RHS to see this: $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}\,(\underline{a} \cdot \underline{c}) - \underline{c}\,(\underline{a} \cdot \underline{b})$.

1.6 The Kronecker delta symbol δ_{ij}

Define the symbol δ_{ij} (*in words "delta i j"*), where i and j can take on the values 1, 2, 3:

$$\boxed{\begin{aligned} \delta_{ij} &= 1 \quad \text{when } i = j \\ &= 0 \quad \text{when } i \neq j \end{aligned}}$$

i.e. $\delta_{11} = \delta_{22} = \delta_{33} = 1$ and $\delta_{12} = \delta_{13} = \delta_{23} = \cdots = 0$

The orthornormality equations (1) satisfied by the *orthonormal basis vectors* $\{\underline{e}_i\}$, with $i = 1, 2, 3$ can now be written succinctly as

$$\boxed{\underline{e}_i \cdot \underline{e}_j \;=\; \delta_{ij}}$$

This shorthand notation will save us a lot of writing later.

2 Level surfaces, gradient and directional derivative

2.1 Level surfaces/equipotentials of a scalar field

If $\phi(\underline{r})$ is a non-constant scalar field, then the equation $\phi(\underline{r}) = c$ where c is a *constant*, defines a *level surface* or *equipotential* of the field. Different level surfaces do not intersect, or ϕ would be multi-valued at the point of intersection.

Familiar examples in two dimensions, where they are level curves rather than level surfaces, are contours of constant height on a geographical map, $h(x_1, x_2) = c$. Similarly, isobars on a weather map are level curves of pressure $P(x_1, x_2) = c$.

Examples in three dimensions:

(i) Suppose that

$$\phi(\underline{r}) = r^2 \equiv x_1^2 + x_2^2 + x_3^2 \qquad (= x^2 + y^2 + z^2 \text{ in '}xyz\text{' notation})$$

The level surface $\phi(\underline{r}) = c$ is the surface of a *sphere* of radius $\sqrt{c}$ centred on the origin.

If we vary c, we obtain a family of level surfaces or equipotentials which are concentric spheres.[5]

[5]It may seem strange to call the surface of a sphere a *level* surface! The point is that the scalar field is equal to a *constant* everywhere on the level surface, and it is in this sense that the surface is said to be level.

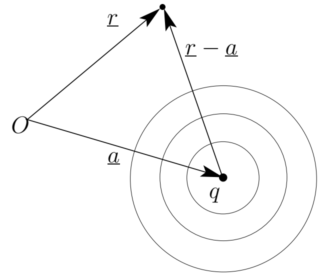
(ii) The electrostatic potential at $\underline{r}$ due to a point charge q situated at the point $\underline{a}$ is

$$\phi(\underline{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\underline{r} - \underline{a}|}$$

where $|\underline{r} - \underline{a}|$ denotes the length of the vector $\underline{r} - \underline{a}$:

$$|\underline{r} - \underline{a}| = \sqrt{(\underline{r} - \underline{a}) \cdot (\underline{r} - \underline{a})} = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2}$$

The equipotentials or level surfaces are concentric spheres centred on the point $\underline{a}$, as shown in the figure.



If $\underline{a} = 0$, the equipotentials are centered on the origin.

(iii) Let $\phi(\underline{r}) = \underline{k} \cdot \underline{r}$.

The level surfaces are planes $\underline{k} \cdot \underline{r} = constant$, with $\underline{k}$ normal to the planes.

(iv) Let $\phi(\underline{r}) = \exp(i\underline{k} \cdot \underline{r})$, which is a complex scalar field.

Since $\underline{k} \cdot \underline{r} = constant$ is the equation for a plane, the level surfaces are again planes.

2.2 Gradient of a scalar field

How do we describe mathematically the variation of a scalar field as a function of small changes in position?

As an example, think of a 2-d contour map of the height $h = h(x_1, x_2)$ of a hill. $h(x_1, x_2)$ is a scalar field. If we are on the hill and move in the x_1–x_2 plane then the change in height will depend on the direction in which we move (unless the hill is completely flat!) For example there will be a direction in which the height increases most steeply: 'straight up the hill.'

We now introduce a formalism to describe how a scalar field $\phi(\underline{r})$ changes as a function of $\underline{r}$. We begin by recalling Taylor's theorem and the definition of partial derivatives.

Taylor's theorem: Recall that if $f(x)$ is a function of a single variable x, Taylor's theorem states that $f(x + \delta x)$ can be expanded in powers of δx

$$f(x + \delta x) = f(x) + \delta x\, \frac{df(x)}{dx} + \frac{(\delta x)^2}{2!} \frac{d^2 f(x)}{dx^2} + \cdots + \frac{(\delta x)^n}{n!} \frac{d^n f(x)}{dx^n} + \cdots$$

If δx is very small, we may approximate $f(x + \delta x) = f(x) + \delta x\, \dfrac{df(x)}{dx} + O\left((\delta x)^2\right)$

Partial derivatives: If $f(x_1, x_2, x_3)$ is a function of the three *independent* variables x_1, x_2 and x_3, then the *partial derivative*

$$\frac{\partial f(x_1, x_2, x_3)}{\partial x_1}$$

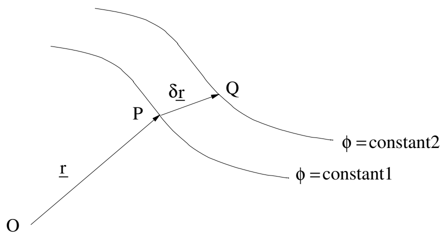
is obtained by differentiating $f(x_1, x_2, x_3)$ with respect to x_1, whilst keeping x_2 and x_3 *fixed*. Similarly for the partial derivatives with respect to x_2 and x_3.

Mathematical aside: A scalar field $\phi(\underline{r}) = \phi(x_1, x_2, x_3)$ is said to be *continuously differentiable* in a region R if its partial derivatives

$$\frac{\partial \phi(\underline{r})}{\partial x_1}, \qquad \frac{\partial \phi(\underline{r})}{\partial x_2} \qquad \text{and} \qquad \frac{\partial \phi(\underline{r})}{\partial x_3}$$

exist, and are *continuous* at every point $\underline{r} \in R$. We will generally assume scalar fields are continuously differentiable.

Let $\phi(\underline{r})$ be a scalar field, and consider 2 nearby points: P with position vector $\underline{r}$, and Q with position vector $\underline{r} + \delta\underline{r}$, where $\delta\underline{r}$ has components $(\delta x_1, \delta x_2, \delta x_3)$. Assume P and Q lie on *different* level surfaces as shown:



Now use Taylor's theorem to first order in each of the 3 variables x_1, x_2 and x_3 to evaluate the change in ϕ as we move from P to Q

$$\begin{aligned}
\delta\phi &\equiv \phi(\underline{r} + \delta\underline{r}) - \phi(\underline{r}) \\
&= \phi(x_1 + \delta x_1,\, x_2 + \delta x_2,\, x_3 + \delta x_3) - \phi(x_1,\, x_2,\, x_3) \\
&= \left[\phi(x_1,\, x_2,\, x_3) + \frac{\partial \phi(\underline{r})}{\partial x_1}\,\delta x_1 + \frac{\partial \phi(\underline{r})}{\partial x_2}\,\delta x_2 + \frac{\partial \phi(\underline{r})}{\partial x_3}\,\delta x_3 + O(\delta x_i\,\delta x_j)\right] - \phi(x_1,\, x_2,\, x_3) \\
&= \frac{\partial \phi(\underline{r})}{\partial x_1}\,\delta x_1 + \frac{\partial \phi(\underline{r})}{\partial x_2}\,\delta x_2 + \frac{\partial \phi(\underline{r})}{\partial x_3}\,\delta x_3 + O(\delta x_i\,\delta x_j)
\end{aligned}$$

where we assumed that the higher order partial derivatives exist. For sufficiently small $\delta\underline{r}$ we can neglect these higher order terms, and we have

$$\boxed{\delta\phi = \underline{\nabla}\,\phi \cdot \delta\underline{r}}$$

where the 3 quantities $\left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3}\right)$ form the Cartesian components of a *vector field* $\underline{\nabla}\,\phi(\underline{r})$, which we can write as

$$\boxed{\underline{\nabla}\,\phi(\underline{r}) \equiv \frac{\partial \phi}{\partial x_1}\,\underline{e}_1 + \frac{\partial \phi}{\partial x_2}\,\underline{e}_2 + \frac{\partial \phi}{\partial x_3}\,\underline{e}_3 \;=\; \sum_{i=1}^{3} \frac{\partial \phi}{\partial x_i}\,\underline{e}_i}$$

In 'xyz' notation

$$\underline{\nabla}\,\phi = \frac{\partial \phi}{\partial x}\,\underline{e}_x + \frac{\partial \phi}{\partial y}\,\underline{e}_y + \frac{\partial \phi}{\partial z}\,\underline{e}_z$$

The *vector field* $\underline{\nabla}\,\phi(\underline{r})$ is called the *gradient* of $\phi(\underline{r})$, and is pronounced *'grad phi'*.

Example: Calculate the gradient of the scalar field $\phi(\underline{r}) = r^2 = x_1^2 + x_2^2 + x_3^2$.

First, recall that the *partial derivative* $\partial\phi/\partial x_1$ is the derivative of $\phi(\underline{r}) = \phi(x_1, x_2, x_2)$ with respect to x_1, keeping x_2 and x_3 *fixed, etc.*

$$\text{So} \quad \frac{\partial x_1}{\partial x_1} = 1 \quad \text{and} \quad \frac{\partial x_2}{\partial x_1} = 0\,. \quad \text{Similarly} \quad \frac{\partial x_1^2}{\partial x_1} = 2x_1 \quad \text{and} \quad \frac{\partial x_2^2}{\partial x_1} = 0\,, \quad \text{etc.}$$

The first component of $\underline{\nabla}\,r^2$ is then

$$\frac{\partial}{\partial x_1}(x_1^2 + x_2^2 + x_3^2) = 2x_1 + 0 + 0$$

Similarly for the 2nd and 3rd components (exercise), and hence

$$\underline{\nabla}\,r^2 = 2x_1\,\underline{e}_1 + 2x_2\,\underline{e}_2 + 2x_3\,\underline{e}_3 = 2\underline{r}$$

The vector $\underline{\nabla}\,r^2 = 2\underline{r}$ points radially outwards from the origin with *magnitude* $2r$. The level surfaces, $r^2 =$ constant, are spheres centred on the origin.

Example: Calculate the gradient of $\phi(\underline{r}) = \sin x_1 + 2x_1 x_2^2$

$$\frac{\partial \phi}{\partial x_1} = \cos x_1 + 2x_2^2\,, \qquad \frac{\partial \phi}{\partial x_2} = 0 + 4x_1 x_2\,, \qquad \frac{\partial \phi}{\partial x_3} = 0 + 0$$

$$\Rightarrow \quad \underline{\nabla}\,\phi = (\cos x_1 + 2x_2^2)\,\underline{e}_1 + 4x_1 x_2\,\underline{e}_2$$

Example: Calculate $\underline{\nabla}\,(\underline{a} \cdot \underline{r})$ where $\underline{a}$ is a constant vector. The first component is

$$\frac{\partial}{\partial x_1}(a_1 x_1 + a_2 x_2 + a_3 x_3) = a_1 + 0 + 0$$

Similarly for the other two components. Hence

$$\underline{\nabla}\,(\underline{a} \cdot \underline{r}) = a_1\,\underline{e}_1 + a_2\,\underline{e}_2 + a_3\,\underline{e}_3 = \underline{a}$$

This is a very important result – as we shall see.

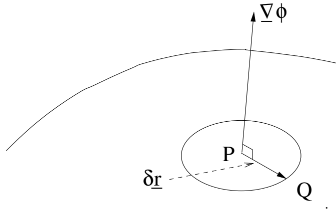
Note: A useful shorthand for partial derivatives is

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \quad \text{which holds for all } i, j = 1, 2, 3.$$

2.3 Interpretation of the gradient

In deriving the expression for $\delta\phi$ above, we assumed that the points P and Q lie on *different* level surfaces. Now consider the situation where P and Q are nearby points on the *same* level surface. In this case, $\delta\phi = 0$ and so

$$\delta\phi = \underline{\nabla}\,\phi \cdot \delta\underline{r} = 0$$



In this case, the infinitesimal vector $\delta\underline{r}$ lies in the level surface at $\underline{r}$, and since the above equation holds for all such $\delta\underline{r}$, we may deduce that

$$\boxed{\underline{\nabla}\,\phi(\underline{r}) \text{ is } \textit{normal} \ (\textit{i.e.} \text{ perpendicular) to the level surface at } \underline{r}}$$

To construct a *unit normal* $\underline{n}(\underline{r})$ to the level surface at $\underline{r}$, we divide $\underline{\nabla}\,\phi$ by its length

$$\underline{n}(\underline{r}) \;=\; \frac{\underline{\nabla}\,\phi(\underline{r})}{|\underline{\nabla}\,\phi(\underline{r})|} \qquad (\text{when} \quad |\underline{\nabla}\,\phi(\underline{r})| \neq 0)$$

2.4 Directional derivative

Consider the change, $\delta\phi$, produced in $\phi(\underline{r})$ by moving a distance δs in the direction of the unit vector $\underline{\hat{s}}$, so that $\delta\underline{r} = \delta s\,\underline{\hat{s}}$. Then

$$\delta\phi = \underline{\nabla}\,\phi \cdot \delta\underline{r} = (\underline{\nabla}\,\phi) \cdot \underline{\hat{s}}\;\delta s$$

Now divide by δs, and then let $\delta s \to 0$. The rate of change of ϕ, with respect to distance, s, in the direction of $\underline{\hat{s}}$, is then

$$\frac{d\phi}{ds} = \underline{\hat{s}} \cdot \underline{\nabla}\,\phi = |\underline{\nabla}\,\phi|\,\cos\theta \tag{6}$$

where θ is the angle between $\underline{\hat{s}}$ and the normal to the level surface at $\underline{r}$.

$$\boxed{\frac{d\phi}{ds} = \underline{\hat{s}} \cdot \underline{\nabla}\,\phi \text{ is called the } \textit{directional derivative} \text{ of the scalar field } \phi \text{ in the direction of } \underline{\hat{s}}}$$

Equivalently, $d\phi/ds$ is the component of $\underline{\nabla}\,\phi$ in the $\underline{\hat{s}}$ direction.

From equation (6), the directional derivative has its *maximum* value, $|\underline{\nabla}\,\phi|$, when $\underline{\hat{s}}$ is *parallel* to $\underline{\nabla}\,\phi$, and is *zero* when $\delta s\,\underline{\hat{s}}$ lies *in* the level surface (where ϕ is constant.)

Therefore

$$\boxed{\underline{\nabla}\,\phi \text{ points in the direction of the } \textit{maximum} \text{ rate of increase in } \phi}$$

Recall that this direction is normal (perpendicular) to the level surface. A familiar example is that of contour lines on a map: the steepest direction is *perpendicular* to the contour lines, *i.e.* straight up the hill.

Example: Find the directional derivative of $\phi(\underline{r}) = xy(x+z)$ at the point $(1, 2, -1)$ in the direction of the unit vector $(\underline{e}_x + \underline{e}_y)/\sqrt{2}$.

$$\underline{\nabla}\,\phi = (2xy + yz)\,\underline{e}_x + x(x+z)\,\underline{e}_y + xy\,\underline{e}_z \qquad = \; 2\underline{e}_x + 2\underline{e}_z \quad \text{at} \quad (1, 2, -1)$$

Thus, at this point, the directional derivative in the direction $(\underline{e}_x + \underline{e}_y)/\sqrt{2}$ is

$$\frac{1}{\sqrt{2}}(\underline{e}_x + \underline{e}_y) \cdot \underline{\nabla}\,\phi = \sqrt{2}$$

Physical example: Let $T(\underline{r})$ be the temperature of the atmosphere at the point $\underline{r}$. An object flies through the atmosphere with velocity $\underline{v} \equiv d\underline{r}/dt$. Obtain an expression for the rate of change of temperature experienced by the object.

As the object moves from $\underline{r}$ to $\underline{r} + \delta\underline{r}$ in time δt, it experiences a change in temperature

$$\delta T \;=\; \underline{\nabla}\,T \cdot \delta\underline{r} \;=\; \left(\underline{\nabla}\,T \cdot \frac{\delta\underline{r}}{\delta t}\right)\delta t$$

Dividing by δt and taking the limit $\delta t \to 0$, we obtain

$$\frac{dT(\underline{r})}{dt} = \underline{v} \cdot \underline{\nabla}\,T(\underline{r})$$

3 More on gradient, the operator *del*

3.1 Examples of the gradient in physical laws

3.1.1 Gravitational force due to the Earth

The potential energy of a particle of mass m at a modest height, z, above the Earth's surface is $V = mgz$. The force due to gravity can be written as

$$\underline{F} = -\underline{\nabla}\,V = -mg\,\underline{e}_z$$

Exercise: Show that this last expression is correct.

Note that we *choose* to put a minus sign in the expression $\underline{F} = -\underline{\nabla}\,V$ so that the force acts *down* the potential energy gradient – as observed in nature!

3.1.2 More examples on *grad*

Before looking at Newton's Universal Law of Gravitation, we consider two straightforward (but important) examples of gradients.

(i) Calculate $\underline{\nabla}\,\phi$ for $\phi(\underline{r}) = r = \sqrt{x_1^2 + x_2^2 + x_3^2}$

Using the chain rule, the first component of $\underline{\nabla}\,r$ is

$$\frac{\partial}{\partial x_1}(x_1^2 + x_2^2 + x_3^2)^{1/2} \;=\; \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)^{-1/2}\,2x_1 \;=\; \frac{x_1}{r}$$

Similarly, the second and third components are:

$$\frac{\partial}{\partial x_2}(x_1^2 + x_2^2 + x_3^2)^{1/2} = \frac{x_2}{r} \qquad \text{and} \qquad \frac{\partial}{\partial x_3}(x_1^2 + x_2^2 + x_3^2)^{1/2} = \frac{x_3}{r}$$

Putting these together, we get $\underline{\nabla}\,r \;=\; (x_1\,\underline{e}_1 + x_2\,\underline{e}_2 + x_3\,\underline{e}_3)\,\dfrac{1}{r}$

Hence

$$\boxed{\underline{\nabla}\,r \;=\; \frac{1}{r}\,\underline{r} \;=\; \underline{\hat{r}}}$$

We conclude that for $\phi(\underline{r}) = r$ = *"the length of the position vector at position $\underline{r}$"*, the gradient $\underline{\nabla}\,\phi$ is just the unit vector $\underline{\hat{r}}$ which points *radially outwards* from the origin.[6] It has the same *magnitude* everywhere, but its *direction* is normal to the level surfaces which are spheres centered on the origin.

(ii) Calculate $\underline{\nabla}\,\phi$ for $\phi(\underline{r}) = \dfrac{1}{r} = \dfrac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$

The first component of $\underline{\nabla}\,(1/r)$ is

$$\frac{\partial}{\partial x_1}(x_1^2 + x_2^2 + x_3^2)^{-1/2} \;=\; -\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)^{-3/2}\,2x_1 \;=\; -\frac{x_1}{r^3}$$

Similarly for the second and third components:

$$\frac{\partial}{\partial x_2}(x_1^2 + x_2^2 + x_3^2)^{-1/2} \;=\; -\frac{x_2}{r^3} \qquad \text{and} \qquad \frac{\partial}{\partial x_3}(x_1^2 + x_2^2 + x_3^2)^{-1/2} \;=\; -\frac{x_3}{r^3}$$

Hence

$$\boxed{\underline{\nabla}\,(1/r) \;=\; -\frac{1}{r^3}\,\underline{r} \;=\; -\frac{1}{r^2}\,\underline{\hat{r}}}$$

For $\phi(\underline{r}) = 1/r$ = *"the inverse of the length of the position vector"*, the gradient $\underline{\nabla}\,\phi$ points *radially inwards* towards the origin, with magnitude $1/r^2$. The level surfaces are again spheres centered on the origin.

[6] We use the notations $\underline{\hat{r}}$ and $\underline{e}_r$ for a unit vector in the direction of the position $\underline{r}$.

3.1.3 Newton's Law of Gravitation:

Armed with this result, we now return to the force $\underline{F}(\underline{r})$ on a mass m_1 at $\underline{r}$ due to a mass m at the origin:

$$\underline{F}(\underline{r}) \;=\; -\frac{Gm_1 m}{r^3}\,\underline{r}$$

Using the result $\underline{\nabla}\,(1/r) = -\underline{r}/r^3$ derived in (ii) above, we may write this as

$$\underline{F}(\underline{r}) \;=\; -\,\underline{\nabla}\,V(\underline{r})$$

where the gravitational *potential energy* is $V(\underline{r}) = -Gm_1 m/r$.

Similarly, we can write the *gravitational field*

$$\underline{G}(\underline{r}) \;=\; -\frac{Gm}{r^3}\,\underline{r} \;=\; -\,\underline{\nabla}\,\phi(\underline{r})$$

where the gravitational *potential* is $\phi(\underline{r}) = -Gm/r$.

Similar results hold for the electrostatic force and its associated electric field..

We shall show later that there is a very general class of vector fields that can be written as the gradient of a scalar field known as a *scalar potential.*

3.2 Identities for gradients

Thus far, we have calculated gradients in a rather tedious and repetitive fashion – we worked out each example from scratch, and we calculated each of the three components of $\underline{\nabla}\,\phi$ in each example. This was deliberate.... We shall now see what we can gain by becoming a little more sophisticated.

We shall derive several *identities* which hold for the gradient of any scalar field, and which we may use to speed up the evaluation of the gradient of more complicated scalar fields.

If $\phi(\underline{r})$ and $\psi(\underline{r})$ are scalar fields, then:

(i) **Distributive law**

$$\boxed{\underline{\nabla}\,(\phi + \psi) \;=\; \underline{\nabla}\,\phi + \underline{\nabla}\,\psi}$$

Proof: For the first component

$$\left(\underline{\nabla}\,(\phi\,+\,\psi)\right)_1 \;\equiv\; \frac{\partial}{\partial x_1}\,(\phi\,+\,\psi) \;=\; \frac{\partial\phi}{\partial x_1} + \frac{\partial\phi}{\partial x_1} \;=\; (\underline{\nabla}\,\phi)_1 + (\underline{\nabla}\,\psi)_1$$

Similarly for the second and third components, and the result then follows for the gradient.

We may remove some repetition from the proof by evaluating the i^{th} component of the gradient of the sum, for each of $i = 1, 2, 3$:

$$\left(\underline{\nabla}\,(\phi\,+\,\psi)\right)_i \;\equiv\; \frac{\partial}{\partial x_i}\,(\phi\,+\,\psi) \;=\; \frac{\partial\phi}{\partial x_i} + \frac{\partial\phi}{\partial x_i} \;=\; (\underline{\nabla}\,\phi)_i + (\underline{\nabla}\,\psi)_i$$

(ii) **Product rule** $$\boxed{\underline{\nabla}\,(\phi\,\psi) \;=\; (\underline{\nabla}\,\phi)\,\psi + \phi\,(\underline{\nabla}\,\psi)}$$

Proof: Using the product rule of ordinary calculus, the first component of $\underline{\nabla}\,(\phi\,\psi)$ is

$$\begin{aligned}(\underline{\nabla}\,(\phi\psi))_1 &= \frac{\partial}{\partial x_1}\,(\phi\,\psi) = \left(\frac{\partial\phi}{\partial x_1}\right)\psi + \phi\left(\frac{\partial\psi}{\partial x_1}\right)\\ &= (\underline{\nabla}\,\phi)_1\,\psi + \phi\,(\underline{\nabla}\,\psi)_1\end{aligned}$$

Similarly for the second and third components, so the product rule holds for grad.

Again, we may save some time by evaluating the i^{th} component for each of $i = 1, 2, 3$:

$$\begin{aligned}(\underline{\nabla}\,(\phi\psi))_i &= \frac{\partial}{\partial x_i}\,(\phi\,\psi) = \left(\frac{\partial\phi}{\partial x_i}\right)\psi + \phi\left(\frac{\partial\psi}{\partial x_i}\right)\\ &= (\underline{\nabla}\,\phi)_i\,\psi + \phi\,(\underline{\nabla}\,\psi)_i\end{aligned}$$

Since this holds for each of the three components $i = 1, 2, 3$ of the gradient, the product rule must hold for the gradient operation itself.

If you're not comfortable with evaluating the i^{th} component, then put $i = 1$ to recover the expression in the first derivation. Then put $i = 2$, then $i = 3$. After a while, it seems perfectly natural to consider the i^{th} component from the beginning.

(iii) **Chain rule:** If $F(\phi(\underline{r}))$ is a scalar field, then

$$\boxed{\underline{\nabla}\,F\,(\phi) \;=\; \frac{dF(\phi)}{d\phi}\,\underline{\nabla}\,\phi}$$

Proof: The i^{th} component is

$$(\underline{\nabla}\,F(\phi))_i = \frac{\partial}{\partial x_i}(F(\phi)) = \frac{dF(\phi)}{d\phi}\,\frac{\partial\phi}{\partial x_i} = \frac{dF(\phi)}{d\phi}\,(\underline{\nabla}\,\phi)_i$$

where we used the ordinary chain rule to get the second-last expression. Again, if you're not comfortable with evaluating the i^{th} component, just set $i = 1, 2, 3$, in turn.

Example of chain rule: If $\phi(\underline{r}) = r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ we can use result (i) from section (3.1.2), $\underline{\nabla}\,r = \underline{r}/r$, to get

$$\underline{\nabla}\,F(r) = \frac{dF(r)}{dr}\,\underline{\nabla}\,r = \frac{F'(r)}{r}\,\underline{r}$$

where, as usual, $F'(r)$ denotes the derivative of the function $F(r)$ with respect to r.

If $F\,(\phi(\underline{r})) = r^n$, we have $\phi(\underline{r}) = r$ as in the previous example, and we obtain the important result

$$\boxed{\underline{\nabla}\,(r^n) \;=\; \frac{d\,r^n}{dr}\,(\underline{\nabla}\,r) \;=\; \left(n\,r^{n-1}\right)\frac{1}{r}\,\underline{r} \;=\; \left(n\,r^{n-2}\right)\underline{r}}$$

NB: This is a *much* quicker way of evaluating $\underline{\nabla}\,(r^n)$ than writing out the components – as you were asked to do in Homework Problem 1.8.

Setting $n = -1$ gives

$$\underline{\nabla}\left(\frac{1}{r}\right) = -\frac{\underline{r}}{r^3}$$

which reproduces result (ii) from section (3.1.2).

Example: Calculate $\underline{\nabla}\,\phi$ when $\phi(\underline{r}) = r^n\,(\underline{a}\cdot\underline{r})^m$.

$$\begin{aligned}\underline{\nabla}\,\{r^n\,(\underline{a}\cdot\underline{r})^m\} &= (\underline{\nabla}\,r^n)\,(\underline{a}\cdot\underline{r})^m + r^n\,\{\underline{\nabla}\,(\underline{a}\cdot\underline{r})^m\} && \text{(using the product rule)}\\ &= (\underline{\nabla}\,r^n)\,(\underline{a}\cdot\underline{r})^m + r^n\,m\,(\underline{a}\cdot\underline{r})^{m-1}\,\{\underline{\nabla}\,(\underline{a}\cdot\underline{r})\} && \text{(using the chain rule)}\\ &= n\,r^{n-2}\,(\underline{a}\cdot\underline{r})^m\,\underline{r} + mr^n\,(\underline{a}\cdot\underline{r})^{m-1}\,\underline{a}\end{aligned}$$

where, in the last line, we used $\underline{\nabla}\,r^n = n\,r^{n-2}\underline{r}$ and $\underline{\nabla}\,(\underline{a}\cdot\underline{r}) = \underline{a}$.

This example demonstrates the "toolkit" approach to evaluating gradients of complicated expressions. Here, we combined the product rule for grad with *known* gradients of "simple" scalar fields $(\underline{a}\cdot\underline{r})$ and r^n, which you should *know*, *and* be able to work out from first principles.

3.3 The operator *del*

We can think of $\underline{\nabla}$ as a *vector operator*, called *del*, which acts on the *scalar field* $\phi(\underline{r})$ to produce the *vector field* $\underline{\nabla}\,\phi(\underline{r})$, which is pronounced *grad phi*.

In Cartesians: $$\boxed{\underline{\nabla} \;\equiv\; \underline{e}_1\frac{\partial}{\partial x_1} + \underline{e}_2\frac{\partial}{\partial x_2} + \underline{e}_3\frac{\partial}{\partial x_3} \;\equiv\; \sum_{i=1}^{3}\underline{e}_i\,\frac{\partial}{\partial x_i}}$$

We call $\underline{\nabla}$ an *operator* since it operates on something to its *right*. It is a *vector* operator because it produces a *vector* field when it operates on a *scalar* field.

We have seen how $\underline{\nabla}$ acts on a scalar field to produce a vector field. We can take products of the vector operator $\underline{\nabla}$ with other vector quantities to produce new operators and fields in the same way we can take scalar and vector products of two vectors.

For example, the directional derivative of ϕ in the direction $\hat{\underline{s}}$, is given by $\hat{\underline{s}}\cdot\underline{\nabla}\,\phi$.

More generally, if $\underline{a}$ is any vector (or any vector field), we can interpret $\underline{a}\cdot\underline{\nabla}$ as a *scalar operator*

$$(\underline{a}\cdot\underline{\nabla}\,) = a_1\,\frac{\partial}{\partial x_1} + a_2\,\frac{\partial}{\partial x_2} + a_3\,\frac{\partial}{\partial x_3} = \sum_{i=1}^{3} a_i\,\frac{\partial}{\partial x_i}$$

The operator $\underline{a}\cdot\underline{\nabla}$ acts on a *scalar* field to its *right* to produce another *scalar field*

$$(\underline{a}\cdot\underline{\nabla}\,)\,\phi = a_1\,\frac{\partial\phi}{\partial x_1} + a_2\,\frac{\partial\phi}{\partial x_2} + a_3\,\frac{\partial\phi}{\partial x_3} = \sum_{i=1}^{3} a_i\,\frac{\partial\phi}{\partial x_i} = \underline{a}\cdot(\underline{\nabla}\,\phi)$$

We can also act with this operator on a *vector field* $\underline{b}(\underline{r})$ to get another *vector field*,

$$(\underline{a}\cdot\underline{\nabla}\,)\,\underline{b} = \underline{e}_1\,(\underline{a}\cdot\underline{\nabla}\,)\,b_1 + \underline{e}_2\,(\underline{a}\cdot\underline{\nabla}\,)\,b_2 + \underline{e}_3\,(\underline{a}\cdot\underline{\nabla}\,)\,b_3$$

The alternative expression $\underline{a}\cdot(\underline{\nabla}\,\underline{b})$ is *undefined* because $\underline{\nabla}\,\underline{b}$ doesn't make sense – just like $\underline{a}\,\underline{b}$ doesn't make sense.

(For this reason, the parentheses are sometimes omitted, and $\underline{a}\cdot\underline{\nabla}\,\underline{b}$ is taken to mean $(\underline{a}\cdot\underline{\nabla}\,)\,\underline{b}$, but I wouldn't recommend doing this because it often leads to errors.)

NB Great care is required with the *order* in products since, in general, products involving operators are not commutative. For example, if $\underline{a}(\underline{r})$ is a vector field

$$\underline{a}\cdot\underline{\nabla} \;\neq\; \underline{\nabla}\cdot\underline{a}$$

The quantity $\underline{a}\cdot\underline{\nabla}$ is a scalar differential *operator*, whereas $\underline{\nabla}\cdot\underline{a}$ is a *scalar* field called the *divergence* of $\underline{a}$ – see later.

Example: If $\underline{a}(\underline{r})$ is a vector field, show that $(\underline{a}\cdot\underline{\nabla})\,\underline{r} = \underline{a}$. This is left as an (important) tutorial exercise for the student.

Examples: In section (2.2) we showed that, for small displacements $\delta\underline{r}$, we have

$$\phi(\underline{r}+\delta\underline{r}) \;=\; \phi(\underline{r}) + (\underline{\nabla}\,\phi)\cdot\delta\underline{r} + O\left((\delta r)^2\right) \;=\; \phi(\underline{r}) + \delta\underline{r}\cdot\underline{\nabla}\,\phi + O\left((\delta r)^2\right) \qquad (7)$$

If we set $\delta\underline{r} = \underline{a}$, where $\underline{a}$ is an arbitrary (but small) constant vector, we have[7]

$$\boxed{\phi(\underline{r}+\underline{a}) \;=\; \phi(\underline{r}) + \underline{a}\cdot\underline{\nabla}\,\phi(\underline{r}) + O\left(a^2\right)} \qquad (8)$$

As we shall see, this expression is is very useful when $\phi(\underline{r})$ is the electrostatic potential.

We can expand a *vector field* about some point $\underline{r}$ in exactly the same way. For example, let $\underline{E}(\underline{r})$ be the electric field at $\underline{r}$. If we take the result of equation (7), and simply replace $\phi(\underline{r})$ by (say) the first component of the electric field $E_1(\underline{r})$, we obtain:

$$E_1(\underline{r}+\delta\underline{r}) \;=\; E_1(\underline{r}) + (\delta\underline{r}\cdot\underline{\nabla})\,E_1(\underline{r}) + O\left((\delta r)^2\right)$$

Doing the same for the second and third components of the electric field, and again setting $\delta\underline{r} = \underline{a}$, gives

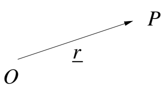
$$\boxed{\underline{E}(\underline{r}+\underline{a}) \;=\; \underline{E}(\underline{r}) + (\underline{a}\cdot\underline{\nabla})\,\underline{E}(\underline{r}) + O\left(a^2\right)} \qquad (9)$$

Equations (8) and (9) arise in the study of dipoles in electrostatics.

3.4 Equations of points, lines and planes

(This section may be revision.)

3.4.1 The position vector

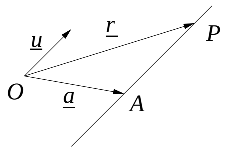


The equation for a point is simply $\underline{r} = \underline{a}$ where $\underline{a}$ is some constant vector.

[7]We can write this equation as $\phi(\underline{r}+\underline{a}) \;=\; \phi(\underline{r}) + a\frac{d\phi}{da} + O\left(a^2\right)$ where $\frac{d\phi}{da}$ is the directional derivative in the direction of the unit vector $\hat{\underline{a}}$, and $a = |\underline{a}|$, but this is rarely done in practice - possibly because its aparent simplicity leads to errors!

3.4.2 The equation of a line

Suppose that the point P lies on a line which passes through a point A, which has a position vector $\underline{a}$ with respect to an origin O. Let P have position vector $\underline{r}$ relative to O, and let $\underline{u}$ be a vector through the origin in a direction parallel to the line.



From the figure, $\underline{r} = \overrightarrow{OA} + \overrightarrow{AP}$, which, for some λ, we may write as

$$\underline{r} = \underline{a} + \lambda\underline{u}$$

This is the *explicit* or *parametric equation of the line*, *i.e.* as we vary the parameter λ from $-\infty$ to ∞, $\underline{r}$ describes all points on the line.

Rearranging and using $\underline{u}\times\underline{u} = 0$, we can also write this as

$$(\underline{r}-\underline{a})\times\underline{u} \;=\; 0$$

or

$$\boxed{\underline{r}\times\underline{u} = \underline{c}}$$

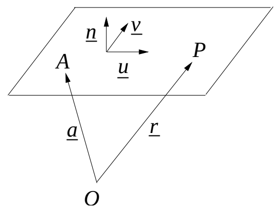
where $\underline{c} = \underline{a}\times\underline{u}$ is normal to the plane containing the line and the origin.

Notes

(i) $\underline{r}\times\underline{u} = \underline{c}$ is an *implicit* equation for a line parallel to the vector $\underline{u}$.

(ii) $\underline{r}\times\underline{u} = 0$ is the equation of a line through the origin ($\underline{a}$ and $\underline{u}$ are parallel in this case.)

3.4.3 The equation of a plane



$\underline{r}$ is the position vector of an *arbitrary* point P on the plane;
$\underline{a}$ is the position vector of a *fixed* point A in the plane;
$\underline{u}$ and $\underline{v}$ are any vectors parallel to the plane, but non-collinear: $\underline{u}\times\underline{v} \neq 0$.

We can express the vector $\overrightarrow{AP}$ in terms of $\underline{u}$ and $\underline{v}$, so that

$$\underline{r} = \underline{a} + \overrightarrow{AP} = \underline{a} + \lambda\underline{u} + \mu\underline{v} \qquad (10)$$

for some λ and μ. This is the *parametric equation of the plane.*

Now define the unit normal to the plane

$$\underline{n} = \frac{\underline{u}\times\underline{v}}{|\underline{u}\times\underline{v}|}\ .$$

Clearly $\underline{u}\cdot\underline{n} = \underline{v}\cdot\underline{n} = 0$. Rearranging equation (10), and using these results, we find the *implicit equation for the plane*

$$(\underline{r}-\underline{a})\cdot\underline{n} = 0\ .$$

Alternatively, we can write this as

$$\boxed{\underline{r} \cdot \underline{n} \;=\; p}$$

where $p = \underline{a} \cdot \underline{n}$ is the perpendicular distance of the plane from the origin.

This is a very important equation which you must be able to recognise.

Note: $\underline{r} \cdot \underline{n} = 0$ is the equation for a plane through the origin (with unit normal $\underline{n}$).

Example: Recall that $\underline{\nabla}\,(\underline{a} \cdot \underline{r}) = \underline{a}$ when $\underline{a}$ is a constant vector. Thus the level surfaces of the scalar field $\underline{a} \cdot \underline{r}$ are planes orthogonal to $\underline{a}$.

4 Div, curl and the Laplacian

We now combine the vector operator $\underline{\nabla}$ *(del)* with a *vector* field to define two new operations *div* and *curl*. Then we define the *Laplacian*.

4.1 Divergence

We define the *divergence* of a vector field $\underline{a}(\underline{r})$ (pronounced *'div a'*) by

$$\boxed{\operatorname{div} \underline{a}(\underline{r}) \;\equiv\; \underline{\nabla} \cdot \underline{a}(\underline{r})}$$

In Cartesian coordinates

$$\begin{aligned} \underline{\nabla} \cdot \underline{a} &\equiv \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \\ &\equiv \sum_{i=1}^{3} \frac{\partial a_i}{\partial x_i} \end{aligned}$$

$\underline{\nabla} \cdot \underline{a}$ is a single number at each point $\underline{r}$, so it's a *scalar field*.

Example: If $\underline{a}(\underline{r}) = \underline{r}$ then $\boxed{\underline{\nabla} \cdot \underline{r} \;=\; 3}$ which is a very important and useful result.

Explicitly: $$\underline{\nabla} \cdot \underline{r} = \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} + \frac{\partial x_3}{\partial x_3} = 1 + 1 + 1 = 3$$

Example: In *'xyz'* notation, let $\underline{a}(\underline{r}) = x^2 z\,\underline{e}_x - 2y^3 z^2\,\underline{e}_y + xy^2 z\,\underline{e}_z$

$$\begin{aligned} \underline{\nabla} \cdot \underline{a} &= \frac{\partial}{\partial x}\,(x^2 z) - \frac{\partial}{\partial y}\,(2y^3 z^2) + \frac{\partial}{\partial z}\,(xy^2 z) \\ &= 2xz - 6y^2 z^2 + xy^2 \end{aligned}$$

Then, at the point $(1, 1, 1)$ for instance, $\underline{\nabla} \cdot \underline{a} = 2 - 6 + 1 = -3$.

4.2 Curl

We define the *curl* of a vector field $\underline{a}(\underline{r})$ by

$$\boxed{\operatorname{curl} \underline{a}(\underline{r}) \;\equiv\; \underline{\nabla} \times \underline{a}(\underline{r})}$$

$\underline{\nabla} \times \underline{a}$ is a *vector* field.[8]

We can write the curl in determinant form, as we did for the ordinary vector product:

$$\boxed{\underline{\nabla} \times \underline{a} \;=\; \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ a_1 & a_2 & a_3 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} \underline{e}_x & \underline{e}_y & \underline{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}}$$

More explicitly, the components of $\underline{\nabla} \times \underline{a}$ are:

$$(\underline{\nabla} \times \underline{a})_1 = \frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \qquad (\underline{\nabla} \times \underline{a})_2 = \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1} \qquad (\underline{\nabla} \times \underline{a})_3 = \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}$$

Example: If $\underline{a}(\underline{r}) = \underline{r}$ then $\boxed{\underline{\nabla} \times \underline{r} \;=\; 0}$ which is another very important and useful result.

Proof: Using the determinant formula, we get $\underline{\nabla} \times \underline{r} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ x_1 & x_2 & x_3 \end{vmatrix} \equiv 0$

Explicitly:

$$\underline{\nabla} \times \underline{r} \;=\; \underline{e}_1 \left(\frac{\partial x_3}{\partial x_2} - \frac{\partial x_2}{\partial x_3} \right) \;+\; \underline{e}_2 \left(\frac{\partial x_1}{\partial x_3} - \frac{\partial x_3}{\partial x_1} \right) \;+\; \underline{e}_3 \left(\frac{\partial x_2}{\partial x_1} - \frac{\partial x_1}{\partial x_2} \right) \;=\; 0$$

[8] More precisely, $\underline{\nabla} \times \underline{a}$ is a pseudo-vector field, if $\underline{a}$ is a vector field.

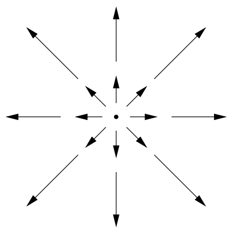
Example: Compute the curl of $\underline{a} = x^2y\,\underline{e}_x + y^2x\,\underline{e}_y + xyz\,\underline{e}_z$

$$\underline{\nabla} \times \underline{a} \;=\; \begin{vmatrix} \underline{e}_x & \underline{e}_y & \underline{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & y^2x & xyz \end{vmatrix} = (xz-0)\,\underline{e}_x - (yz-0)\,\underline{e}_y + (y^2-x^2)\,\underline{e}_z$$

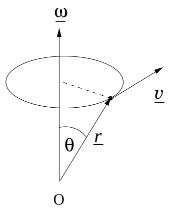
4.3 Geometrical/physical interpretation of *div* and *curl*

A full interpretation of the divergence and curl of a vector field is best left until after we have studied the *divergence theorem* and *Stokes' theorem* respectively. However, we can gain some intuitive understanding by looking at simple examples where div and/or curl vanish.

Example: Consider the radial field $\underline{a}(\underline{r}) = \underline{r}$. We have just shown that $\underline{\nabla} \cdot \underline{r} = 3$ and $\underline{\nabla} \times \underline{r} = 0$. We may sketch the vector field $\underline{a}(\underline{r})$ by drawing vectors of the appropriate direction and magnitude at selected points. These give the tangents of 'flow lines'. Roughly speaking, in this example the divergence is positive because bigger arrows come out of any point than go into it. So the field 'diverges'. (Once the concept of flux of a vector field is understood this will make more sense.)

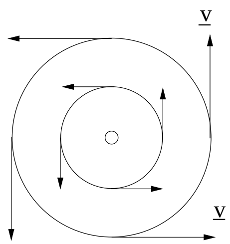


Angular velocity: Consider a point in a rigid body rotating with *angular velocity* $\underline{\omega} = \omega\,\hat{\underline{\omega}}$. The magnitude $\omega = |\underline{\omega}|$ is the angular speed of rotation measured in radians per second, and the unit vector $\hat{\underline{\omega}}$ lies along the axis of rotation. Let the position vector of the point with respect to an origin O on the axis of rotation be $\underline{r}$.



You should convince yourself that the velocity of the point is given by $\underline{v} = \underline{\omega} \times \underline{r}$ by checking that this gives the right direction for $\underline{v}$; that it is perpendicular to the plane of $\underline{\omega}$ and $\underline{r}$; that the magnitude $|\underline{v}| = \omega r \sin\theta = \omega\rho$, where ρ is the radius of the circle in which the point is travelling.

Example: Consider the field $\underline{v}(\underline{r}) = \underline{\omega} \times \underline{r}$ where $\underline{\omega}$ is a *constant* vector. One can think of $\underline{v}(\underline{r})$ as the velocity of a point in a rigid rotating body. The sketch shows a cross-section of the field $\underline{v}(\underline{r})$ with $\underline{\omega}$ chosen to point out of the page.



To evaluate $\underline{\nabla} \times (\underline{\omega} \times \underline{r})$ and $\underline{\nabla} \cdot (\underline{\omega} \times \underline{r})$, first note that

$$\underline{\omega} \times \underline{r} \;=\; \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ x_1 & x_2 & x_3 \end{vmatrix} \;=\; \underline{e}_1\,(\omega_2x_3 - \omega_3x_2) + \underline{e}_2\,(\omega_3x_1 - \omega_1x_3) + \underline{e}_3\,(\omega_1x_2 - \omega_2x_1)$$

Then

$$\underline{\nabla} \times (\underline{\omega} \times \underline{r}) \;=\; \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ (\underline{\omega}\times\underline{r})_1 & (\underline{\omega}\times\underline{r})_2 & (\underline{\omega}\times\underline{r})_3 \end{vmatrix}$$

The first component of $\underline{\nabla} \times (\underline{\omega} \times \underline{r})$ is

$$\begin{aligned} (\underline{\nabla} \times (\underline{\omega} \times \underline{r}))_1 &= \frac{\partial}{\partial x_2}(\underline{\omega}\times\underline{r})_3 - \frac{\partial}{\partial x_3}(\underline{\omega}\times\underline{r})_2 \\ &= \frac{\partial}{\partial x_2}(\omega_1x_2 - \omega_2x_1) - \frac{\partial}{\partial x_3}(\omega_3x_1 - \omega_1x_3) \\ &= (\omega_1 - 0) - (0 - \omega_1) \;=\; 2\omega_1 \end{aligned}$$

The second and third components of $\underline{\nabla} \times (\underline{\omega} \times \underline{r})$ are $2\omega_2$ and $2\omega_3$ respectively (exercise).

Hence $\underline{\nabla} \times (\underline{\omega} \times \underline{r}) = 2\underline{\omega}$

Similarly,

$$\begin{aligned} \underline{\nabla} \cdot (\underline{\omega} \times \underline{r}) &= \frac{\partial}{\partial x_1}(\underline{\omega}\times\underline{r})_1 + \frac{\partial}{\partial x_2}(\underline{\omega}\times\underline{r})_2 + \frac{\partial}{\partial x_3}(\underline{\omega}\times\underline{r})_3 \\ &= \frac{\partial}{\partial x_1}(\omega_2x_3 - \omega_3x_2) + \frac{\partial}{\partial x_2}(\omega_3x_1 - \omega_1x_3) + \frac{\partial}{\partial x_3}(\omega_1x_2 - \omega_2x_1) \\ &= 0 + 0 + 0 \;=\; 0 \end{aligned}$$

Thus we have two more important and useful results, which hold for all *constant* vectors $\underline{\omega}$

$$\boxed{\underline{\nabla} \times (\underline{\omega} \times \underline{r}) = 2\underline{\omega} \qquad \text{and} \qquad \underline{\nabla} \cdot (\underline{\omega} \times \underline{r}) = 0}$$

To understand intuitively the non-zero curl of $\underline{v}(\underline{r}) = \underline{\omega} \times \underline{r}$, imagine that the flow lines are those of a rotating fluid with a ball centred on a flow line of the field. The centre of the ball will follow the flow line. However the effect of the neighbouring flow lines is to make the ball rotate. Therefore the field has non-zero 'curl', and the axis of rotation gives the direction that the field rotates or 'curls' around.

In the previous example, $\underline{a}(\underline{r}) = \underline{r}$, the ball would just move away from the origin without rotating, therefore the field $\underline{r}$ has zero curl. Colloquially, the vector field $\underline{a}(\underline{r}) = \underline{r}$, doesn't "curl around" any point at all.

A shortcut: We may evaluate $\underline{\nabla} \times (\underline{\omega} \times \underline{r})$ by careful use of the vector triple product formula

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}\cdot\underline{c})\,\underline{b} - (\underline{a}\cdot\underline{b})\,\underline{c} = (\underline{a}\cdot\underline{c})\,\underline{b} - (\underline{b}\cdot\underline{a})\,\underline{c}\,. \tag{11}$$

which holds for all vectors and vector fields $\underline{a}$, $\underline{b}$, $\underline{c}$.

Here we wish to evaluate

$$\underline{\nabla} \times (\underline{\omega} \times \underline{r})$$

where the derivatives in the vector *operator* $\underline{\nabla}$ act on everything to their right, namely on both $\underline{\omega}$ and $\underline{r}$, so we must be careful with ordering when using equation (11).

In this case $\underline{\omega}$ is a *constant*, so the only non-zero derivatives are those in which $\underline{\nabla}$ acts on $\underline{r}$. Using the expression after the second equals sign in equation (11), so that $\underline{\nabla}$ acts only on $\underline{r}$, we have

$$\underline{\nabla} \times (\underline{\omega} \times \underline{r}) \;=\; (\underline{\nabla} \cdot \underline{r})\, \underline{\omega} - (\underline{\omega} \cdot \underline{\nabla})\, \underline{r} \;=\; 3\underline{\omega} - \underline{\omega} \;=\; 2\underline{\omega}$$

Terminology: For any vector field $\underline{a}(\underline{r})$

(i) If $\underline{\nabla} \cdot \underline{a} = 0$ in some region R, $\underline{a}$ is said to be *solenoidal* in R.

(ii) If $\underline{\nabla} \times \underline{a} = 0$ in some region R, $\underline{a}$ is said to be *irrotational* in R.

4.4 The Laplacian operator ∇^2

Consider the *divergence* of the *gradient* of the scalar field $\phi(\underline{r})$. Recall that $\underline{\nabla}\,\phi$ has components

$$\left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3}\right)$$

in Cartesian coordinates. Therefore

$$\begin{aligned}
\underline{\nabla} \cdot (\underline{\nabla}\,\phi) \;&=\; \frac{\partial}{\partial x_1}\left(\frac{\partial \phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial \phi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial \phi}{\partial x_3}\right) \\
&=\; \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} \;\equiv\; \nabla^2 \phi
\end{aligned}$$

∇^2 is called the *Laplacian operator*, pronounced *'del-squared'*. In Cartesian coordinates

$$\boxed{\nabla^2 \phi \;=\; \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} \quad \text{or} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}}$$

We may write $\nabla^2\phi$ in terms of indices as

$$\underline{\nabla} \cdot (\underline{\nabla}\,\phi) \;=\; \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial x_i}\right) \;=\; \sum_{i=1}^{3} \frac{\partial^2 \phi}{\partial x_i^2}$$

The Laplacian ∇^2 is a *scalar operator*

$$\nabla^2 \;=\; \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \;=\; \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$$

which acts on the *scalar field* $\phi(\underline{r})$ to produce another *scalar field* $\nabla^2\phi(\underline{r})$.

NB Although we have written it here as an operator, in any actual calculation the Laplacian *must* always act on some quantity to its right.

Example: Evaluate $\nabla^2 r^2$ and $\nabla^2 (\underline{a} \cdot \underline{r})$ where $\underline{a}$ is a *constant* vector.

$$\begin{aligned}
\nabla^2 r^2 \;&=\; \underline{\nabla} \cdot \left(\underline{\nabla}\, r^2\right) \;=\; \underline{\nabla} \cdot (2\underline{r}) \;=\; 2 \times 3 \;=\; 6\,. \\
\nabla^2 (\underline{a} \cdot \underline{r}) \;&=\; \underline{\nabla} \cdot (\underline{\nabla}\,(\underline{a} \cdot \underline{r})) \;=\; \underline{\nabla} \cdot \underline{a} \;=\; 0 \qquad (\text{because } \underline{a} \text{ is a } \textit{constant})
\end{aligned}$$

where we used the basic results $\underline{\nabla}\, r^2 = 2\underline{r}$, $\underline{\nabla} \cdot \underline{r} = 3$ and $\underline{\nabla}\,(\underline{a} \cdot \underline{r}) = \underline{a}$, derived previously.

We could have performed these calculations using components, but that would take longer.

Example: Evaluate $\nabla^2 r^n$.

$$\nabla^2 r^n \;=\; \underline{\nabla} \cdot (\underline{\nabla}\, r^n) \;=\; \underline{\nabla} \cdot \left(n r^{n-2} \underline{r}\right) \;=\; ?$$

We could evaluate the last expression by writing out its components, but we shall be patient and work out a better way of doing it in the next section.

In *Cartesian coordinates only*, the effect of the Laplacian on a vector field $\underline{a}(\underline{r})$ is *defined* to be

$$\nabla^2 \underline{a} \;=\; \frac{\partial^2}{\partial x_1^2}\,\underline{a} + \frac{\partial^2}{\partial x_2^2}\,\underline{a} + \frac{\partial^2}{\partial x_3^2}\,\underline{a} \;=\; \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\,\underline{a}$$

The Laplacian acts on a *vector field* to produce another *vector field*.

5 Vector operator identities

There are many identities involving *div*, *grad*, *curl* and the Laplacian. It is not necessary to know *all* of these, but you should know and be able to use the product and chain rules for gradients (see Section (3.2), together with the product laws for *div* and *curl* given below. Fortunately, most of these are almost almost obvious...

You should be *familiar* with the rest and to be able to *derive* and *use* them when necessary.

It is also extremely useful to *know* and be able to derive the results for elementary quantities such as $\underline{\nabla}\, r$, $\underline{\nabla}\, r^n$, $\underline{\nabla} \cdot \underline{r}$, $\underline{\nabla} \times \underline{r}$, $(\underline{a} \cdot \underline{\nabla})\underline{r}$, $\underline{\nabla}\,(\underline{a} \cdot \underline{r})$, $\underline{\nabla} \times (\underline{a} \times \underline{r})$ and $\underline{\nabla} \cdot (\underline{a} \times \underline{r})$ where $\underline{a}$ is a constant vector. This is similar to learning and understanding multiplication tables, or knowing the derivative of elementary functions such as $\sin x$.

We shall use vector identities and these elementary results to enable us to evaluate *div*, *grad* and *curl* of complicated expression with a minimum of effort – which is surely a good thing!

Most importantly you should be at ease with *div*, *grad* and *curl*. This only comes through practice and deriving the various identities gives you just that.

In what follows $\phi(\underline{r})$, $\underline{a}(\underline{r})$ and $\underline{b}(\underline{r})$ are all continuously-differentiable scalar and vector fields.

5.1 Distributive laws

1. $\underline{\nabla} \cdot (\underline{a} + \underline{b}) \;=\; \underline{\nabla} \cdot \underline{a} + \underline{\nabla} \cdot \underline{b}$
2. $\underline{\nabla} \times (\underline{a} + \underline{b}) \;=\; \underline{\nabla} \times \underline{a} + \underline{\nabla} \times \underline{b}$

The proofs of these are straightforward using components and they follow from the fact that *div* and *curl* are linear operations.

5.2 Product laws: one scalar field and one vector field

The results of taking the *div* or *curl* of *products* of vector and scalar fields are the most useful. They are predictable but they need a little care.

3. $\underline{\nabla} \cdot (\phi\, \underline{a}) \;=\; (\underline{\nabla}\, \phi) \cdot \underline{a} \,+\, \phi\, (\underline{\nabla} \cdot \underline{a})$
4. $\underline{\nabla} \times (\phi\, \underline{a}) \;=\; (\underline{\nabla}\, \phi) \times \underline{a} \,+\, \phi\, (\underline{\nabla} \times \underline{a})$

Proof of (3): [Tutorial exercise]

Proof of (4): Using the determinant formula, we have $\underline{\nabla} \times (\phi\, \underline{a}) = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \phi a_1 & \phi a_2 & \phi a_3 \end{vmatrix}$

The first component is

$$\begin{aligned} \{\underline{\nabla} \times (\phi\, \underline{a})\}_1 \;&=\; \frac{\partial(\phi a_3)}{\partial x_2} - \frac{\partial(\phi a_2)}{\partial x_3} \;=\; \left(\frac{\partial \phi}{\partial x_2}\right) a_3 - \left(\frac{\partial \phi}{\partial x_3}\right) a_2 + \phi \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) \\ &=\; \{(\underline{\nabla}\, \phi) \times \underline{a}\}_1 + \phi\, (\underline{\nabla} \times \underline{a})_1 \end{aligned}$$

Similarly, for the second and third components, hence the result.

Although we have used Cartesian coordinates in all our proofs, the identities hold in all coordinate systems (the concept of a vector is coordinate-independent).

5.3 Identities with indices – a temporary step-up in sophistication

We have worked out most results for *div*, *grad* and *curl* component-by-component, although we introduced the idea of evaluating the i^{th} component of the gradient in Section (3.2). In this section we extend this idea and apply it to *div* and *curl*.

We have already written $\underline{\nabla}\, \phi(\underline{r})$ as a sum of "derivatives multiplied by unit vectors":

$$\underline{\nabla}\, \phi(\underline{r}) \;\equiv\; \underline{e}_1 \frac{\partial \phi}{\partial x_1} + \underline{e}_2 \frac{\partial \phi}{\partial x_2} + \underline{e}_3 \frac{\partial \phi}{\partial x_3} \;=\; \sum_{i=1}^{3} \underline{e}_i \frac{\partial \phi}{\partial x_i}$$

We now show that *div* and *curl* can be written as:

$$\boxed{\underline{\nabla} \cdot \underline{a} = \sum_{i=1}^{3} \underline{e}_i \cdot \frac{\partial \underline{a}}{\partial x_i} \qquad \text{and} \qquad \underline{\nabla} \times \underline{a} = \sum_{i=1}^{3} \underline{e}_i \times \frac{\partial \underline{a}}{\partial x_i}} \tag{12}$$

where $\dfrac{\partial \underline{a}}{\partial x_i}$ is the partial derivative of the vector field $\underline{a}(\underline{r})$ with respect to the i^{th} component of the position vector x_i, *i.e.* it has components $\left(\dfrac{\partial a_1}{\partial x_i}, \dfrac{\partial a_2}{\partial x_i}, \dfrac{\partial a_3}{\partial x_i}\right)$.

Equations (12) follow from the linearity of *div* and *curl*, but we shall derive them explicitly.

The idea is to separate the vector properties of *div* and *curl* (namely the basis vectors $\underline{e}_i$ and the "dot" or "cross" products) from the derivatives $\partial \underline{a}/\partial x_i$.

(i) The RHS of the first of equations (12) is

$$\sum_{i=1}^{3} \underline{e}_i \cdot \frac{\partial \underline{a}}{\partial x_i} \;=\; \sum_{i=1}^{3} \underline{e}_i \cdot \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{3} \underline{e}_j\, a_j\right) \;=\; \sum_{i,j=1}^{3} \left(\underline{e}_i \cdot \underline{e}_j\right) \frac{\partial a_j}{\partial x_i} \;=\; \sum_{i=1}^{3} \frac{\partial a_i}{\partial x_i} \;=\; \underline{\nabla} \cdot \underline{a}$$

In the second-last step we used orthonormality of the basis vectors, $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$ (recall that δ_{ij} is 1 if $i = j$, and 0 otherwise). Therefore we can set $j = i$, and we need only sum over i (with $j = i$).

(ii) The RHS of the second of equations (12) is

$$\sum_{i=1}^{3} \underline{e}_i \times \frac{\partial \underline{a}}{\partial x_i} \;=\; \sum_{i=1}^{3} \underline{e}_i \times \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{3} \underline{e}_j\, a_j\right) \;=\; \sum_{i,j=1}^{3} \left(\underline{e}_i \times \underline{e}_j\right) \frac{\partial a_j}{\partial x_i}$$

The terms in the sum obtained by allowing each of i, j to take on the values $1, 2$ are

$$(\underline{e}_1 \times \underline{e}_1) \frac{\partial a_1}{\partial x_1} + (\underline{e}_1 \times \underline{e}_2) \frac{\partial a_2}{\partial x_1} + (\underline{e}_2 \times \underline{e}_1) \frac{\partial a_1}{\partial x_2} + (\underline{e}_2 \times \underline{e}_2) \frac{\partial a_2}{\partial x_2} \;=\; \underline{e}_3 \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)$$

which is the third component of $\underline{\nabla} \times \underline{a}$. Similarly, the other two pairs of values of i, j give the first and second components of $\underline{\nabla} \times \underline{a}$ (exercise).

Proof of (4) [revisited]:

$$\begin{aligned} \underline{\nabla} \times (\phi\, \underline{a}) \;&=\; \sum_{i=1}^{3} \underline{e}_i \times \frac{\partial}{\partial x_i}(\phi\, \underline{a}) \;=\; \sum_{i=1}^{3} \underline{e}_i \times \left(\frac{\partial \phi}{\partial x_i}\, \underline{a} \,+\, \phi\, \frac{\partial \underline{a}}{\partial x_i}\right) \qquad \text{(product rule)} \\ &=\; \sum_{i=1}^{3} \underline{e}_i\, \frac{\partial \phi}{\partial x_i} \times \underline{a} \,+\, \phi \sum_{i=1}^{3} \underline{e}_i \times \frac{\partial \underline{a}}{\partial x_i} \;=\; (\underline{\nabla}\, \phi) \times \underline{a} \,+\, \phi\, (\underline{\nabla} \times \underline{a}) \end{aligned}$$

To get the second line we used

$$\underline{e}_i \times \frac{\partial \phi}{\partial x_i}\, \underline{a} = \underline{e}_i\, \frac{\partial \phi}{\partial x_i} \times \underline{a}$$

which holds because $\frac{\partial \phi}{\partial x_i}$ is a function (not a vector), so it can be moved past the cross-product sign.

5.4 Product laws: two vector fields

The following identities are extremely useful but less obvious.

5. $\underline{\nabla} \cdot (\underline{a} \times \underline{b}) \;=\; \underline{b} \cdot (\underline{\nabla} \times \underline{a}) - \underline{a} \cdot (\underline{\nabla} \times \underline{b})$
6. $\underline{\nabla} \times (\underline{a} \times \underline{b}) \;=\; (\underline{\nabla} \cdot \underline{b})\, \underline{a} + (\underline{b} \cdot \underline{\nabla})\, \underline{a} - (\underline{\nabla} \cdot \underline{a})\, \underline{b} - (\underline{a} \cdot \underline{\nabla})\, \underline{b}$
7. $\underline{\nabla}\, (\underline{a} \cdot \underline{b}) \;=\; (\underline{a} \cdot \underline{\nabla})\, \underline{b} + (\underline{b} \cdot \underline{\nabla})\, \underline{a} + \underline{a} \times (\underline{\nabla} \times \underline{b}) + \underline{b} \times (\underline{\nabla} \times \underline{a})$

Identity (5) can be proved (with some effort) using components, identity (6) is a bit more work, and (7) is much longer. We shall use the sophisticated index method here and leave the component proofs to the tutorials.

Proof of (5):

$$
\begin{aligned}
\underline{\nabla}\cdot(\underline{a}\times\underline{b}) &= \sum_{i=1}^{3}\underline{e}_i\cdot\frac{\partial}{\partial x_i}(\underline{a}\times\underline{b}) \;=\; \sum_{i=1}^{3}\underline{e}_i\cdot\left(\frac{\partial\underline{a}}{\partial x_i}\times\underline{b} \;+\; \underline{a}\times\frac{\partial\underline{b}}{\partial x_i}\right) \qquad \text{(product rule)}\\
&= \sum_{i=1}^{3}\underline{b}\cdot\left(\underline{e}_i\times\frac{\partial\underline{a}}{\partial x_i}\right) \;-\; \sum_{i=1}^{3}\underline{a}\cdot\left(\underline{e}_i\times\frac{\partial\underline{b}}{\partial x_i}\right)\\
&= \underline{b}\cdot\left(\sum_{i=1}^{3}\underline{e}_i\times\frac{\partial\underline{a}}{\partial x_i}\right) \;-\; \underline{a}\cdot\left(\sum_{i=1}^{3}\underline{e}_i\times\frac{\partial\underline{b}}{\partial x_i}\right)\\
&= \underline{b}\cdot(\underline{\nabla}\times\underline{a})-\underline{a}\cdot(\underline{\nabla}\times\underline{b})
\end{aligned}
$$

where we used $\underline{A}\cdot(\underline{B}\times\underline{C})=\underline{C}\cdot(\underline{A}\times\underline{B})=-\underline{B}\cdot(\underline{A}\times\underline{C})$ to get the second line, and the second of equations (12) to get the last line.

Proof of (6):

$$
\begin{aligned}
\underline{\nabla}\times(\underline{a}\times\underline{b}) &= \sum_{i=1}^{3}\underline{e}_i\times\frac{\partial}{\partial x_i}(\underline{a}\times\underline{b}) \;=\; \sum_{i=1}^{3}\underline{e}_i\times\left(\frac{\partial\underline{a}}{\partial x_i}\times\underline{b} \;+\; \underline{a}\times\frac{\partial\underline{b}}{\partial x_i}\right) \qquad \text{(product rule)}\\
&= \sum_{i=1}^{3}\left\{(\underline{e}_i\cdot\underline{b})\frac{\partial\underline{a}}{\partial x_i} \;-\; \left(\underline{e}_i\cdot\frac{\partial\underline{a}}{\partial x_i}\right)\underline{b} \;+\; \left(\underline{e}_i\cdot\frac{\partial\underline{b}}{\partial x_i}\right)\underline{a} \;-\; (\underline{e}_i\cdot\underline{a})\frac{\partial\underline{b}}{\partial x_i}\right\}\\
&= \sum_{i=1}^{3}\left\{\left(\underline{b}\cdot\underline{e}_i\frac{\partial}{\partial x_i}\right)\underline{a} \;-\; \left(\underline{e}_i\cdot\frac{\partial\underline{a}}{\partial x_i}\right)\underline{b} \;+\; \left(\underline{e}_i\cdot\frac{\partial\underline{b}}{\partial x_i}\right)\underline{a} \;-\; \left(\underline{a}\cdot\underline{e}_i\frac{\partial}{\partial x_i}\right)\underline{b}\right\}\\
&= (\underline{b}\cdot\underline{\nabla})\,\underline{a} \;-\; (\underline{\nabla}\cdot\underline{a})\,\underline{b} \;+\; (\underline{\nabla}\cdot\underline{b})\,\underline{a} \;-\; (\underline{a}\cdot\underline{\nabla})\,\underline{b}\\
&= (\underline{\nabla}\cdot\underline{b})\,\underline{a} \;+\; (\underline{b}\cdot\underline{\nabla})\,\underline{a} \;-\; (\underline{\nabla}\cdot\underline{a})\,\underline{b} \;-\; (\underline{a}\cdot\underline{\nabla})\,\underline{b}
\end{aligned}
$$

where we used $\underline{A}\times(\underline{B}\times\underline{C})=(\underline{A}\cdot\underline{C})\,\underline{B}-(\underline{A}\cdot\underline{B})\,\underline{C}$ to get the second line, and the first of equations (12) to get the second-last line. The last line is just a re-ordering of the result.

Alternative proof of (6): Consider the expression

$$\underline{\nabla}\times(\underline{a}\times\underline{b})$$

The operator $\underline{\nabla}$ acts on *both* $\underline{a}$ and $\underline{b}$. So we will get the wrong answer if we naively replace the vector $\underline{c}$ by the operator $\underline{\nabla}$ in either one of the expressions after the equals signs in the usual expansion:

$$\underline{c}\times(\underline{a}\times\underline{b}) \;=\; (\underline{c}\cdot\underline{b})\,\underline{a}-(\underline{c}\cdot\underline{a})\,\underline{b} \;=\; (\underline{b}\cdot\underline{c})\,\underline{a}-(\underline{a}\cdot\underline{c})\,\underline{b}\,.$$

Since $\underline{\nabla}$ acts on *both* $\underline{a}$ and $\underline{b}$, the product rule tells us that we must have *both* orderings in the scalar products on the RHS

$$\underline{\nabla}\times(\underline{a}\times\underline{b}) \;=\; (\underline{\nabla}\cdot\underline{b})\,\underline{a}+(\underline{b}\cdot\underline{\nabla})\,\underline{a}-(\underline{\nabla}\cdot\underline{a})\,\underline{b}-(\underline{a}\cdot\underline{\nabla})\,\underline{b}\,.$$

We can obtain identity (5) in a similar way, which is useful exercise.

Proof of (7): [Tutorial exercise]

Identities (5), (6) & (7) are very useful in explicit calculations. You don't need to memorise them in gory detail, but you should know they exist, and you should understand the derivations. Other results involving one $\underline{\nabla}$ can be derived similarly.

Example: Show that $\underline{\nabla}\cdot(r^{-3}\underline{r})=0$, for $r\neq 0$, where as usual $r=|\underline{r}|$.

Solution: Using identity (3), we have

$$
\begin{aligned}
\underline{\nabla}\cdot\left(r^{-3}\underline{r}\right) &= \left(\underline{\nabla}\,r^{-3}\right)\cdot\underline{r} \;+\; r^{-3}\;(\underline{\nabla}\cdot\underline{r})\\
&= \left(\frac{-3}{r^5}\,\underline{r}\right)\cdot\underline{r}+\frac{3}{r^3} \;=\; \frac{-3}{r^5}r^2+\frac{3}{r^3} \;=\; 0 \qquad \text{(except at } r=0\text{)}
\end{aligned}
$$

where we used the results $\underline{\nabla}\,r^n=n\,r^{n-2}\,\underline{r}$ and $\underline{\nabla}\cdot\underline{r}=3$ that we derived previously.

5.5 Identities involving two ∇s

8. $\underline{\nabla}\times(\underline{\nabla}\,\phi) \;=\; 0$ (*curl grad* ϕ is *always* zero)
9. $\underline{\nabla}\cdot(\underline{\nabla}\times\underline{a}) \;=\; 0$ (*div curl* $\underline{a}$ is *always* zero)
10. $\underline{\nabla}\times(\underline{\nabla}\times\underline{a}) \;=\; \underline{\nabla}\,(\underline{\nabla}\cdot\underline{a})-\nabla^2\underline{a}$

Proofs of (8) and (9) are readily obtained directly in Cartesian coordinates. You should *know* (8) and (9), and knowing (10) is really, *really* useful!

Proof of (8):

$$
\begin{aligned}
\underline{\nabla}\times(\underline{\nabla}\,\phi) &= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3\\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3}\\ \dfrac{\partial\phi}{\partial x_1} & \dfrac{\partial\phi}{\partial x_2} & \dfrac{\partial\phi}{\partial x_3}\end{vmatrix}\\
&= \underline{e}_1\left\{\frac{\partial}{\partial x_2}\left(\frac{\partial\phi}{\partial x_3}\right)-\frac{\partial}{\partial x_3}\left(\frac{\partial\phi}{\partial x_2}\right)\right\} \;+\; \underline{e}_2\left\{\frac{\partial}{\partial x_3}\left(\frac{\partial\phi}{\partial x_1}\right)-\frac{\partial}{\partial x_1}\left(\frac{\partial\phi}{\partial x_3}\right)\right\}\\
&\quad + \underline{e}_3\left\{\frac{\partial}{\partial x_1}\left(\frac{\partial\phi}{\partial x_2}\right)-\frac{\partial}{\partial x_2}\left(\frac{\partial\phi}{\partial x_1}\right)\right\} \;=\; 0
\end{aligned}
$$

where in the last step we used the result that partial derivatives commute

$$\frac{\partial}{\partial x_1}\left(\frac{\partial\phi}{\partial x_2}\right)=\frac{\partial}{\partial x_2}\left(\frac{\partial\phi}{\partial x_1}\right) \qquad \text{etc.}$$

Proof of (9): This is similar to (8) and is left as a tutorial exercise.

Proof of (10): This can be proven using explicit coordinates or by using the second of equations (12) twice [tutorial]. The proofs are easier than you might expect.

Identity (10) is used in curvilinear coordinate systems to *define* the action of the Laplacian on a *vector* field as

$$\nabla^2 \underline{a} \;\equiv\; \underline{\nabla}\,(\underline{\nabla}\cdot\underline{a}) - \underline{\nabla}\times(\underline{\nabla}\times\underline{a})$$

We shall do this at the end the course.

A mnemonic for the Laplacian acting on a vector field is *GDMCC – Grad-Div Minus Curl-Curl*, pronounced g(i)d(u)mcc!

Identities (8), (9) and (10) are extremely important: they're used a lot in electromagnetism; fluid mechanics; elasticity theory and other field theories.

Example: Scalar fields which depend only on r .

In many physical examples the scalar field ϕ depends only on the length of the position vector $r = |\underline{r}|$, and we have

$$\nabla^2\,\phi(r) \;=\; \phi''(r) + \frac{2\phi'(r)}{r} \;=\; \frac{1}{r^2}\left(r^2\phi'(r)\right)'$$

where the prime denotes differentiation with respect to r. Proof of this relation utilises the chain rule

$$\underline{\nabla}\,\phi(r) \;=\; \frac{d\phi(r)}{dr}\,\underline{\nabla}\,r \;=\; \phi'(r)\,\frac{\underline{r}}{r}$$

and is left to the tutorial.

Example: Evaluate $\underline{\nabla}\times\{(\underline{c}\times\underline{r})\,/r^3\}$ where $\underline{c}$ is a constant vector.

Start with the product rule $\underline{\nabla}\times(\phi\,\underline{a}) \;=\; (\underline{\nabla}\,\phi)\times\underline{a} + \phi\,(\underline{\nabla}\times\underline{a})$ where $\phi = 1/r^3$ and $\underline{a} = \underline{c}\times\underline{r}$.

$$\begin{aligned}
\underline{\nabla}\times\left(\frac{\underline{c}\times\underline{r}}{r^3}\right) &= \underline{\nabla}\left(\frac{1}{r^3}\right)\times(\underline{c}\times\underline{r}) + \frac{1}{r^3}\,\underline{\nabla}\times(\underline{c}\times\underline{r}) \\
&= -3\,\frac{\underline{r}}{r^5}\times(\underline{c}\times\underline{r}) + \frac{1}{r^3}\,2\underline{c} \\
&= -3\,\frac{1}{r^5}\left(r^2\underline{c} - (\underline{r}\cdot\underline{c})\,\underline{r}\right) + \frac{2}{r^3}\,\underline{c} \;=\; \frac{3\,(\underline{r}\cdot\underline{c})\,\underline{r}}{r^5} - \frac{\underline{c}}{r^3}
\end{aligned}$$

where we used $\underline{\nabla}\,r^n = nr^{n-2}\underline{r}$ with $n = -3$, and $\underline{\nabla}\times(\underline{c}\times\underline{r}) = 2\underline{c}$

Physical example: The vector potential of an electric dipole has the form $\underline{A}(\underline{r}) = (\underline{c}\times\underline{r})\,/r^3$.

5.6 Summary

Elementary results: We can calculate *div*, *grad*, *curl* and the *Laplacian* of many of the scalar and vector fields that occur in physics using the following elementary results, which you must know and be able to derive:

$$\begin{array}{llll}
\underline{\nabla}\,r = \hat{\underline{r}} & \underline{\nabla}\,r^n = nr^{n-2}\underline{r} & \underline{\nabla}\cdot\underline{r} = 3 & \underline{\nabla}\times\underline{r} = 0 \\
\underline{\nabla}\,(\underline{a}\cdot\underline{r}) = \underline{a} & (\underline{a}\cdot\underline{\nabla}\,)\underline{r} = \underline{a} & \underline{\nabla}\times(\underline{a}\times\underline{r}) = 2\underline{a} & \underline{\nabla}\cdot(\underline{a}\times\underline{r}) = 0
\end{array}$$

where $\underline{r}$ is the position vector, $r = |\underline{r}|$ is its magnitude (length), and $\underline{a}$ is a *constant* vector.

The identity $(\underline{a}\cdot\underline{\nabla}\,)\underline{r} = \underline{a}$ holds also for vector *fields* $\underline{a}(\underline{r})$, because no derivatives act on $\underline{a}$.

Identities for scalar fields: You must know, and be able to derive, all of the following for scalar fields $\phi(\underline{r})$ and $\psi(\underline{r})$:

(i) Distributive law: $\underline{\nabla}\,(\phi+\psi) \;=\; \underline{\nabla}\,\phi + \underline{\nabla}\,\psi$

(ii) Product rule: $\underline{\nabla}\,(\phi\,\psi) \;=\; (\underline{\nabla}\,\phi)\,\psi + \phi\,(\underline{\nabla}\,\psi)$

(iii) Chain rule: If $F(\phi(\underline{r}))$ is a scalar field, then $\underline{\nabla}\,F(\phi) \;=\; \dfrac{dF(\phi)}{d\phi}\,\underline{\nabla}\,\phi(\underline{r})$

Important example: $\underline{\nabla}\,f(r) = (f'(r)/r)\,\underline{r}$

Identities for vector fields: For vector fields $\underline{a}(\underline{r})$ and $\underline{b}(\underline{r})$, and scalar fields $\phi(\underline{r})$:

1. $\underline{\nabla}\cdot(\underline{a}+\underline{b}) \;=\; \underline{\nabla}\cdot\underline{a} + \underline{\nabla}\cdot\underline{b}$
2. $\underline{\nabla}\times(\underline{a}+\underline{b}) \;=\; \underline{\nabla}\times\underline{a} + \underline{\nabla}\times\underline{b}$
3. $\underline{\nabla}\cdot(\phi\,\underline{a}) \;=\; (\underline{\nabla}\,\phi)\cdot\underline{a} \,+\, \phi\,(\underline{\nabla}\cdot\underline{a})$
4. $\underline{\nabla}\times(\phi\,\underline{a}) \;=\; (\underline{\nabla}\,\phi)\times\underline{a} \,+\, \phi\,(\underline{\nabla}\times\underline{a})$
5. $\underline{\nabla}\cdot(\underline{a}\times\underline{b}) \;=\; \underline{b}\cdot(\underline{\nabla}\times\underline{a}) - \underline{a}\cdot(\underline{\nabla}\times\underline{b})$
6. $\underline{\nabla}\times(\underline{a}\times\underline{b}) \;=\; (\underline{\nabla}\cdot\underline{b})\,\underline{a} + (\underline{b}\cdot\underline{\nabla}\,)\,\underline{a} - (\underline{\nabla}\cdot\underline{a})\,\underline{b} - (\underline{a}\cdot\underline{\nabla}\,)\,\underline{b}$
7. $\underline{\nabla}\,(\underline{a}\cdot\underline{b}) \;=\; (\underline{a}\cdot\underline{\nabla}\,)\,\underline{b} + (\underline{b}\cdot\underline{\nabla}\,)\,\underline{a} + \underline{a}\times(\underline{\nabla}\times\underline{b}) + \underline{b}\times(\underline{\nabla}\times\underline{a})$
8. $\underline{\nabla}\times(\underline{\nabla}\,\phi) \;=\; 0$ (*curl grad* ϕ is *always* zero)
9. $\underline{\nabla}\cdot(\underline{\nabla}\times\underline{a}) \;=\; 0$ (*div curl* $\underline{a}$ is *always* zero)
10. $\underline{\nabla}\times(\underline{\nabla}\times\underline{a}) \;=\; \underline{\nabla}\,(\underline{\nabla}\cdot\underline{a}) - \nabla^2\underline{a}$

You must *know* and be able to use & derive identities 1-4 and 8-9. You must be familiar with the others, and be able to use them in calculations.

6 Line integrals

Having completed our study of differential vector calculus using Cartesian coordinates, we now embark on integral vector calculus.

We begin by revising the standard definition of the integral of a function of a single variable. We then introduce line integrals, surface integrals and volume integrals. We shall assume familiarity with integrals over plane areas (double integrals) using Cartesian and plane polar coordinates, as covered in *Linear Algebra and Several Variable Calculus* or the School of Mathematics course *Several Variable Calculus and Differential Equations*.

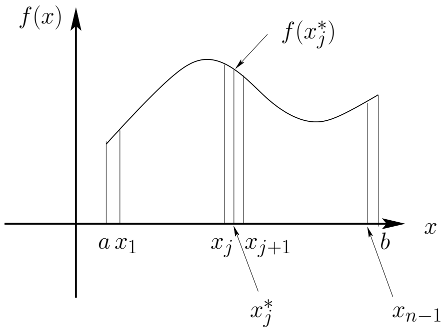
6.1 Revision of ordinary integrals

We start with some formal definitions and discuss some limits, but we shall not be rigorous.

Let $f(x)$ be a continuous real-valued function defined on the interval $a \le x \le b$. Begin by subdividing this interval into n subintervals:

$$a = x_0 < x_1 < x_2 \ldots < x_n = b\,.$$

In interval j, pick an arbitrary point x_j^* with $x_j \le x_j^* \le x_{j+1}$, as illustrated in the figure.



Define the Riemann sum

$$S_n = \sum_{j=0}^{n-1} f(x_j^*)\,(x_{j+1} - x_j)$$

It can be shown that $S_n \to$ *unique limit* as we let $n \to \infty$ and $(x_{j+1} - x_j) \to 0$ for all j, with $\sum_{j=0}^{n-1} (x_{j+1} - x_j) = (b - a)$ kept fixed. [9]

In this limit

$$S_n \to \int_a^b f(x)\,\mathrm{d}x$$

One can then prove the usual properties of integrals. Note that we also define

$$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$$

[9] We assume that $f(x)$ is sufficiently well behaved that the limit actually exists.

6.2 Motivation and formal definition of line integrals

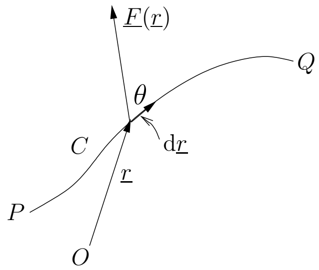
Motivation: As a physical example, consider a particle constrained to move along a wire under the influence of a force $\underline{F}(\underline{r})$.

Only the component of the force *along* the wire does work.

Thus the work, $\mathrm{d}W$, done by the force in moving the particle along the wire from $\underline{r}$ to $\underline{r} + \mathrm{d}\underline{r}$ is

$$\mathrm{d}W \;=\; (|\underline{F}|\cos\theta)\;|\mathrm{d}\underline{r}| \;=\; \underline{F}\cdot\mathrm{d}\underline{r}$$

$|\underline{F}|\cos\theta$ is the component of $\underline{F}$ along $\mathrm{d}\underline{r}$, where θ is the angle between $\underline{F}(\underline{r})$ and $\mathrm{d}\underline{r}$.



The *total* work done in moving the particle along a wire which follows some curve C from point P to point Q is the sum of the (infinitesimal) $\mathrm{d}W$ factors, which is an integral

$$W_C = \int_P^Q \mathrm{d}W = \int_C \underline{F}(\underline{r})\cdot\mathrm{d}\underline{r}$$

This is called a *line integral* along the curve C.

Formal definition: Let $\underline{a}(\underline{r})$ be a vector field defined in the region R, and let C be a curve in R between two points P and Q. Let $\underline{r}$ be the position vector at some point on the curve, and let $d\underline{r}$ be an infinitesimal vector *along* the curve at $\underline{r}$.

The *magnitude* of $\mathrm{d}\underline{r}$ is the infinitesimal *arc length*: $\mathrm{d}s = \sqrt{\mathrm{d}\underline{r}\cdot\mathrm{d}\underline{r}}$

If $\underline{t}$ is the *unit vector* tangent to the curve at $\underline{r}$ (*i.e.* $\underline{t}$ points in the direction of $\mathrm{d}\underline{r}$ at $\underline{r}$), then

$$\mathrm{d}\underline{r} \;=\; \underline{t}\,\mathrm{d}s$$

The line integral is defined formally as a Riemann sum by dividing the curve into n intervals

$$\int_C \underline{a}\cdot\mathrm{d}\underline{r} \;=\; \int_C \underline{a}\cdot\underline{t}\;\mathrm{d}s \;\equiv\; \lim_{\substack{\delta s^{(i)} \to 0 \\ n \to \infty}} \sum_{i=0}^{n-1} \left(\underline{a}\left(\underline{r}^{(i)}\right)\cdot\underline{t}^{(i)}\right)\delta s^{(i)}$$

the i^{th} interval having length $\delta s^{(i)}$, unit tangent vector $\underline{t}^{(i)}$, *etc.* It can be shown that the limit is unique for sufficiently smooth vector fields $\underline{a}(\underline{r})$.

The integrand $\underline{a}\cdot\underline{t}$ is the component of $\underline{a}$ along $\mathrm{d}\underline{r}$ at the point $\underline{r}$.

In Cartesian coordinates, we have

$$\int_C \underline{a}\cdot\mathrm{d}\underline{r} \;=\; \int_C (a_1\mathrm{d}x_1 + a_2\mathrm{d}x_2 + a_3\mathrm{d}x_3) \;=\; \int_C \sum_{i=1}^{3} a_i\,\mathrm{d}x_i \qquad (13)$$

NB: In general, *the line integral depends on the path joining P and Q*. For example, the a_1 component is $a_1(x_1, x_2, x_3)$ and all three coordinates will generally change at once along the path. Therefore, you can *not* compute $\int a_1\,\mathrm{d}x_1$ as an ordinary integral over x_1 holding x_2 and x_3 constant. That would only be correct if x_2 and x_3 were *constant* along the path, *i.e.* if the path were parallel to the x_1 axis. This is a common source of mistakes.

6.3 Parametric representation of line integrals

The definition above was rather formal. What follows is much more useful.

A smooth curve in 3D can be parameterised by a single parameter. For example, if the curve is the trajectory of a particle, then a natural parameter is the time t. Sometimes the parameter is chosen to be the arc-length s along the curve C.

If we parameterise the curve by the parameter λ (varying from λ_P to λ_Q), we can write the coordinates x_i as functions of λ

$$x_1 = x_1(\lambda)\,, \quad x_2 = x_2(\lambda)\,, \quad x_3 = x_3(\lambda)\,, \quad \text{with } \lambda_P \le \lambda \le \lambda_Q$$

and

$$\int_C \underline{a}\cdot \mathrm{d}\underline{r} \;=\; \int_{\lambda_P}^{\lambda_Q}\left(\underline{a}\cdot\frac{\mathrm{d}\underline{r}}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda \;=\; \int_{\lambda_P}^{\lambda_Q}\left(a_1\frac{\mathrm{d}x_1}{\mathrm{d}\lambda}+a_2\frac{\mathrm{d}x_2}{\mathrm{d}\lambda}+a_3\frac{\mathrm{d}x_3}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda$$

If necessary, the curve C may be subdivided into sections, each with a different parameterisation (piecewise smooth curve).

Example: Let $\underline{a}(\underline{r}) = -ky\,\underline{e}_x + kx\,\underline{e}_y$, where k is a positive constant.

Evaluate $\int_C \underline{a}\cdot\mathrm{d}\underline{r}$ between the points with Cartesian coordinates $(1,0,0)$ and $(0,1,0)$ along the curve C: $(x=\cos\lambda,\ y=\sin\lambda,\ z=0)$, where $0\le\lambda\le\pi/2$.

Convince yourself that C is one quarter of a unit circle. Sketch the curve C and the field $\underline{a}(\underline{r})$ on C.

On the curve C, we have

$$\begin{aligned}
\underline{a}(\underline{r}) &= -ky\,\underline{e}_x + kx\,\underline{e}_y \;=\; -k\sin\lambda\,\underline{e}_x + k\cos\lambda\,\underline{e}_y\\
\underline{r} &= \cos\lambda\,\underline{e}_x + \sin\lambda\,\underline{e}_y\\
\frac{\mathrm{d}\underline{r}}{\mathrm{d}\lambda} &= \left(-\sin\lambda\,\underline{e}_x + \cos\lambda\,\underline{e}_y\right)
\end{aligned}$$

Therefore

$$\int_C \underline{a}\cdot\mathrm{d}\underline{r} \;=\; \int_0^{\pi/2}\left(\underline{a}\cdot\frac{\mathrm{d}\underline{r}}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda \;=\; \int_0^{\pi/2} k\left(\sin^2\lambda+\cos^2\lambda\right)\mathrm{d}\lambda \;=\; \int_0^{\pi/2} k\,\mathrm{d}\lambda \;=\; \frac{k\pi}{2}$$

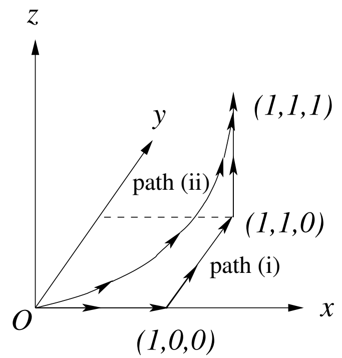
In this example, the result is simple because the field $\underline{a}$ is parallel to $\dfrac{\mathrm{d}\underline{r}}{\mathrm{d}\lambda}$ and hence to $\mathrm{d}\underline{r}$.

Example: Let $\underline{a}(\underline{r}) = (3x^2+6y)\,\underline{e}_x - 14yz\,\underline{e}_y + 20xz^2\,\underline{e}_z$.

Evaluate $\int_C \underline{a}\cdot\mathrm{d}\underline{r}$ between the points with Cartesian coordinates $(0,0,0)$ and $(1,1,1)$, along the two paths C:

(i) $(0,0,0)\to(1,0,0)\to(1,1,0)\to(1,1,1)$
(These are 3 contiguous straight lines parallel to the x, y & z axes respectively.)

(ii) $(x=\lambda,\ y=\lambda^2,\ z=\lambda^3)$ from $\lambda=0$ to $\lambda=1$.



(i) (a) Along the line from $(0,0,0)$ to $(1,0,0)$, we have $y=z=0$, so $\mathrm{d}y=\mathrm{d}z=0$, hence $\mathrm{d}\underline{r}=\underline{e}_x\,\mathrm{d}x$ and $\underline{a}=3x^2\,\underline{e}_x$ (here the parameter is just x itself), and

$$\int_{(0,0,0)}^{(1,0,0)}\underline{a}\cdot\mathrm{d}\underline{r} \;=\; \int_{x=0}^{x=1}3x^2\,\mathrm{d}x \;=\; \left[x^3\right]_0^1 \;=\; 1$$

(b) Along the line from $(1,0,0)$ to $(1,1,0)$, we have $x=1$, $\mathrm{d}x=0$, $z=\mathrm{d}z=0$, so $\mathrm{d}\underline{r}=\underline{e}_y\,\mathrm{d}y$ (here the parameter is y), and

$$\underline{a} \;=\; \left(3x^2+6y\right)\Big|_{x=1}\underline{e}_x \;=\; (3+6y)\,\underline{e}_x$$

$$\Rightarrow\quad \int_{(1,0,0)}^{(1,1,0)}\underline{a}\cdot\mathrm{d}\underline{r} \;=\; \int_{y=0}^{y=1}(3+6y)\ \underline{e}_x\cdot\underline{e}_y\ \mathrm{d}y \;=\; 0$$

(c) Along the line from $(1,1,0)$ to $(1,1,1)$, we have $x=y=1$, $\mathrm{d}x=\mathrm{d}y=0$, and hence $\mathrm{d}\underline{r}=\underline{e}_z\,\mathrm{d}z$ and $\underline{a}=9\,\underline{e}_x-14z\,\underline{e}_y+20z^2\,\underline{e}_z$. Therefore

$$\int_{(1,1,0)}^{(1,1,1)}\underline{a}\cdot\mathrm{d}\underline{r} \;=\; \int_{z=0}^{z=1}20z^2\,\mathrm{d}z \;=\; \left[\frac{20}{3}z^3\right]_0^1 \;=\; \frac{20}{3}$$

Adding up the 3 contributions, we get

$$\int_C \underline{a}\cdot\mathrm{d}\underline{r} \;=\; 1+0+\frac{20}{3} \;=\; \frac{23}{3} \qquad \text{along path (i)}$$

(ii) To integrate $\underline{a}=(3x^2+6y)\,\underline{e}_x-14yz\underline{e}_y+20xz^2\underline{e}_z$ along path (ii), we parameterise

$$\begin{aligned}
\underline{r} &= \lambda\,\underline{e}_x+\lambda^2\,\underline{e}_y+\lambda^3\,\underline{e}_z\\
\frac{\mathrm{d}\underline{r}}{\mathrm{d}\lambda} &= \underline{e}_x+2\lambda\,\underline{e}_y+3\lambda^2\,\underline{e}_z\\
\underline{a} &= \left(3\lambda^2+6\lambda^2\right)\underline{e}_x-14\lambda^5\,\underline{e}_y+20\lambda^7\,\underline{e}_z \quad \text{so that}
\end{aligned}$$

$$\int_C\left(\underline{a}\cdot\frac{\mathrm{d}\underline{r}}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda \;=\; \int_{\lambda=0}^{\lambda=1}\left(9\lambda^2-28\lambda^6+60\lambda^9\right)\mathrm{d}\lambda \;=\; \left[3\lambda^3-4\lambda^7+6\lambda^{10}\right]_0^1 \;=\; 5$$

Hence $\displaystyle\int_C \underline{a}\cdot\mathrm{d}\underline{r} \;=\; 5 \qquad \text{along path (ii)}$

In this case, the integral of $\underline{a}$ from $(0,0,0)$ to $(1,1,1)$ depends on the path taken.

Notes:

(i) The line integral $\int_C \underline{a}\cdot \mathrm{d}\underline{r}$ is a *scalar* quantity. Another *scalar* line integral is $\int_C f\,\mathrm{d}s$ where $f(\underline{r})$ is a scalar field and $\mathrm{d}s = \sqrt{\mathrm{d}\underline{r}\cdot\mathrm{d}\underline{r}}$ is the infinitesimal arc-length introduced earlier.

(ii) A line integral around a *simple* (doesn't intersect itself) *closed* curve C is denoted by the symbol $\oint_C$

$$\text{Example:}\quad \oint_C \underline{a}\cdot\mathrm{d}\underline{r} \;\equiv\; \text{the } \textit{circulation} \text{ of } \underline{a} \text{ around } C$$

Example: Let $f(\underline{r}) = ax^2 + by^2$. Evaluate $\oint_C f\,\mathrm{d}s$ around the unit circle C centred on the origin in the $x{-}y$ plane:

$$x = \cos\phi,\; y = \sin\phi,\; z = 0; \quad 0 \le \phi \le 2\pi.$$

On the curve C:

$$\begin{aligned}
f(\underline{r}) &= ax^2 + by^2 \;=\; a\cos^2\phi + b\sin^2\phi \\
\underline{r} &= \cos\phi\,\underline{e}_x + \sin\phi\,\underline{e}_y \\
\mathrm{d}\underline{r} &= \left(-\sin\phi\,\underline{e}_x + \cos\phi\,\underline{e}_y\right)\mathrm{d}\phi \\
\Rightarrow\quad \mathrm{d}s &= \sqrt{\mathrm{d}\underline{r}\cdot\mathrm{d}\underline{r}} \;=\; \left(\sin^2\phi+\cos^2\phi\right)^{1/2}\mathrm{d}\phi \;=\; \mathrm{d}\phi
\end{aligned}$$

Therefore, in this example,

$$\oint_C f\,\mathrm{d}s \;=\; \int_0^{2\pi}\left(a\cos^2\phi + b\sin^2\phi\right)\mathrm{d}\phi \;=\; \pi\,(a+b)$$

The *length* s of a curve C is given by $s = \int_C \mathrm{d}s$. In this example $s = \int_0^{2\pi}\mathrm{d}\phi = 2\pi$.

We can also define *vector* line integrals, whose *result* is a vector:

(i) $\int_C \underline{a}\,\mathrm{d}s \;=\; \int_C (\underline{e}_1\,a_1 + \underline{e}_2\,a_2 + \underline{e}_3\,a_3)\,\mathrm{d}s$ in Cartesian coordinates = a vector.

(ii) $\int_C \underline{a}\times\mathrm{d}\underline{r} \;=\; \int_C [\,\underline{e}_1\,(a_2\,\mathrm{d}x_3 - a_3\,\mathrm{d}x_2) + \underline{e}_2\,(a_3\,\mathrm{d}x_1 - a_1\,\mathrm{d}x_3) + \underline{e}_3\,(a_1\,\mathrm{d}x_2 - a_2\,\mathrm{d}x_1)\,]$

in Cartesian coordinates. The parametric form is simply $\int_C \underline{a}\times\mathrm{d}\underline{r} \;=\; \int_C\left(\underline{a}\times\frac{\mathrm{d}\underline{r}}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda$

(iii) $\int_C f\,\mathrm{d}\underline{r} \;=\; \int_C f\;(\underline{e}_1\,\mathrm{d}x_1 + \underline{e}_2\,\mathrm{d}x_2 + \underline{e}_3\,\mathrm{d}x_3)$ in Cartesian coordinates. In parametric form, this becomes $\int_C f\,\mathrm{d}\underline{r} \;=\; \int_C f\,\frac{\mathrm{d}\underline{r}}{\mathrm{d}\lambda}\,\mathrm{d}\lambda$

6.4 Current loop in a magnetic field

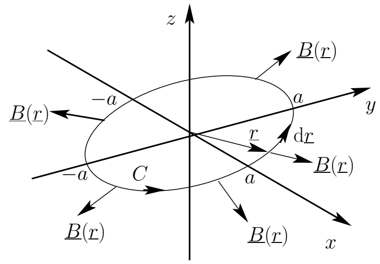
Consider an electric current of magnitude I flowing along a thin wire in the shape of a closed path C.

The magnetic force on an element $\mathrm{d}\underline{r}$ of the wire at $\underline{r}$ due to an external magnetic field $\underline{B}(\underline{r})$ is given by the Lorentz force

$$\mathrm{d}\underline{F}(\underline{r}) \;=\; I\,\mathrm{d}\underline{r}\times\underline{B}(\underline{r})\,.$$

The *total* force $\underline{F}$ on the wire is the vector sum of the forces on the individual elements, which is given by the line integral of $\mathrm{d}\underline{F}$ around the closed curve C.

$$\underline{F} \;=\; \oint_C \mathrm{d}\underline{F} \;=\; \oint_C I\,\mathrm{d}\underline{r}\times\underline{B}(\underline{r}) \;=\; -I\oint_C \underline{B}(\underline{r})\times\mathrm{d}\underline{r}$$



Example: For the case where the external magnetic field is $\underline{B}(\underline{r}) = B_0(x\,\underline{e}_x + y\,\underline{e}_y)$, evaluate the total force on a circular current loop of radius a which lies in the $x{-}y$ plane and is centred on the origin.

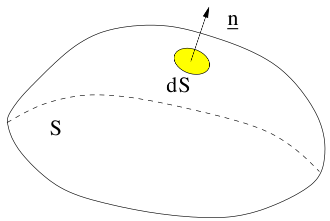
We parameterise the curve by the angle ϕ (as in plane polars), so that on the curve C, we have

$$\begin{aligned}
\underline{r} &= a\cos\phi\,\underline{e}_x + a\sin\phi\,\underline{e}_y \\
\mathrm{d}\underline{r} &= (-\,a\sin\phi\,\underline{e}_x + a\cos\phi\,\underline{e}_y)\,\mathrm{d}\phi \\
\underline{B} &= B_0(a\cos\phi\,\underline{e}_x + a\sin\phi\,\underline{e}_y)
\end{aligned}$$

$$\Rightarrow\quad \oint_C \underline{B}\times\mathrm{d}\underline{r} \;=\; B_0\int_0^{2\pi}(a^2\cos^2\phi + a^2\sin^2\phi)\,\underline{e}_z\,\mathrm{d}\phi \;=\; B_0\,\underline{e}_z\,a^2\int_0^{2\pi}\mathrm{d}\phi \;=\; 2\pi a^2 B_0\,\underline{e}_z$$

$$\text{So}\quad \underline{F} \;=\; -2\pi a^2 B_0 I\,\underline{e}_z \quad \text{which is in a vertically downward direction.}$$

7 Surface integrals



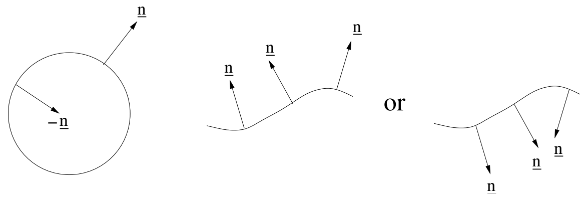
Let S be a two-sided surface in three-dimensional space as shown. If an infinitesimal element of surface with (scalar) area $\mathrm{d}S$ has unit normal $\underline{n}$, then the infinitesimal *vector element of area* is *defined* by

$$\boxed{\mathrm{d}\underline{S} \;=\; \underline{n}\,\mathrm{d}S}$$

Example: If S lies in the (x, y) plane, then in Cartesian coordinates the infinitesimal *scalar* element of area is $\mathrm{d}S = \mathrm{d}x\,\mathrm{d}y$, and the infinitesimal *vector* element of area is $\mathrm{d}\underline{S} = \underline{e}_z\,\mathrm{d}x\,\mathrm{d}y$.

Geometrical interpretation: $\underline{m} \cdot \mathrm{d}\underline{S}$ gives the projected (scalar) element of area onto the plane with unit normal $\underline{m}$. See later for more details.

For *closed* surfaces (*e.g.* a sphere) we always *choose* $\underline{n}$ to be the *outward* normal. For *open* surfaces (*e.g.* the curved surface of a hemisphere), the sense of $\underline{n}$ is arbitrary – except that it is chosen in the *same* sense for *all* elements of the surface.



If $\underline{a}(\underline{r})$ is a vector field defined on the surface S, we define the (normal) *surface integral* formally by the Riemann sum

$$\int_S \underline{a} \cdot \mathrm{d}\underline{S} \;=\; \int_S \underline{a} \cdot \underline{n}\, \mathrm{d}S \;=\; \lim_{\substack{m\to\infty \\ \delta S^{(i)}\to 0}} \sum_{i=0}^{m-1} \left(\underline{a}\left(\underline{r}^{(i)}\right) \cdot \underline{n}^{(i)} \right) \delta S^{(i)}$$

where we divided the surface S into m small areas, the i^{th} area having vector area $\delta \underline{S}^{(i)}$.

The quantity $\underline{a}(\underline{r}^{(i)}) \cdot \underline{n}^{(i)}$ is the component of $\underline{a}$ *normal* to the surface at the point $\underline{r}^{(i)}$. This is similar to the definition of planar integrals given in *Linear Algebra and Several Variable Calculus*, except that the (scalar) area elements $\delta S^{(i)}$ are not constrained to lie in a plane.

- We use the notation $\int_S \underline{a} \cdot \mathrm{d}\underline{S}$ for both *open* and *closed* surfaces. Sometimes the integral over a *closed* surface is denoted by $\oint_S \underline{a} \cdot \mathrm{d}\underline{S}$ (*not* used here).
- The integral over S is an ordinary *double integral* in each case.[10]

Example: Let S be the surface of a unit cube. Note that S is the sum of *all six faces* of the cube.

On the *front* face, parallel to the (y, z) plane, at $x = 1$,

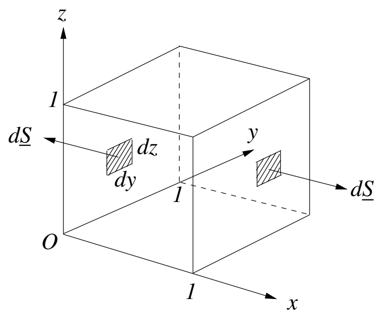
$$\mathrm{d}\underline{S} \;=\; \underline{n}\, \mathrm{d}S \;=\; \underline{e}_x\, \mathrm{d}y\, \mathrm{d}z$$

On the *back* face at $x = 0$ in the (y, z) plane,

$$\mathrm{d}\underline{S} \;=\; \underline{n}\, \mathrm{d}S \;=\; -\,\underline{e}_x\, \mathrm{d}y\, \mathrm{d}z$$

Similarly for the other four faces.

In each case, the unit normal $\underline{n}$ is an *outward* normal because S is a *closed* surface.



[10] Surface integrals are sometimes denoted by $\iint_S \underline{a} \cdot \mathrm{d}\underline{S}$ (*not* used here).

If $\underline{a}(\underline{r})$ is a vector field, the integral $\int_S \underline{a} \cdot \mathrm{d}\underline{S}$ over the front face, where $\mathrm{d}\underline{S} = \underline{e}_x\, \mathrm{d}y\, \mathrm{d}z$, is

$$\int_{z=0}^{z=1}\int_{y=0}^{y=1} \underline{a} \cdot \left(\underline{e}_x\, \mathrm{d}y\, \mathrm{d}z\right) \;=\; \int_{z=0}^{z=1} \mathrm{d}z \int_{y=0}^{y=1} \underline{a} \cdot \underline{e}_x\, \mathrm{d}y \;=\; \int_{z=0}^{z=1} \mathrm{d}z \int_{y=0}^{y=1} a_x\Big|_{x=1}\, \mathrm{d}y$$

where $a_x|_{x=1} \equiv a_x(1, y, z)$. The integral over y and z is an ordinary double integral over a square of side 1.

Similarly, the integral over the back face, where $\mathrm{d}\underline{S} = -\underline{e}_x\, \mathrm{d}y\, \mathrm{d}z$ is

$$\int_{z=0}^{z=1}\int_{y=0}^{y=1} \underline{a} \cdot \left(-\underline{e}_x\, \mathrm{d}y\, \mathrm{d}z\right) \;=\; -\int_{z=0}^{z=1} \mathrm{d}z \int_{y=0}^{y=1} \underline{a} \cdot \underline{e}_x\, \mathrm{d}y \;=\; -\int_{z=0}^{z=1} \mathrm{d}z \int_{y=0}^{y=1} a_x\Big|_{x=0}\, dy$$

where $a_x|_{x=0} \equiv a_x(0, y, z)$. Again, this is an ordinary double integral over a square of side 1.

The total integral over S is the sum of integrals over all 6 faces of the cube. See Tutorial 6 for an explicit example.

7.1 Parametric form of the surface integral

We now introduce a parametric representation for surface integrals.[11]

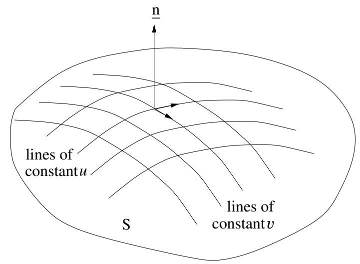
Suppose the points on a surface S can be specified by two real parameters u and v, so that the position vector on the surface may be written as

$$\underline{r} \;=\; \underline{r}(u, v) \;=\; x(u, v)\, \underline{e}_x \,+\, y(u, v)\, \underline{e}_y \,+\, z(u, v)\, \underline{e}_z$$

Then

- the lines $\underline{r}(u, v)$ for fixed u, variable v, and
- the lines $\underline{r}(u, v)$ for fixed v, variable u

are *parametric lines* and form a *grid* on the surface S as shown.



If we change u by $\mathrm{d}u$, and v by $\mathrm{d}v$, then $\underline{r}$ changes by $\mathrm{d}\underline{r}$, where the infinitesimal vector $\mathrm{d}\underline{r}$ lies in the surface, and is given by

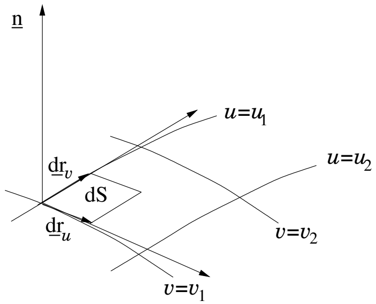
$$\mathrm{d}\underline{r} \;=\; \frac{\partial \underline{r}}{\partial u}\, \mathrm{d}u + \frac{\partial \underline{r}}{\partial v}\, \mathrm{d}v$$

[11] This is the extension to 3D of the representation used for planar integrals in *LA&SVC* Section (18.4).

Along the curves $v = constant$, we have $\mathrm{d}v = 0$, so $\mathrm{d}\underline{r}$ is just

$$\mathrm{d}\underline{r}_u \;=\; \frac{\partial \underline{r}}{\partial u}\,\mathrm{d}u$$

The vector $\partial\underline{r}/\partial u$ is tangent to the surface at $\underline{r}$, and tangent to the lines $v = constant$.



Similarly, for $u = constant$, we have

$$\mathrm{d}\underline{r}_v \;=\; \frac{\partial \underline{r}}{\partial v}\,\mathrm{d}v$$

so $\partial\underline{r}/\partial v$ is tangent to the surface at $\underline{r}$, and tangent to the lines $u = constant$, as shown in the figure.

The infinitesimal vectors $\mathrm{d}\underline{r}_u$ and $\mathrm{d}\underline{r}_v$ lie *in the surface* at $\underline{r}$.

The vectors $\dfrac{\partial \underline{r}}{\partial u}$ and $\dfrac{\partial \underline{r}}{\partial v}$ lie in the *tangent plane* to the surface at $\underline{r}$.

We can construct a *unit vector* $\underline{n}$, *normal* to the surface at $\underline{r}$

$$\underline{n} \;=\; \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right) \Big/ \left|\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right|$$

Since the vector element of area, $\mathrm{d}\underline{S}$, has magnitude equal to the area of the infinitesimal parallelogram shown in the figure above, and it points in the direction of $\underline{n}$, we can write

$$\mathrm{d}\underline{S} \;=\; \mathrm{d}\underline{r}_u \times \mathrm{d}\underline{r}_v \;=\; \left(\frac{\partial \underline{r}}{\partial u}\,\mathrm{d}u\right) \times \left(\frac{\partial \underline{r}}{\partial v}\,\mathrm{d}v\right) \;=\; \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v$$

$$\boxed{\mathrm{d}\underline{S} = \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v}$$

Finally, our integral is parameterised as

$$\boxed{\int_S \underline{a}\cdot \mathrm{d}\underline{S} \;=\; \int_v\!\int_u \underline{a}\cdot\left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v}$$

We use two integral signs when writing surface integrals in terms of *explicit* parameters u and v. The limits for the integrals over u and v must be chosen appropriately for the surface.

This was a little abstract and a little complicated...

Fortunately, in most practical cases we dont need the detailed form of this expression, since we tend to use orthogonal curvilinear coordinates, where the vectors $\partial\underline{r}/\partial u$ and $\partial\underline{r}/\partial v$ are perpendicular to each other. It is normally clear from the geometry of the situation whether this is the case, as it is for cylindrical coordinates and spherical polar coordinates - as we shall see.

8 Curvilinear coordinates, flux and surface integrals

8.1 Curvilinear coordinates

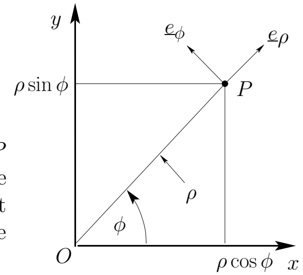
It is often convenient to work with coordinate systems other than the Cartesian coordinates $(x_1,\, x_2,\, x_3)$ or $(x,\, y,\, z)$.

8.1.1 Plane polar coordinates

We begin by revising a familiar example in the $x{-}y$ plane.

Define two new variables ρ, ϕ as

$$\rho = \sqrt{x^2+y^2} \quad\text{and}\quad \tan\phi = \frac{y}{x}$$



with $0 \le \rho \le \infty$ and $0 \le \phi < 2\pi$.

Clearly, ρ is the distance from the origin O to the point P which has Cartesian coordinates (x, y), and ϕ is the angle between the x-axis and the line from the origin to the point P (measured in an anti-clockwise direction from the positive x-axis.)

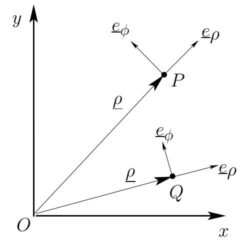
The inverse transformation is clearly

$$x = \rho\cos\phi \quad\text{and}\quad y = \rho\sin\phi$$

The variables ρ and ϕ are *plane polar coordinates.* Any point in the plane can be specified by its Cartesian coordinates (x, y), *or* by its plane polar coordinates (ρ, ϕ). Clearly, $\rho = 0$ at the origin, but ϕ is *undefined* there.

Basis vectors: Thus far, we have used only Cartesian basis vectors $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$, also known as $\{\underline{e}_x, \underline{e}_y, \underline{e}_z\}$. These point in the *same* direction *everywhere* in space.

Restricting ourselves to vectors in the $(x{-}y)$ plane for now, we can define two new orthonormal basis vectors in the plane.



Define the unit vector $\underline{e}_\rho$ *parallel* to the position vector $\overrightarrow{OP} \equiv \underline{\rho}$.

Then define the unit vector $\underline{e}_\phi$ *orthogonal* to $\underline{e}_\rho$ and pointing in the direction of *increasing* ϕ.

Note that $\underline{e}_\rho$ and $\underline{e}_\phi$ point in *different* directions at different points in the plane, as shown in the figure for two points P and Q.

Since $\underline{\rho} = x\,\underline{e}_x + y\,\underline{e}_y = \rho\left(\cos\phi\,\underline{e}_x + \sin\phi\,\underline{e}_y\right)$ then

$$\underline{e}_\rho \;=\; \cos\phi\,\underline{e}_x + \sin\phi\,\underline{e}_y \quad\text{and}\quad \underline{e}_\phi \;=\; -\sin\phi\,\underline{e}_x + \cos\phi\,\underline{e}_y \quad (14)$$

Components of vectors in plane polars: Any vector $\underline{a}$ which lies in the $x-y$ plane may be expressed in the original Cartesian basis $\{\underline{e}_x,\ \underline{e}_y\}$, or in the plane polar basis $\{\underline{e}_\rho,\ \underline{e}_\phi\}$

$$\underline{a} \;=\; a_x\,\underline{e}_x \;+\; a_y\,\underline{e}_y \;=\; a_\rho\,\underline{e}_\rho \;+\; a_\phi\,\underline{e}_\phi \tag{15}$$

The quantities a_ρ and a_ϕ are the *components* of the vector $\underline{a}$ in the basis $\{\underline{e}_\rho,\ \underline{e}_\phi\}$.

Example: The position vector of a point in the $x-y$ plane (which we call $\underline{\rho}$ to distinguish it from the position vector $\underline{r}$ in 3-d) is

$$\underline{\rho} \;=\; x\,\underline{e}_x + y\,\underline{e}_y \;=\; \rho\,\cos\phi\;\underline{e}_x \;+\; \rho\,\sin\phi\;\underline{e}_y \;=\; \rho\,\underline{e}_\rho \tag{16}$$

NB: $\underline{\rho} \neq \rho\,\underline{e}_\rho + \phi\,\underline{e}_\phi$. This is a common mistake and it's very important not to make it! *By definition*, $\underline{\rho}$ has no component in the direction of $\underline{e}_\phi$, as can be seen in the figure.

8.1.2 Cylindrical coordinates

This is the simplest extension of plane polars to three dimensions.[12] The usual plane polar coordinates ρ and ϕ replace the x and y coordinates, but the Cartesian z coordinate is retained.

The cylindrical coordinates (ρ, ϕ, z) of a point are therefore related to its Cartesian coordinates (x, y, z) by

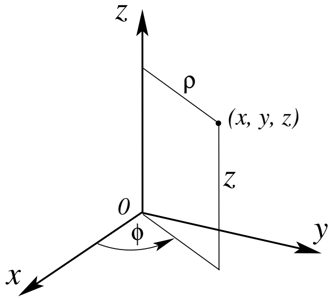
$$x = \rho\cos\phi\,, \quad y = \rho\sin\phi\,, \quad z = z$$

where $0 \le \rho < \infty\,,\ \ 0 \le \phi < 2\pi\,,$ and $-\infty < z < \infty\,.$

The position vector of the point with Cartesian coordinates (x, y, z) is

$$\underline{r} \;=\; \rho\cos\phi\;\underline{e}_x + \rho\sin\phi\;\underline{e}_y + z\,\underline{e}_z$$

where $\{\underline{e}_x,\ \underline{e}_y,\ \underline{e}_z\}$ are the usual Cartesian basis vectors.



The basis vectors for cylindrical coordinates are the plane polar basis vectors, plus the third Cartesian basis vector

$$\underline{e}_\rho \;=\; \cos\phi\;\underline{e}_x + \sin\phi\;\underline{e}_y\,, \quad \underline{e}_\phi \;=\; -\sin\phi\;\underline{e}_x + \cos\phi\;\underline{e}_y\,, \quad \underline{e}_z \;=\; \underline{e}_z$$

In this basis, the position vector for the point with cylindrical coordinates (ρ, ϕ, z) is[13]

$$\underline{r} \;=\; \rho\,\underline{e}_\rho + z\,\underline{e}_z$$

A general vector $\underline{a}$ has components $(a_x,\ a_y,\ a_z)$ in the Cartesian basis, and components $(a_\rho,\ a_\phi,\ a_z)$ in the cylindrical basis, so that

$$\underline{a} \;=\; a_x\,\underline{e}_x + a_y\,\underline{e}_y + a_z\,\underline{e}_z \;=\; a_\rho\,\underline{e}_\rho + a_\phi\,\underline{e}_\phi + a_z\,\underline{e}_z$$

Cylindrical coordinates are useful for problems with cylindrical symmetry, and also for problems involving cones and paraboloids.

[12] Cylindrical coordinates are also known as *circular cylindrical coordinates* or *cylindrical polar coordinates*.

[13] Note that $\underline{r} \neq \rho\,\underline{e}_\rho + \phi\,\underline{e}_\phi + z\,\underline{e}_z$

8.1.3 Spherical polar coordinates

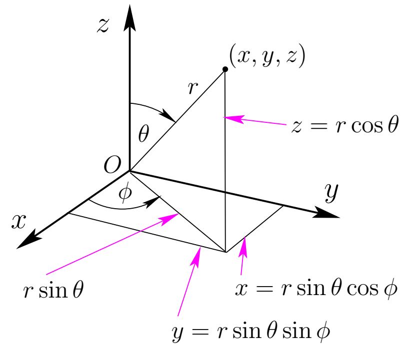
The spherical polar coordinates (r, θ, ϕ) of a point with Cartesian coordinates (x, y, z) are defined by

$$x \;=\; r\sin\theta\,\cos\phi\,, \quad y \;=\; r\sin\theta\,\sin\phi\,, \quad z \;=\; r\cos\theta$$

with $0 \le r < \infty\,, \quad 0 \le \theta \le \pi\,, \quad 0 \le \phi < 2\pi\,.$

The *azimuthal angle* ϕ runs from 0 to 2π, but since $\theta = \pi$ describes a point on the negative z axis, the *polar angle* θ runs from 0 to π only, so that θ and ϕ are specified uniquely.

Note that ϕ is not defined anywhere on the z axis, and θ is not defined at the origin.



We may write the position vector as

$$\underline{r} \;=\; r\sin\theta\cos\phi\;\underline{e}_1 \;+\; r\sin\theta\sin\phi\;\underline{e}_2 \;+\; r\cos\theta\;\underline{e}_3 \;=\; r\,\underline{e}_r$$

$$(\;=\; r\sin\theta\,(\cos\phi\;\underline{e}_1 \;+\; \sin\phi\;\underline{e}_2) \;+\; r\cos\theta\;\underline{e}_3\;)$$

where $\{\underline{e}_1,\ \underline{e}_2,\ \underline{e}_3\}$ are the usual Cartesian basis vectors, and $\underline{e}_r$ is a unit vector in the direction of $\underline{r}$,

$$\underline{e}_r = \sin\theta\cos\phi\;\underline{e}_1 \;+\; \sin\theta\sin\phi\;\underline{e}_2 \;+\; \cos\theta\;\underline{e}_3$$

Unit vectors $\underline{e}_\theta$ and $\underline{e}_\phi$ will be derived later.

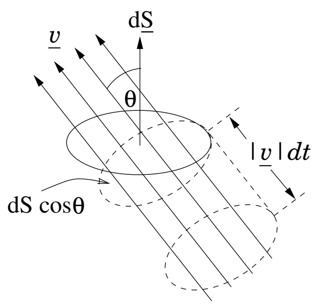
We may invert the expressions for (x, y, z) in spherical polars $(r,\ \theta,\ \phi)$ to obtain

$$r \;=\; \sqrt{x^2 + y^2 + z^2}\,, \quad \theta \;=\; \cos^{-1}\left\{\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right\}, \quad \phi \;=\; \tan^{-1}\left(\frac{y}{x}\right)\,.$$

NB with our conventions, the angle ϕ is the *same* in each of plane polar coordinates, cylindrical coordinates and spherical polar coordinates, and r is always the length of the position vector in three dimensions. Beware of other conventions!

Note that $r\sin\theta$ in spherical polars is equal to the coordinate ρ in cylindrical coordinates.

8.2 Flux of a vector field through a surface



Let $\underline{v}(\underline{r})$ be the velocity at a point $\underline{r}$ in a moving fluid.

In a small region, where $\underline{v}$ is approximately constant, the *volume* of fluid crossing the element of vector area $\mathrm{d}\underline{S} = \underline{n}\,\mathrm{d}S$ in time $\mathrm{d}t$ is

$$(|\underline{v}|\ \mathrm{d}t)\ (\mathrm{d}S\ \cos\theta) \;=\; (\underline{v}\cdot\mathrm{d}\underline{S})\ \mathrm{d}t$$

This is just the distance, $|\underline{v}|\,\mathrm{d}t$, travelled by the fluid, multiplied by the scalar area *normal* to the direction of motion, that it flows through.

This scalar area is $\mathrm{d}S\cos\theta = \hat{\underline{v}}\cdot\mathrm{d}\underline{S}$, where $\hat{\underline{v}}$ is a unit vector in the direction of $\underline{v}$.

Therefore $\quad \underline{v}\cdot \mathrm{d}\underline{S} \;=\;$ *volume per unit time* of fluid crossing $\mathrm{d}\underline{S}$

$\Rightarrow \quad \int_S \underline{v}\cdot \mathrm{d}\underline{S} \;=\;$ *volume per unit time* of fluid crossing a finite surface S

More generally, for a vector field $\underline{a}(\underline{r})$,

$$\boxed{\text{The surface integral } \int_S \underline{a}\cdot \mathrm{d}\underline{S} \text{ is called the \textit{flux} of } \underline{a} \text{ through the surface } S}$$

The concept of flux is useful in many different contexts *e.g.* flux of molecules in a flow of gas; electric or magnetic flux through a surface, *etc.*

Example: Let S be the surface of the sphere $x_1^2 + x_2^2 + x_3^2 = R^2$, with radius R. Find the unit normal $\underline{n}$, the vector element of area $\mathrm{d}\underline{S}$, and evaluate the total flux of the vector field $\underline{a}(\underline{r}) = \underline{r}/r^3$ out of the sphere.

An arbitrary point $\underline{r}$ on the surface S may be parameterised using the spherical polar coordinates θ and ϕ as

$$\begin{aligned}
\underline{r} &= R\sin\theta\cos\phi\,\underline{e}_1 + R\sin\theta\sin\phi\,\underline{e}_2 + R\cos\theta\,\underline{e}_3 \qquad \{0\le\theta\le\pi,\ 0\le\phi<2\pi\}\\
\text{so}\quad \frac{\partial\underline{r}}{\partial\theta} &= R\cos\theta\cos\phi\,\underline{e}_1 + R\cos\theta\sin\phi\,\underline{e}_2 - R\sin\theta\,\underline{e}_3\\
\text{and}\quad \frac{\partial\underline{r}}{\partial\phi} &= -R\sin\theta\sin\phi\,\underline{e}_1 + R\sin\theta\cos\phi\,\underline{e}_2 + 0\,\underline{e}_3
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial\underline{r}}{\partial\theta}\times\frac{\partial\underline{r}}{\partial\phi} &= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3\\ R\cos\theta\cos\phi & R\cos\theta\sin\phi & -R\sin\theta\\ -R\sin\theta\sin\phi & R\sin\theta\cos\phi & 0\end{vmatrix}\\
&= R^2\sin^2\theta\cos\phi\,\underline{e}_1 + R^2\sin^2\theta\sin\phi\,\underline{e}_2 + R^2\sin\theta\cos\theta\left(\cos^2\phi+\sin^2\phi\right)\underline{e}_3\\
&= R^2\sin\theta\,(\sin\theta\cos\phi\,\underline{e}_1 + \sin\theta\sin\phi\,\underline{e}_2 + \cos\theta\,\underline{e}_3)\\
&= R^2\sin\theta\,\underline{e}_r
\end{aligned}$$

where $\underline{e}_r$ is the unit vector ($\underline{e}_r\cdot\underline{e}_r = 1$) in the direction of $\underline{r}$ that we introduced previously

$$\underline{e}_r \;=\; \sin\theta\cos\phi\,\underline{e}_1 + \sin\theta\sin\phi\,\underline{e}_2 + \cos\theta\,\underline{e}_3$$

Hence the unit normal $\underline{n} = \underline{e}_r$. Note: $\underline{e}_r = \underline{e}_r(\theta,\phi)$ is a function of θ and ϕ, but not of r.

The vector element of area on the surface of the sphere is then

$$\mathrm{d}\underline{S} \;=\; \frac{\partial\underline{r}}{\partial\theta}\times\frac{\partial\underline{r}}{\partial\phi}\,\mathrm{d}\theta\,\mathrm{d}\phi \;=\; R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\underline{e}_r$$

On S, we have $r = R$, so the vector field $\underline{a}(\underline{r})$ on the surface S is $\underline{a} = (R\,\underline{e}_r)/R^3 = \underline{e}_r/R^2$.

The flux of $\underline{a}$ through the closed surface S is then

$$\int_S \underline{a}\cdot\mathrm{d}\underline{S} \;=\; \int_S \frac{\underline{r}}{r^3}\cdot\mathrm{d}\underline{S} \;=\; \int_0^\pi \mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\,\left(\frac{\underline{e}_r}{R^2}\right)\cdot\left(R^2\sin\theta\,\underline{e}_r\right) \;=\; \int_0^\pi\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi \;=\; 4\pi$$

In this example, the result of the integral is the surface area of a unit sphere. This is important in physics, as we now show…

Physics example: The electric field $\underline{E}(\underline{r})$ at $\underline{r}$ due to a point charge q situated at the origin is

$$\underline{E}(\underline{r}) \;=\; \frac{q}{4\pi\epsilon_0}\,\frac{\underline{r}}{r^3}$$

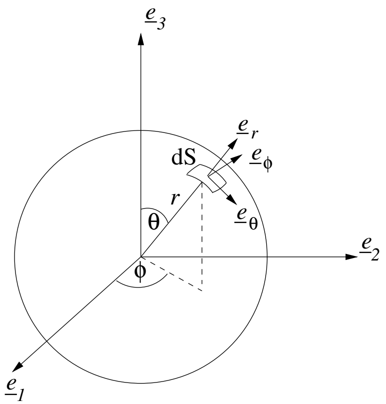
Evaluate the total flux of $\underline{E}(\underline{r})$ out of the sphere $x_1^2 + x_2^2 + x_3^2 = R^2$.

Noting that the electric field $\underline{E}(\underline{r}) = q/\,(4\pi\epsilon_0)\;\underline{a}(\underline{r})$ in the example above, we can use our previous result to obtain

$$\int_S \underline{E}\cdot\mathrm{d}\underline{S} \;=\; \frac{q}{4\pi\epsilon_0}\int_S\frac{\underline{r}\cdot\mathrm{d}\underline{S}}{r^3} \;=\; \frac{q}{4\pi\epsilon_0}\,4\pi \;=\; \frac{q}{\epsilon_0}$$

This is an example of Gauss' Law of Electromagnetism. We shall show later that the total electric flux through *any* closed surface is the total charge enclosed by the surface, divided by the constant ϵ_0.

Basis vectors for spherical polars:



In spherical polars, the position vector at the point $\underline{r}$ with spherical polar coordinates (r,θ,ϕ) is

$$\underline{r} = r\sin\theta\cos\phi\,\underline{e}_1 + r\sin\theta\sin\phi\,\underline{e}_2 + r\cos\theta\,\underline{e}_3$$

Consider the change in $\underline{r}$ as we let $r\to r+\mathrm{d}r$

$$\mathrm{d}\underline{r}_r \;\equiv\; \underline{r}(r+\mathrm{d}r,\theta,\phi) - \underline{r}(r,\theta,\phi) \;=\; \frac{\partial\underline{r}}{\partial r}\,\mathrm{d}r$$

Clearly $\mathrm{d}\underline{r}_r$ and hence $\partial\underline{r}/\partial r$ are *parallel* to $\underline{r}$.

Similarly, we define the infinitesimal vectors $\mathrm{d}\underline{r}_\theta$ and $\mathrm{d}\underline{r}_\phi$.

Then the normalised vectors

$$\underline{e}_r = \frac{\partial\underline{r}}{\partial r}\bigg/\left|\frac{\partial\underline{r}}{\partial r}\right|, \qquad \underline{e}_\theta = \frac{\partial\underline{r}}{\partial\theta}\bigg/\left|\frac{\partial\underline{r}}{\partial\theta}\right|, \qquad \underline{e}_\phi = \frac{\partial\underline{r}}{\partial\phi}\bigg/\left|\frac{\partial\underline{r}}{\partial\phi}\right|,$$

are [important exercise for the student]

$$\begin{aligned}
\underline{e}_r &= \sin\theta\cos\phi\,\underline{e}_1 + \sin\theta\sin\phi\,\underline{e}_2 + \cos\theta\,\underline{e}_3\\
\underline{e}_\theta &= \cos\theta\cos\phi\,\underline{e}_1 + \cos\theta\sin\phi\,\underline{e}_2 - \sin\theta\,\underline{e}_3\\
\underline{e}_\phi &= -\sin\phi\,\underline{e}_1 + \cos\phi\,\underline{e}_2
\end{aligned}$$

These form an *orthonormal* basis, *i.e.* $\underline{e}_r\cdot\underline{e}_r = \underline{e}_\theta\cdot\underline{e}_\theta = \underline{e}_\phi\cdot\underline{e}_\phi = 1$, and $\underline{e}_r\cdot\underline{e}_\theta = \underline{e}_\theta\cdot\underline{e}_\phi = \underline{e}_\phi\cdot\underline{e}_r = 0$ [important exercise]

The unit vector $\underline{e}_\theta$ points in the direction of increasing θ, with r and ϕ fixed – as illustrated in the diagram above. Similarly for $\underline{e}_r$ and $\underline{e}_\phi$.

The basis vectors $\underline{e}_r$, $\underline{e}_\theta$ and $\underline{e}_\phi$ depend on θ and ϕ, but not on r. They point in different directions at different points. We say they form a non-Cartesian or *curvilinear* basis.

8.3 Other surface integrals

If $f(\underline{r})$ is a scalar field, we may define a scalar surface integral

$$\int_S f \, \mathrm{d}S$$

For example, the (scalar) *area* of the surface S is just

$$S \;=\; \int_S \mathrm{d}S \;=\; \int_S |\mathrm{d}\underline{S}| \;=\; \int_v \int_u \left| \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right| \mathrm{d}u \, \mathrm{d}v$$

We may also define vector surface integrals

$$\int_S f \, \mathrm{d}\underline{S} \qquad \int_S \underline{a} \, \mathrm{d}S \qquad \int_S \underline{a} \times \mathrm{d}\underline{S}$$

Each of these is a double integral, and is evaluated in a way similar to the scalar integrals, the result being a vector in each case.

The *vector area* of a surface is defined as $\underline{S} = \int_S \mathrm{d}\underline{S}$.

Example: The *vector area* $\underline{S}$ of the (open) hemisphere, $x_1^2 + x_2^2 + x_3^2 = R^2$, $(x_3 \geq 0)$, of radius R, is, using spherical polars,

$$\underline{S} \;\equiv\; \int_S \mathrm{d}\underline{S} \;=\; \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} R^2 \sin\theta \, \underline{e}_r \, \mathrm{d}\theta \, \mathrm{d}\phi \, .$$

NB Since $\underline{e}_r = \sin\theta\cos\phi \, \underline{e}_1 + \sin\theta\sin\phi \, \underline{e}_2 + \cos\theta \, \underline{e}_3$ is not a constant (it depends on θ and ϕ), we can't take it out of the integral. So we have

$$\begin{aligned}
\underline{S} \;&=\; \underline{e}_1 R^2 \int_0^{\pi/2} \sin^2\theta \, \mathrm{d}\theta \int_0^{2\pi} \cos\phi \, \mathrm{d}\phi \;+\; \underline{e}_2 R^2 \int_0^{\pi/2} \sin^2\theta \, \mathrm{d}\theta \int_0^{2\pi} \sin\phi \, \mathrm{d}\phi \;+ \\
&\quad\; \underline{e}_3 R^2 \int_0^{\pi/2} \sin\theta\cos\theta \, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \\
&=\; 0 + 0 + \pi R^2 \, \underline{e}_3
\end{aligned}$$

Notes

The vector surface integral over the *full* sphere $x_1^2 + x_2^2 + x_3^2 = R^2$ is *zero* because the contributions from the upper and lower hemispheres cancel. Similarly, the vector area of a *closed* hemisphere is zero because the vector area of the bottom face, which is a circular disc of radius R in the x_1–x_2 plane, is $-\pi R^2 \, \underline{e}_3$. This is just the projection of the sum of infinitesimal vector areas onto the base of the hemisphere.

In fact, for *any closed surface*,

$$\int_S \mathrm{d}\underline{S} = 0$$

To show this, it is simplest to use the divergence theorem – see later.

9 Volume integrals

Volume integrals are conceptually simpler than line and surface integrals because the element of volume $\mathrm{d}V$ is a *scalar* quantity.

We define integrals over a volume V in the standard way using the Riemann sum.

9.1 Integrals over scalar fields

Let $f(\underline{r})$ be a scalar field defined in a volume V. Divide V into n small volumes $\delta V^{(i)}$. Then

$$\int_V f(\underline{r}) \, \mathrm{d}V \;=\; \lim_{\substack{n\to\infty \\ \delta V^{(i)}\to 0}} \sum_{i=0}^{n-1} f(\underline{r}^{(i)}) \, \delta V^{(i)}$$

where $f(\underline{r}^{(i)})$ is the value of the function f at some point $\underline{r}^{(i)}$ in the element of volume $\delta V^{(i)}$.

In Cartesian coordinates

$$\int_V f(\underline{r}) \, \mathrm{d}V \;=\; \int_z \int_y \int_x f(x,y,z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

where the integrals over x, y and z have appropriate limits. Note that $\int_V f(\underline{r}) \, \mathrm{d}V$ is a *triple* integral, but we use three integral signs only when writing it in terms of explicit integration variables, namely x, y, z in this case.

Example: Integrate $f(x,y,z) = xyz^2$ over the cuboid: $\{0 \leq x < a, \; 0 \leq y < b, \; 0 \leq z < c\}$.

We *choose* to perform the integral over x first, keeping y and z fixed; then the integral over y, keeping z fixed; and finally the integral over z:

$$\begin{aligned}
I \;&=\; \int_V f(\underline{r}) \, \mathrm{d}V \;=\; \int_{z=0}^{z=c} \int_{y=0}^{y=b} \int_{x=0}^{x=a} xyz^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&=\; \int_{z=0}^{z=c} \mathrm{d}z \int_{y=0}^{y=b} \mathrm{d}y \, \frac{a^2}{2} yz^2 \;=\; \frac{a^2}{2} \int_{z=0}^{z=c} \mathrm{d}z \, \frac{b^2}{2} z^2 \;=\; \frac{a^2}{2} \frac{b^2}{2} \frac{c^3}{3} \;=\; \frac{a^2 b^2 c^3}{12}
\end{aligned}$$

The volume of the cuboid is simply

$$V \;=\; \int_V \mathrm{d}V \;=\; \int_{z=0}^{z=c} \int_{y=0}^{y=b} \int_{x=0}^{x=a} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \;=\; abc$$

The integrals may be performed in any order, but the limits on the integrals must be chosen appropriately to cover the volume V.

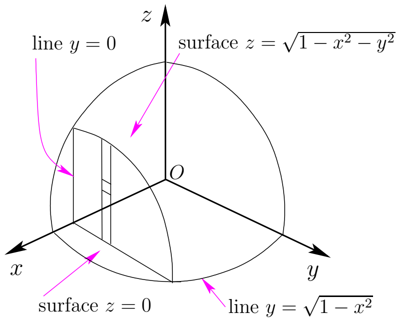
For example, if we choose to perform the z integral first, its limits may depend on x and y. The limits on the second integral over y may then depend on x (but not on z, because we have already integrated over z). The limits on the last integral over x can't depend on either y or z (because we have already integrated over them.)

Example:

$$\int_0^1 \mathrm{d}x \int_0^{x^2} \mathrm{d}y \int_{xy}^1 2x^2 z \, \mathrm{d}z \;=\; \frac{28}{165} \quad \text{(exercise)}$$

We now illustrate how to find the limits in an explicit example, albeit a rather complicated (and slightly masochistic) one.

Example: Use Cartesian coordinates to evaluate
$$\int_V (x+y+z)\,\mathrm{d}V$$
where V is the positive octant of the unit sphere:
$$x^2+y^2+z^2 \le 1, \qquad x \ge 0,\ y\ge 0,\ z\ge 0$$
If we *choose* to perform the z integral first:



(i) For *fixed* (x, y), we first integrate with respect to z, from the point $(x, y, 0)$ (*i.e.* $z = 0$), to the point (x, y, z) with $z = \sqrt{1-x^2-y^2}$. This is the integral up the strip shown.

(ii) Then, for fixed x, we integrate wrt y from the vertical strip at $y = 0$ to the vertical strip at $y = \sqrt{1-x^2}$. This sums over all such strips in the planar quadrant shown.

(iii) Finally, we integrate from $x = 0$ to $x = 1$, which sums over all planes from $x = 0$ to 1.

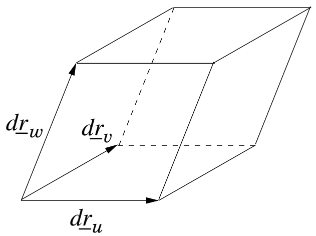
$$\Rightarrow \quad \int_V (x+y+z)\,\mathrm{d}V \;=\; \int_0^1 \mathrm{d}x \int_0^{\sqrt{1-x^2}} \mathrm{d}y \int_0^{\sqrt{1-x^2-y^2}} (x+y+z)\,\mathrm{d}z$$

9.2 Parametric form of volume integrals

The example above was very complicated – because we used Cartesians. It's *much* easier using a *parametric* representation. (We would use spherical polars in this example.)

Suppose we can write the position vector $\underline{r}$ in terms of three real parameters u, v and w, so that $\underline{r} = \underline{r}(u, v, w)$. If we make a small change in each of these parameters, then $\underline{r}$ changes by
$$\mathrm{d}\underline{r} \;=\; \frac{\partial \underline{r}}{\partial u}\,du + \frac{\partial \underline{r}}{\partial v}\,dv + \frac{\partial \underline{r}}{\partial w}\,dw$$
Along the curves $\{v = \text{constant},\ w = \text{constant}\}$, we have $dv = dw = 0$, so $\mathrm{d}\underline{r}$ is simply
$$\mathrm{d}\underline{r}_u \;=\; \frac{\partial \underline{r}}{\partial u}\,du$$
with $\mathrm{d}\underline{r}_v$ and $\mathrm{d}\underline{r}_w$ having similar definitions.

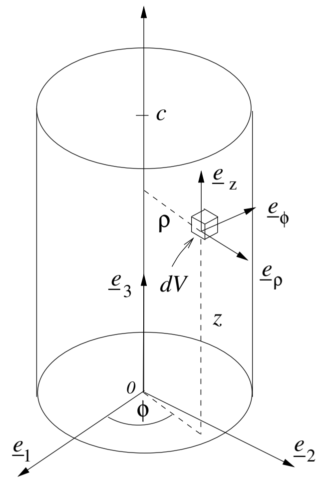
As shown in the figure, the vectors $\mathrm{d}\underline{r}_u$, $\mathrm{d}\underline{r}_v$ and $\mathrm{d}\underline{r}_w$ form the sides of an infinitesimal parallelepiped of volume
$$\mathrm{d}V \;=\; \left|\mathrm{d}\underline{r}_u \cdot \left(\mathrm{d}\underline{r}_v \times \mathrm{d}\underline{r}_w\right)\right|$$
$$\boxed{\Rightarrow \quad \mathrm{d}V \;=\; \left|\frac{\partial \underline{r}}{\partial u} \cdot \left(\frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial w}\right)\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w}$$



We take the *magnitude* of the scalar triple product so that the element of volume $\mathrm{d}V$ is always *positive*.

Jacobians revisited: We may write the volume element in terms of the Jacobian J, $\mathrm{d}V \;=\; |J|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w\,,$ where
$$J \;\equiv\; \frac{\partial(x,y,z)}{\partial(u,v,w)} \;\equiv\; \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{pmatrix}$$
This is the three dimensional version of the 2×2 Jacobian derived in *LA&SVC* Section 18 and (presumably) in *SVC&DE*. It generalises to higher dimensions in a straightforward way.

Example: Consider a circular cylinder of radius a, height c. We can parameterise $\underline{r}$ using cylindrical coordinates. Within the cylinder, we have
$$\underline{r} \;=\; \rho\cos\phi\,\underline{e}_1 + \rho\sin\phi\,\underline{e}_2 + z\,\underline{e}_3 \quad \text{with} \quad \{0\le\rho\le a,\ 0\le\phi\le 2\pi,\ 0\le z\le c\}$$
Then
$$\frac{\partial \underline{r}}{\partial \rho} \;=\; \cos\phi\,\underline{e}_1 + \sin\phi\,\underline{e}_2$$
$$\frac{\partial \underline{r}}{\partial \phi} \;=\; -\rho\,\sin\phi\,\underline{e}_1 + \rho\,\cos\phi\,\underline{e}_2$$
$$\frac{\partial \underline{r}}{\partial z} \;=\; \underline{e}_3$$
And hence (easy exercise)
$$\mathrm{d}V \;=\; \left|\frac{\partial \underline{r}}{\partial \rho} \cdot \left(\frac{\partial \underline{r}}{\partial \phi} \times \frac{\partial \underline{r}}{\partial z}\right)\right| \mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z \;=\; \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z$$
The *volume* of the cylinder is
$$V \;=\; \int_V \mathrm{d}V \;=\; \int_{z=0}^{z=c}\int_{\phi=0}^{\phi=2\pi}\int_{\rho=0}^{\rho=a} \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,dz \;=\; \pi\,a^2 c.$$



Cylindrical basis: the normalised vectors
$$\underline{e}_\rho \;=\; \frac{\partial \underline{r}}{\partial \rho}\Big/\left|\frac{\partial \underline{r}}{\partial \rho}\right| \quad;\quad \underline{e}_\phi \;=\; \frac{\partial \underline{r}}{\partial \phi}\Big/\left|\frac{\partial \underline{r}}{\partial \phi}\right| \quad;\quad \underline{e}_z \;=\; \frac{\partial \underline{r}}{\partial z}\Big/\left|\frac{\partial \underline{r}}{\partial z}\right|$$
(shown in the figure) are the orthonormal basis vectors that we wrote down earlier for cylindrical coordinates (exercise).

Exercise: For spherical polars: $\underline{r} = r\sin\theta\cos\phi\,\underline{e}_1 + r\sin\theta\sin\phi\,\underline{e}_2 + r\cos\theta\,\underline{e}_3\,,$ show that
$$\mathrm{d}V \;=\; \left|\frac{\partial \underline{r}}{\partial r} \cdot \left(\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi}\right)\right| \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \;=\; r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

9.3 Integrals over vector fields

The integral of a vector field $\underline{a}(\underline{r})$ over a volume V is

$$\int_V \underline{a}\,\mathrm{d}V \;=\; \sum_{i=1}^{3} \underline{e}_i \int_V a_i\,\mathrm{d}V \;=\; \sum_{i=1}^{3} \underline{e}_i \int_z\int_y\int_x a_i\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \quad \text{in Cartesian coordinates}$$

The result of the integral is a *vector*, and we must evaluate 3 triple integrals for *each* of the 3 components of the vector.

Example: Consider a solid hemisphere of radius a centered on the $\underline{e}_3$ axis, with its bottom face at $x_3 = 0$. If the mass density (mass/unit volume), a scalar field, is $\rho(r) = \rho_0/r$ where ρ_0 is a constant, what is the total mass, M, of the hemisphere?

It is most convenient to use spherical polar coordinates. Then $\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$ and

$$M \;=\; \int_V \rho(r)\,\mathrm{d}V \;=\; \int_0^a \rho(r)\,r^2\,\mathrm{d}r \int_0^{\pi/2}\sin\theta\,\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\phi \;=\; 2\pi\rho_0\int_0^a r\,\mathrm{d}r \;=\; \pi\rho_0\,a^2$$

Now consider the *centre of mass vector* $\underline{R} = (X\,\underline{e}_1 + Y\,\underline{e}_2 + Z\,\underline{e}_3)$, defined by

$$M\underline{R} \;\equiv\; \int_V \rho(r)\,\underline{r}\,\mathrm{d}V$$

We integrate each component of the vector field $\rho(r)\underline{r}$ in turn using

$$\underline{r} \;=\; r\sin\theta\cos\phi\,\underline{e}_1 + r\sin\theta\sin\phi\,\underline{e}_2 + r\cos\theta\,\underline{e}_3 \quad \text{and} \quad \mathrm{d}V \;=\; r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

which gives

$$MX \;=\; \int_0^a \rho(r)\,r^3\,\mathrm{d}r \int_0^{\pi/2}\sin^2\theta\,\mathrm{d}\theta \int_0^{2\pi}\cos\phi\,\mathrm{d}\phi \;=\; 0 \quad (\text{since the } \phi \text{ integral gives } 0)$$

$$MY \;=\; \int_0^a \rho(r)\,r^3\,\mathrm{d}r \int_0^{\pi/2}\sin^2\theta\,\mathrm{d}\theta \int_0^{2\pi}\sin\phi\,\mathrm{d}\phi \;=\; 0 \quad (\text{since the } \phi \text{ integral gives } 0)$$

$$\begin{aligned} MZ \;&=\; \int_0^a \rho(r)\,r^3\,\mathrm{d}r \int_0^{\pi/2}\sin\theta\cos\theta\,\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\phi \;=\; 2\pi\rho_0\int_0^a r^2\mathrm{d}r\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} \\ &=\; 2\pi\rho_0\,\frac{a^3}{3}\,\frac{1}{2} \;=\; \frac{\pi\rho_0\,a^3}{3} \end{aligned}$$

Hence

$$\underline{R} \;=\; \frac{\pi\rho_0\,a^3/3}{\pi\rho_0\,a^2}\,\underline{e}_3 \;=\; \frac{a}{3}\,\underline{e}_3 \qquad (\text{independent of } \rho_0)$$

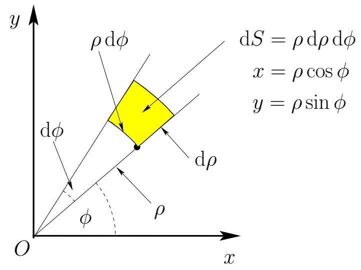
Note that the integrals over ϕ in the first two components of $M\underline{R}$ are zero. Watch out for integrals that are zero – spotting them can save you a lot of un-necessary work!

9.4 Summary of polar coordinate systems

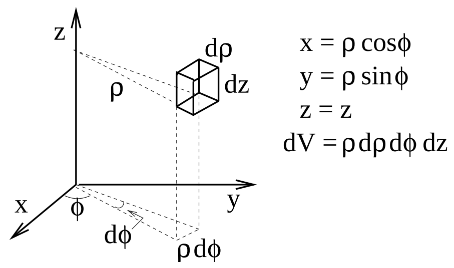
To conclude this section, we give a brief summary of polar coordinate systems.

In the figures below, $\mathrm{d}S$ indicates an area element and $\mathrm{d}V$ a volume element. We also sketch geometrical "derivations" of the infinitesimal elements of area and volume.

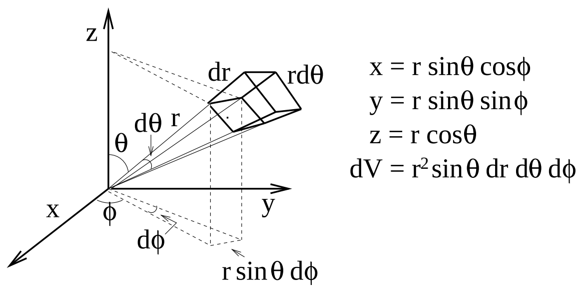
Plane polar coordinates: $(\rho,\,\phi)$



Cylindrical coordinates: $(\rho,\,\phi,\,z)$



Spherical polar coordinates: $(r,\,\theta,\,\phi)$



10 The divergence theorem

10.1 Integral definition of divergence

Let $\underline{a}$ be a vector field in the region R, and let P be a point in R, then the divergence of $\underline{a}$ at P may be *defined* by

$$\boxed{\operatorname{div}\underline{a} \;=\; \lim_{\delta V\to 0}\frac{1}{\delta V}\int_{\delta S}\underline{a}\cdot \mathrm{d}\underline{S}}$$

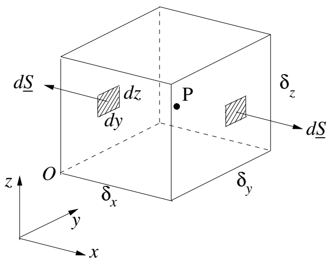
where δS is a *closed* surface in R which encloses the volume δV. The limit must be taken so that the point P is within δV. It can be shown that the limit is independent of the *shape* of δV.

This definition of div $\underline{a}$ is also *basis independent*, *i.e.* the result doesn't depend on whether we evaluate it in Cartesian coordinates, spherical polars, *etc.*

We now show that our original definition of $\underline{\nabla}\cdot\underline{a}$ is recovered in Cartesian co-ordinates.

Let P be a point with Cartesian coordinates $(x_0,\, y_0,\, z_0)$ situated at the *centre* of a *small* rectangular block of size $\delta_x\times\delta_y\times\delta_z$, with volume $\delta V = \delta_x\,\delta_y\,\delta_z$.

- On the *front* face of the block, parallel to the $y-z$ plane at $x = x_0+\delta_x/2$, we have *outward* normal $\underline{n} = \underline{e}_x$ and so $\mathrm{d}\underline{S} = \underline{e}_x\,\mathrm{d}y\,\mathrm{d}z$
- On the *back* face of the block, parallel to the $y-z$ plane at $x = x_0-\delta_x/2$, we have *outward* normal $\underline{n} = -\underline{e}_x$ and so $\mathrm{d}\underline{S} = -\underline{e}_x\,\mathrm{d}y\,\mathrm{d}z$



Hence $\underline{a}\cdot\mathrm{d}\underline{S} = \pm\, a_x\,\mathrm{d}y\,\mathrm{d}z$ on these two faces. Let us denote the union (sum) of the two faces orthogonal to the x axis by δS_x.

The contribution of these two surfaces to the integral $\displaystyle\int_S \underline{a}\cdot\mathrm{d}\underline{S}$ is given by

$$\begin{aligned}
\int_{\delta S_x}\underline{a}\cdot\mathrm{d}\underline{S} &= \int_z\!\int_y\Big\{a_x(x_0+\delta_x/2,y,z) - a_x(x_0-\delta_x/2,y,z)\Big\}\,\mathrm{d}y\,\mathrm{d}z\\
&= \int_z\!\int_y\Bigg\{\left[a_x(x_0,y,z) \;+\; \frac{\delta_x}{2}\left.\frac{\partial a_x}{\partial x}\right|_{(x_0,y,z)} + O(\delta_x^2)\right]\\
&\qquad\quad - \left[a_x(x_0,y,z) \;-\; \frac{\delta_x}{2}\left.\frac{\partial a_x}{\partial x}\right|_{(x_0,y,z)} + O(\delta_x^2)\right]\Bigg\}\,\mathrm{d}y\,\mathrm{d}z\\
&= \int_z\!\int_y \delta_x \left.\frac{\partial a_x}{\partial x}\right|_{(x_0,y,z)}\mathrm{d}y\,\mathrm{d}z
\end{aligned}$$

where we Taylor-expanded $a_x(x,y,z)$ about the point (x_0,y,z), and we have dropped the $O(\delta_x^2)$ terms which will vanish when we divide by δV at the end.

So

$$\frac{1}{\delta V}\int_{\delta S_x}\underline{a}\cdot\mathrm{d}\underline{S} \;=\; \frac{1}{\delta_y\,\delta_z}\int_z\!\int_y\left.\frac{\partial a_x}{\partial x}\right|_{(x_0,y,z)}\mathrm{d}y\,\mathrm{d}z$$

As we take the limit $\delta_x,\delta_y,\delta_z\to 0$, the integrand is approximately *constant* in the volume δV, so that

$$\int_z\!\int_y\left.\frac{\partial a_x}{\partial x}\right|_{(x_0,y,z)}\mathrm{d}y\,\mathrm{d}z \;\to\; \delta_y\,\delta_z\left.\frac{\partial a_x}{\partial x}\right|_{(x_0,y_0,z_0)}$$

and hence

$$\lim_{\delta V\to 0}\frac{1}{\delta V}\int_{\delta S_x}\underline{a}\cdot\mathrm{d}\underline{S} \;=\; \left.\frac{\partial a_x}{\partial x}\right|_{(x_0,y_0,z_0)}$$

Adding similar contributions from the other 4 faces, we find

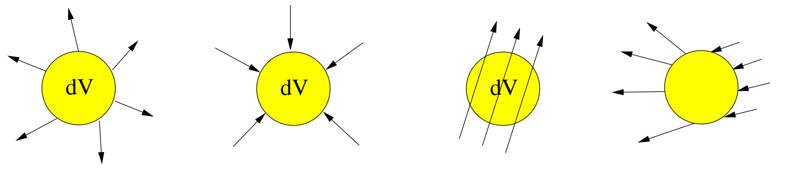
$$\operatorname{div}\underline{a} \;=\; \frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z} \;=\; \underline{\nabla}\cdot\underline{a}$$

in agreement with our original definition in Cartesian co-ordinates. Thus we can use the notation $\underline{\nabla}\cdot\underline{a}$ for both the differential and the integral definitions of the divergence of a vector field $\underline{a}(\underline{r})$.

The integral definition

$$\operatorname{div}\underline{a} \;=\; \underline{\nabla}\cdot\underline{a} \;=\; \lim_{\delta V\to 0}\frac{1}{\delta V}\int_{\delta S}\underline{a}\cdot\mathrm{d}\underline{S}$$

provides a precise intuitive understanding of divergence as the (net) flux per unit volume leaving a small volume around a point $\underline{r}$. In pictures, for an infinitesimal volume $\mathrm{d}V$,



div $\underline{a}$ > 0   div $\underline{a}$ < 0   div $\underline{a}$ = 0 (flux out = flux in)   div $\underline{a}$ > 0 (flux out > flux in)

10.2 The divergence theorem (Gauss' theorem)

Let $\underline{a}$ be a vector field in a volume V, and let S be the closed surface bounding V, then

$$\boxed{\int_V \underline{\nabla}\cdot\underline{a}\;\mathrm{d}V \;=\; \int_S \underline{a}\cdot\mathrm{d}\underline{S}}$$

Proof: We derive the divergence theorem using the integral definition of $\underline{\nabla}\cdot\underline{a}$

$$\underline{\nabla}\cdot\underline{a} \;=\; \lim_{\delta V\to 0}\frac{1}{\delta V}\int_{\delta S}\underline{a}\cdot\mathrm{d}\underline{S}.$$

Since this *definition* of $\underline{\nabla} \cdot \underline{a}$ is valid for volumes of arbitrary shape, we can build a smooth surface S from a large number, N, of small blocks of volume $\delta V^{(i)}$ and area $\delta S^{(i)}$. For small, but not infinitesimal, $\delta S^{(i)}$ we may write

$$\underline{\nabla} \cdot \underline{a}(\underline{r}^{(i)}) \;=\; \frac{1}{\delta V^{(i)}} \int_{\delta S^{(i)}} \underline{a} \cdot \mathrm{d}\underline{S} + \epsilon^{(i)}$$

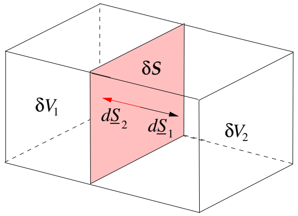
where $\epsilon^{(i)} \to 0$ as $\delta V^{(i)} \to 0$. Now multiply both sides by $\delta V^{(i)}$ and sum over all i

$$\sum_{i=1}^{N} \underline{\nabla} \cdot \underline{a}(\underline{r}^{(i)})\, \delta V^{(i)} \;=\; \sum_{i=1}^{N} \int_{\delta S^{(i)}} \underline{a} \cdot \mathrm{d}\underline{S} \;+\; \sum_{i=1}^{N} \epsilon^{(i)}\, \delta V^{(i)} \qquad (17)$$

On the RHS the contributions from surface elements *interior* to S *cancel.* This is because where two blocks touch, the outward normals are in *opposite* directions, implying that the contributions to the respective integrals cancel.

To illustrate this, consider two adjacent blocks, 1 and 2.

The volume of block 1 is δV_1 and its surface is δS_1. Similarly for block 2. Denote the shaded surface common to the two blocks by $\delta\mathcal{S}$, then



$$\begin{aligned} &\int_{\delta\mathcal{S}} \underline{a} \cdot \mathrm{d}\underline{S} \quad \text{regarded as part of block 1} \\ =\; -&\int_{\delta\mathcal{S}} \underline{a} \cdot \mathrm{d}\underline{S} \quad \text{regarded as part of block 2} \end{aligned}$$

because their outward normals $\mathrm{d}\underline{S}_1$ and $\mathrm{d}\underline{S}_2$ are equal and opposite. Therefore

$$\int_{\delta S_1} \underline{a} \cdot \mathrm{d}\underline{S} \;+\; \int_{\delta S_2} \underline{a} \cdot \mathrm{d}\underline{S} \;=\; \int_{\delta S_{1+2}} \underline{a} \cdot \mathrm{d}\underline{S} \quad \text{(because contributions from } \delta\mathcal{S} \text{ cancel)}$$

where we have denoted the *exterior* surface of the *compound block* by δS_{1+2}.

Thus the contributions from all *interior* surface elements cancel *pairwise*. Using this result, and letting $N \to \infty$ in equation (17), the $O(\epsilon^{(i)}\, \delta V^{(i)})$ terms go to zero, and we get

$$\int_V \underline{\nabla} \cdot \underline{a}\, \mathrm{d}V \;=\; \int_S \underline{a} \cdot \mathrm{d}\underline{S}\,.$$

For an alternative proof of the divergence theorem, see *Bourne & Kendall*, Chapter 6.

Note: The divergence theorem is the generalisation to 3-D of

$$\int_a^b \frac{\mathrm{d}f(x)}{\mathrm{d}x}\, \mathrm{d}x \;=\; f(b) - f(a)$$

10.3 Volume of a body using the divergence theorem

The volume of a body is $V = \displaystyle\int_V \mathrm{d}V$

Recalling that $\underline{\nabla} \cdot \underline{r} = 3$, we can write

$$V \;=\; \frac{1}{3} \int_V \underline{\nabla} \cdot \underline{r}\, \mathrm{d}V \;=\; \frac{1}{3} \int_S \underline{r} \cdot \mathrm{d}\underline{S}$$

where we used the divergence theorem in the last step.

Example: Consider the hemisphere $x^2 + y^2 + z^2 \le R^2$ centered on $\underline{e}_3$ with its bottom face in the $x{-}y$ plane. Recalling that the divergence theorem holds for a *closed* surface, the volume of the hemisphere is

$$V \;=\; \frac{1}{3} \left[\int_{S_C} \underline{r} \cdot \mathrm{d}\underline{S} + \int_{S_B} \underline{r} \cdot \mathrm{d}\underline{S} \right]$$

where S_C is the curved surface of the hemisphere and S_B is its bottom. On S_B, we have $\mathrm{d}\underline{S} = -\underline{e}_z\, \mathrm{d}S$ and $z = 0$, so $\underline{r} \cdot \mathrm{d}\underline{S} = -z\, dS = 0$. Therefore the only contribution comes from the (open) curved surface S_C,

$$V \;=\; \frac{1}{3} \int_{S_C} \underline{r} \cdot \mathrm{d}\underline{S}$$

We can evaluate this surface integral using spherical polars. For a hemisphere of radius R we showed previously that $\mathrm{d}\underline{S} \;=\; R^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\, \underline{e}_r$.

On the hemisphere, $\underline{r} \cdot \mathrm{d}\underline{S} = R\, \underline{e}_r \cdot \mathrm{d}\underline{S} = R^3 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$, therefore

$$V \;=\; \frac{1}{3} \int_S \underline{r} \cdot \mathrm{d}\underline{S} \;=\; \frac{R^3}{3} \int_0^{\pi/2} \sin\theta\, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \;=\; \frac{2\pi R^3}{3}$$

as anticipated.

10.4 The continuity equation

Consider a fluid with mass density $\rho(\underline{r}, t)$ and velocity field $\underline{v}(\underline{r}, t)$. We have seen previously that the *volume flux* (volume per unit time) of fluid flowing across a surface S is $\int_S \underline{v} \cdot \mathrm{d}\underline{S}$.

The corresponding *mass flux* (mass per unit time) flowing across the surface is

$$\int_S (\rho \underline{v}\,) \cdot \mathrm{d}\underline{S} \;\equiv\; \int_S \underline{J} \cdot \mathrm{d}\underline{S} \qquad (18)$$

where $\underline{J}(\underline{r}, t) \equiv \rho(\underline{r}, t)\, \underline{v}(\underline{r}, t)$ is the *mass current density.*

Now consider a volume V bounded by the *closed* surface S containing *no sources or sinks* of fluid. Conservation of mass means that the *outward* mass flux through the surface S must be equal to the rate of *decrease* of mass of fluid contained in the volume V,

$$\int_S \underline{J} \cdot \mathrm{d}\underline{S} = -\frac{\partial M}{\partial t}\,, \qquad (19)$$

where M is the total mass in V, which may be written as $M = \int_V \rho\, \mathrm{d}V$. Substituting this into equation (19), we get

$$\frac{\partial}{\partial t} \int_V \rho\, \mathrm{d}V + \int_S \underline{J} \cdot \mathrm{d}\underline{S} = 0\,.$$

Using the divergence theorem to rewrite the second term as a volume integral, we obtain

$$\int_V \left[\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} \right] \mathrm{d}V \;=\; 0$$

Since this holds for arbitrary volumes V, we must have

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0}$$

everywhere. This is the *continuity equation.* In this case, it expresses conservation of mass locally at each point $\underline{r}$.

The continuity equation appears in many different contexts because it holds for *any conserved quantity*. Here we considered mass density ρ and mass current density $\underline{J}$ in a fluid, but equally it could have been thermal energy density and heat-current density, electric charge density and electric current density, or more abstract quantities such as probability density and probability-current density in quantum mechanics [tutorial].

In the case of fluid flow, the continuity equation tells us that

$$\text{if} \quad \underline{\nabla} \cdot \underline{J} > 0 \quad \text{then} \quad \frac{\partial \rho}{\partial t} < 0 \quad \text{and the mass density at } \underline{r} \textit{ decreases}$$

$$\text{if} \quad \underline{\nabla} \cdot \underline{J} < 0 \quad \text{then} \quad \frac{\partial \rho}{\partial t} > 0 \quad \text{and the mass density at } \underline{r} \textit{ increases}$$

If the mass density at each point is constant in time, so that $\partial\rho/\partial t = 0$, the continuity equation tells us that for the mass density to be constant in time, and we must have $\underline{\nabla}\cdot\underline{J} = 0$, *i.e.* the mass flux into a point equals the flux out.

Incompressible flow: If $\rho(\underline{r}, t)$ is a constant (*i.e.* time-independent *and* independent of $\underline{r}$), then

$$\underline{\nabla} \cdot \underline{J} \;=\; \rho\, \underline{\nabla} \cdot \underline{v} \;=\; 0 \qquad \Rightarrow \qquad \underline{\nabla} \cdot \underline{v} = 0$$

Fluid flows with $\rho =$ *constant*, and hence $\underline{\nabla} \cdot \underline{v} = 0$, are said to be *incompressible flows*.

10.5 Sources and sinks

We can generalise these ideas. For a vector field $\underline{a}(\underline{r})$, the quantity

$$\frac{1}{V} \int_S \underline{a} \cdot \mathrm{d}\underline{S}$$

gives us information about whether there are sources or sinks of the vector field $\underline{a}$ within the volume V. If V contains

- a net *source*, then $\displaystyle\int_S \underline{a} \cdot \mathrm{d}\underline{S} \;=\; \int_V \underline{\nabla} \cdot \underline{a}\, \mathrm{d}V \;>\; 0$
- a net *sink*, then $\displaystyle\int_S \underline{a} \cdot \mathrm{d}\underline{S} \;=\; \int_V \underline{\nabla} \cdot \underline{a}\, \mathrm{d}V \;<\; 0$

If S contains neither sources nor sinks, or sources and sinks in equal measure, then $\displaystyle\int_S \underline{a} \cdot \mathrm{d}\underline{S} = 0$.

The quantity

$$\underline{\nabla} \cdot \underline{a} \;=\; \lim_{V \to 0} \frac{1}{V} \int_S \underline{a} \cdot \mathrm{d}\underline{S}$$

is therefore a measure of the *density* of sources or sinks,

$$\underline{\nabla} \cdot \underline{a} \;=\; \text{net } \textit{outward} \text{ flux per unit volume at } \underline{r}.$$

If we have a net *source* or *sink* of a vector field $\underline{a}$ at the point $\underline{r}$, then $\underline{\nabla} \cdot \underline{a} \neq 0$ at $\underline{r}$.

These ideas can be applied to *electric* and *magnetic* fields.

10.6 Electrostatics - Gauss' law and Maxwell's first equation

As a very important example, we consider electrostatics.

The electric field at $\underline{r}$ due to a point charge q at the *origin* is

$$\underline{E}(\underline{r}) \;=\; \frac{q}{4\pi\epsilon_0} \frac{\underline{r}}{r^3}$$

Then, for $r \neq 0$,

$$\underline{\nabla} \cdot \underline{E} \;=\; \frac{q}{4\pi\epsilon_0} \underline{\nabla} \cdot \left(\frac{\underline{r}}{r^3}\right) \;=\; \frac{q}{4\pi\epsilon_0} \left(\underline{\nabla}\left(\frac{1}{r^3}\right) \cdot \underline{r} + \frac{\underline{\nabla} \cdot \underline{r}}{r^3}\right) \;=\; \frac{q}{4\pi\epsilon_0} \left(-\frac{3\underline{r}}{r^5} \cdot \underline{r} + \frac{3}{r^3}\right) \;=\; 0$$

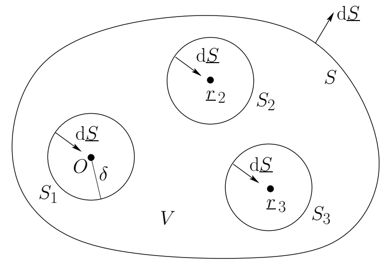
In section (8.2), we showed that

$$\int_{\text{sphere}} \underline{E} \cdot \mathrm{d}\underline{S} \;=\; \frac{q}{4\pi\epsilon_0} \int_{\text{sphere}} \frac{\underline{r} \cdot \mathrm{d}\underline{S}}{r^3} \;=\; \frac{q}{4\pi\epsilon_0} 4\pi \;=\; \frac{q}{\epsilon_0} \tag{20}$$

where the integral is over the surface of a sphere centred on the origin. The key result was $\int_{\text{sphere}} (\underline{r} \cdot \mathrm{d}\underline{S})/r^3 = 4\pi$, independent of the radius of the sphere.

Now consider an *arbitrary* closed surface S which encloses the charge at the origin. Define the volume V to be the region *between* the surfaces S and S_1, where S_1 is a small sphere, radius δ, centred on the origin. The volume V is then bounded by the closed surface $S + S_1$.

(Ignore the spheres S_2 and S_3 in the figure for now.)



Since the volume V does *not* contain the origin, $\underline{r} = 0$, then $\underline{\nabla} \cdot \underline{E} = 0$ *everywhere* in V, and the divergence theorem tells us that

$$\int_{S+S_1} \underline{E} \cdot \mathrm{d}\underline{S} \;=\; \int_S \underline{E} \cdot \mathrm{d}\underline{S} \;+\; \int_{S_1} \underline{E} \cdot \mathrm{d}\underline{S} \;=\; \int_V \underline{\nabla} \cdot \underline{E}\, \mathrm{d}V \;=\; 0 \tag{21}$$

Since the *outward* normal on the sphere S_1 (*i.e.* outward from *within* the volume V) points *towards* the origin, equation (20) gives

$$\int_{S_1} \underline{E} \cdot \mathrm{d}\underline{S} \;=\; -\frac{q}{\epsilon_0}$$

independent of δ, and we may safely take the limit $\delta \to 0$. Equation (21) then becomes

$$\int_S \underline{E} \cdot \mathrm{d}\underline{S} \;=\; \frac{q}{\epsilon_0} \tag{22}$$

This holds for *any* closed surface S which encloses the charge at the origin.

If, instead of charge q at the origin, we have charge q_i at position $\underline{r}_i$ inside S, we can change integration variable from $\underline{r}$ to $\underline{\rho} = \underline{r} - \underline{r}_i$ when integrating over the sphere S_i (with outward normal pointing *towards* its centre), and we get

$$\int_{S_i} \underline{E}_i \cdot \mathrm{d}\underline{S} \;=\; \frac{q_i}{4\pi\epsilon_0} \int_{S_i} \frac{(\underline{r} - \underline{r}_i) \cdot \mathrm{d}\underline{S}}{|\underline{r} - \underline{r}_i|^3} \;=\; \frac{q_i}{4\pi\epsilon_0} \int_{S_i} \frac{\underline{\rho} \cdot \mathrm{d}\underline{S}}{\rho^3} \;=\; -\frac{q_i}{\epsilon_0} \tag{23}$$

Equations (23) and (21) then give

$$\int_S \underline{E}_i \cdot \mathrm{d}\underline{S} \;=\; \frac{q_i}{\epsilon_0} \qquad (24)$$

Now let's replace the single charge by a set of N charges q_i at positions $\underline{r}_i$. Experiment tells us that the *total* electric field $\underline{E}$ is the sum of the electric fields $\underline{E}_i$ due to the individual charges. Therefore, using equation (24), we get

$$\int_S \underline{E} \cdot \mathrm{d}\underline{S} \;=\; \int_S \left(\sum_{i=1}^{N} \underline{E}_i\right) \cdot \mathrm{d}\underline{S} \;=\; \sum_{i=1}^{N} \int_S \underline{E}_i \cdot \mathrm{d}\underline{S} \;=\; \sum_{i=1}^{N} \frac{q_i}{\epsilon_0} \;=\; \frac{Q}{\epsilon_0}$$

where $Q = \sum_{i=1}^{N} q_i$ is *the total charge* enclosed by S. This is *Gauss' Law* of electrostatics.

Generalising further, if we have a *charge density* $\rho(\underline{r})$ (charge/unit volume), then the total charge in a volume V is

$$Q = \int_V \rho(\underline{r})\, \mathrm{d}V$$

Applying the divergence theorem and Gauss' Law (respectively), we get

$$\int_V \underline{\nabla} \cdot \underline{E}\, \mathrm{d}V \;=\; \int_S \underline{E} \cdot \mathrm{d}\underline{S} \;=\; \frac{Q}{\epsilon_0} \;=\; \frac{1}{\epsilon_0} \int_V \rho(\underline{r})\, \mathrm{d}V$$

Since this holds for arbitrary volumes V, we must have

$$\boxed{\underline{\nabla} \cdot \underline{E}(\underline{r}) \;=\; \frac{\rho(\underline{r})}{\epsilon_0}}$$

which holds for all $\underline{r}$. This is *Maxwell's first equation* of electromagnetism.[14]

Evidently, it states that the divergence of the electric field at any point $\underline{r}$ is equal to the charge density at that point divided by (in SI units) the constant ϵ_0.

A positive charge is a *source* of electric field (*i.e.* it creates a positive flux) and a negative charge is a *sink* (*i.e.* it absorbs flux, or, equivalently, creates a negative flux).

10.7 Corollaries of the divergence theorem

We may deduce several immediate consequences of the divergence theorem

$$\int_V \underline{\nabla} \cdot \underline{a}\, \mathrm{d}V \;=\; \int_S \underline{a} \cdot \mathrm{d}\underline{S}$$

(i) Let $\underline{a} = \underline{c}$ where $\underline{c}$ is an arbitrary *constant* vector, then $\underline{\nabla} \cdot \underline{c} = 0$, and hence

$$\int_S \underline{c} \cdot \mathrm{d}\underline{S} = \underline{c} \cdot \int_S \mathrm{d}\underline{S} = 0$$

Since this holds for *all* constant vectors $\underline{c}$, we must have

$$\int_S \mathrm{d}\underline{S} = 0$$

for *any closed surface* S (as claimed previously).

[14] We have derived it for static electric fields, but it also holds in electrodynamics – see Junior Honours *Electromagnetism* or *Electromagnetism and Relativity*.

(ii) Let $\underline{a}(\underline{r}) = p(\underline{r})\,\underline{c}$ where $p(\underline{r})$ is a scalar field and $\underline{c}$ is a constant vector, then (tutorial)

$$\int_V (\underline{\nabla}\, p)\, \mathrm{d}V \;=\; \int_S p\, \mathrm{d}\underline{S}$$

This is used to derive the condition $\rho\,\underline{F} = \underline{\nabla}\, p$, for hydrostatic equilibrium in a fluid or gas of density ρ, subject to force per unit mass $\underline{F}$, at pressure p in tutorial problem (4.6).

(iii) Consider the vector field $\underline{a} \times \underline{c}$, where $\underline{a}(\underline{r})$ is an arbitrary vector field and $\underline{c}$ is a *constant* vector. We have, using a standard identity,

$$\underline{\nabla} \cdot (\underline{a} \times \underline{c}) \;=\; \underline{c} \cdot (\underline{\nabla} \times \underline{a}) - \underline{a} \cdot (\underline{\nabla} \times \underline{c}) \;=\; \underline{c} \cdot (\underline{\nabla} \times \underline{a}) - 0$$

Now apply the divergence theorem to $\underline{a} \times \underline{c}$

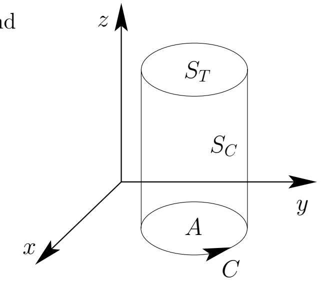
$$\underline{c} \cdot \int_V (\underline{\nabla} \times \underline{a})\, \mathrm{d}V \;=\; \int_V \underline{\nabla} \cdot (\underline{a} \times \underline{c})\, \mathrm{d}V \;=\; \int_S \mathrm{d}\underline{S} \cdot (\underline{a} \times \underline{c}) \;=\; \underline{c} \cdot \int_S \mathrm{d}\underline{S} \times \underline{a}$$

This holds for all constant vectors $\underline{c}$, hence

$$\int_V \underline{\nabla} \times \underline{a}\, \mathrm{d}V \;=\; \int_S \mathrm{d}\underline{S} \times \underline{a}$$

(iv) **Green's theorem in the plane**

Let V be the volume inside the cylinder $0 < z < 1$, and define the vector field $\underline{a}(\underline{r})$ as



$$\underline{a} \;=\; P(x, y)\, \underline{e}_x + Q(x, y)\, \underline{e}_y$$

Then
$$\int_V \underline{\nabla} \cdot \underline{a}\, \mathrm{d}V \;=\; \int_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$
$$\;=\; \int_A \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y$$

where we performed the (trivial) integral over z to get unity, and defined $A = S_B$ to be the bottom surface of the cylinder in the $x{-}y$ plane.

Let $S = S_C + S_T + S_B$ be the closed surface bounding V. On the top and bottom surfaces, S_T and S_B, we have $\mathrm{d}\underline{S} = \pm \mathrm{d}S\, \underline{e}_z$ and therefore $\underline{a} \cdot \mathrm{d}\underline{S} = 0$. On the curved surface, S_C,

$$\mathrm{d}\underline{S} \;=\; \big(\mathrm{d}x\, \underline{e}_x + \mathrm{d}y\, \underline{e}_y\big) \times \underline{e}_z\, \mathrm{d}z \;=\; (\mathrm{d}y\, \underline{e}_x - \mathrm{d}x\, \underline{e}_y)\, \mathrm{d}z$$

Hence

$$\int_S \underline{a} \cdot \mathrm{d}\underline{S} \;=\; \int_{S_C} \underline{a} \cdot \mathrm{d}\underline{S} \;=\; \iint (P\, \mathrm{d}y - Q\, \mathrm{d}x)\, \mathrm{d}z$$

The (trivial) integral over z again gives unity. Using the divergence theorem, we get

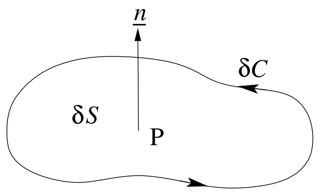
$$\boxed{\int_A \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y \;=\; \oint_C (P\, \mathrm{d}y - Q\, \mathrm{d}x)}$$

which is *Green's theorem in the plane* (sometimes called the *two dimensional divergence theorem*) relating the integral over a planar surface A to the line integral around the closed curve C enclosing A. The theorem applies to *any* surface in the $x{-}y$ plane, because the proof above doesn't rely on the base of the cylinder being circular.

11 Line integral definition of curl, Stokes' theorem

11.1 Line integral definition of curl

Consider a small planar surface with unit normal $\underline{n}$ and (scalar) area δS, bounded by a *closed* curve δC, and containing the point P. Let $\underline{a}$ be a vector field defined on this surface.



The component of $\underline{\nabla} \times \underline{a}$ in the direction of $\underline{n}$ is *defined* to be

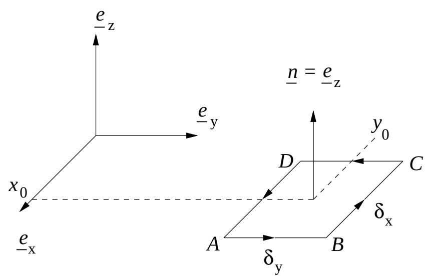
$$\boxed{\underline{n} \cdot (\underline{\nabla} \times \underline{a}) \;=\; \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_{\delta C} \underline{a} \cdot \mathrm{d}\underline{r}}$$

NB: The integral around δC is taken in the right-hand sense with respect to the normal $\underline{n}$ to the surface – as shown in the figure above.

This definition of curl is *independent of the choice of basis.*

Cartesian form of curl: The usual Cartesian form for curl can be recovered from this general definition by considering small rectangles parallel to the $x{-}y$, $y{-}z$, and $z{-}x$ planes respectively.

Let P be a point with Cartesian coordinates (x_0, y_0, z_0) situated at the *centre* of a small rectangle $\delta C = ABCD$ of size $\delta_x \times \delta_y$, area $\delta S = \delta_x\,\delta_y$, parallel to the $x{-}y$ plane.



The line integral around δC is given by the sum of four terms

$$\begin{aligned}
\oint_{\delta C} \underline{a} \cdot \mathrm{d}\underline{r} &= \int_A^B \underline{a} \cdot \mathrm{d}\underline{r} + \int_B^C \underline{a} \cdot \mathrm{d}\underline{r} + \int_C^D \underline{a} \cdot \mathrm{d}\underline{r} + \int_D^A \underline{a} \cdot \mathrm{d}\underline{r} \\
&= \int_A^B \underline{a} \cdot \mathrm{d}\underline{r} - \int_C^B \underline{a} \cdot \mathrm{d}\underline{r} - \int_D^C \underline{a} \cdot \mathrm{d}\underline{r} + \int_D^A \underline{a} \cdot \mathrm{d}\underline{r}
\end{aligned}$$

Since $\underline{r} = x\,\underline{e}_x + y\,\underline{e}_y + z\,\underline{e}_z$ we have $\mathrm{d}\underline{r} = \underline{e}_x\,\mathrm{d}x$ along $D \to A$ and $C \to B$, and $\mathrm{d}\underline{r} = \underline{e}_y\,\mathrm{d}y$ along $A \to B$ and $D \to C$. Therefore

$$\oint_{\delta C} \underline{a} \cdot \mathrm{d}\underline{r} = \int_A^B a_y\,\mathrm{d}y - \int_C^B a_x\,\mathrm{d}x - \int_D^C a_y\,\mathrm{d}y + \int_D^A a_x\,\mathrm{d}x$$

For small δ_x and δ_y, we can Taylor expand the integrands about $y = y_0$,

$$\begin{aligned}
\int_D^A a_x\,\mathrm{d}x &= \int_D^A a_x(x, y_0 - \delta_y/2, z_0)\,\mathrm{d}x \\
&= \int_{x_0-\delta_x/2}^{x_0+\delta_x/2} \left[a_x(x, y_0, z_0) - \frac{\delta_y}{2} \frac{\partial a_x}{\partial y}\bigg|_{(x,y_0,z_0)} + O(\delta_y^2) \right] \mathrm{d}x \\
\int_C^B a_x\,\mathrm{d}x &= \int_C^B a_x(x, y_0 + \delta_y/2, z_0)\,\mathrm{d}x \\
&= \int_{x_0-\delta_x/2}^{x_0+\delta_y/2} \left[a_x(x, y_0, z_0) + \frac{\delta_y}{2} \frac{\partial a_x}{\partial y}\bigg|_{(x,y_0,z_0)} + O(\delta_y^2) \right] \mathrm{d}x
\end{aligned}$$

so

$$\begin{aligned}
\frac{1}{\delta S}\left[\int_D^A \underline{a} \cdot \mathrm{d}\underline{r} + \int_B^C \underline{a} \cdot \mathrm{d}\underline{r} \right] &= \frac{1}{\delta_x\,\delta_y}\left[\int_D^A a_x\,\mathrm{d}x - \int_C^B a_x\,\mathrm{d}x \right] \\
&= \frac{1}{\delta_x \delta_y} \int_{x_0-\delta_x/2}^{x_0+\delta_x/2} \left[-\delta_y \frac{\partial a_x}{\partial y}\bigg|_{(x,y_0,z_0)} + O(\delta_y^2) \right] \mathrm{d}x \\
&\to -\frac{\partial a_x}{\partial y}\bigg|_{(x_0,y_0,z_0)} \quad \text{as} \quad \delta_x,\, \delta_y \to 0
\end{aligned}$$

In the last step, as we take the limit $\delta_x \to 0$, the integrand tends to a constant in the region of integration:

$$\frac{\partial a_x}{\partial y}\bigg|_{(x,y_0,z_0)} \to \frac{\partial a_x}{\partial y}\bigg|_{(x_0,y_0,z_0)}$$

and the integral over x is then trivial.

A similar analysis of the line integrals along $A \to B$ and $C \to D$ gives (exercise)

$$\frac{1}{\delta S}\left[\int_A^B \underline{a} \cdot \mathrm{d}\underline{r} + \int_C^D \underline{a} \cdot \mathrm{d}\underline{r} \right] \to \frac{\partial a_y}{\partial x}\bigg|_{(x_0,y_0,z_0)} \quad \text{as} \quad \delta_x,\, \delta_y \to 0$$

Adding the results gives, for our line integral definition of curl,

$$\underline{e}_z \cdot (\underline{\nabla} \times \underline{a}) = (\underline{\nabla} \times \underline{a})_z = \left[\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right]_{(x_0,\,y_0,\,z_0)}$$

in agreement with our original definition in Cartesian coordinates.

The other components of $\underline{\nabla} \times \underline{a}$ can be obtained from similar rectangles parallel to the $y{-}z$ and $x{-}z$ planes, respectively.

It can be shown that $\underline{\nabla} \times \underline{a}$, when defined in this way, is independent of the *shape* of the infinitesimal area δS.

11.2 Physical/geometrical interpretation of curl

Consider a force field $\underline{F}(\underline{r})$, and let δC be a small rectangular contour which encloses an area δS in the $x-y$ plane – as in the line-integral definition of curl above.

The *work done* on a (point) test particle in moving it around the closed curve δC is

$$\delta W \;=\; \oint_{\delta C} \underline{F}\cdot \mathrm{d}\underline{r} \;=\; \textit{circulation} \text{ of } \underline{F}(\underline{r}) \text{ about } \delta C$$

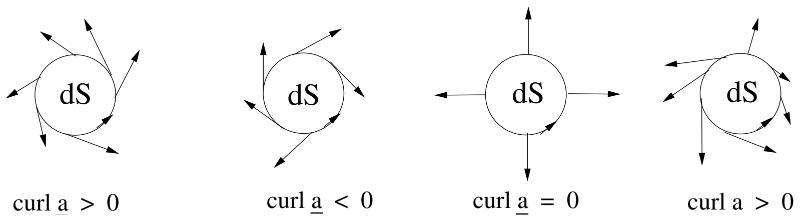
From the integral definition of curl, we know that for small δS

$$\oint_{\delta C} \underline{F}\cdot \mathrm{d}\underline{r} \;\approx\; \left(\underline{\nabla}\times\underline{F}\right)_z \,\delta S$$

Therefore $\left(\underline{\nabla}\times\underline{F}\right)_z \neq 0$ is equivalent to saying that a non-zero amount of work is done in moving the test particle around a small closed path in the $x-y$ plane.

Alternatively one can think of the non-zero circulation of $\underline{F}$ as causing a small test particle to *rotate* about its centre, with the axis of rotation in the direction of $\underline{\nabla}\times\underline{F}$.

More generally, $\underline{n}\cdot(\underline{\nabla}\times\underline{a})$ is a measure of the net *circulation* (per unit area) of the vector field $\underline{a}$ about an *infinitesimal* area $\mathrm{d}S$ with normal $\underline{n}$.

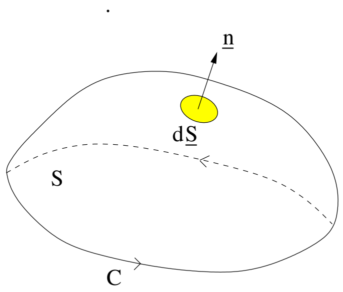


11.3 Stokes' theorem

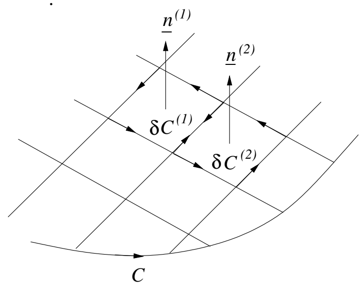
Let S be an *open* surface, bounded by a simple *closed* curve C, and let $\underline{a}$ be a vector field defined on S, then

$$\boxed{\int_S (\underline{\nabla}\times\underline{a})\cdot \mathrm{d}\underline{S} \;=\; \oint_C \underline{a}\cdot \mathrm{d}\underline{r}}$$

where C is traversed in a right-hand sense about $\mathrm{d}\underline{S}$. As usual, $\mathrm{d}\underline{S} = \underline{n}\,\mathrm{d}S$ where $\underline{n}$ is the unit normal to S.



Proof: Divide the surface S into N adjacent small surfaces. Let $\delta\underline{S}^{(i)} = \delta S^{(i)}\,\underline{n}^{(i)}$ be the vector element of area at $\underline{r}^{(i)}$, enclosed by the curve $\delta C^{(i)}$.



Start with the integral definition of curl

$$\underline{n}\cdot(\underline{\nabla}\times\underline{a}) \;=\; \lim_{\delta S\to 0}\frac{1}{\delta S}\oint_{\delta C}\underline{a}\cdot\mathrm{d}\underline{r}\,,$$

For a small but not infinitesimal open surface $\delta S^{(i)}$

$$\left(\underline{\nabla}\times\underline{a}\left(\underline{r}^{(i)}\right)\right)\cdot\underline{n}^{(i)} \;=\; \frac{1}{\delta S^{(i)}}\oint_{\delta C^{(i)}}\underline{a}\cdot\mathrm{d}\underline{r} \;+\; \epsilon^{(i)}$$

where $\epsilon^{(i)}\to 0$ as $\delta S^{(i)}\to 0$.

Multiply by $\delta S^{(i)}$ (before taking the limit), and sum over all i to get

$$\sum_{i=1}^{N}\left(\underline{\nabla}\times\underline{a}\left(\underline{r}^{(i)}\right)\right)\cdot\underline{n}^{(i)}\,\delta S^{(i)} \;=\; \sum_{i=1}^{N}\oint_{\delta C^{(i)}}\underline{a}\cdot\mathrm{d}\underline{r} \;+\; \sum_{i=1}^{N}\epsilon^{(i)}\,\delta S^{(i)}$$

Since each small closed curve $\delta C^{(i)}$ is traversed in the *same* sense, then, from the diagram, all contributions to $\sum_{i=1}^{N}\oint_{\delta C^{(i)}}\underline{a}\cdot\mathrm{d}\underline{r}$ *cancel*, except on those curves where part of $\delta C^{(i)}$ lies on the curve C. For example, the line integrals along the *common* section of the two small closed curves $\delta C^{(1)}$ and $\delta C^{(2)}$ in the figure *cancel exactly*. Therefore

$$\sum_{i=1}^{N}\oint_{\delta C^{(i)}}\underline{a}\cdot\mathrm{d}\underline{r} \;=\; \oint_C\underline{a}\cdot\mathrm{d}\underline{r}$$

Hence, as $N\to\infty$,

$$\oint_C\underline{a}\cdot\mathrm{d}\underline{r} \;=\; \int_S(\underline{\nabla}\times\underline{a})\cdot\mathrm{d}\underline{S} \;=\; \int_S\underline{n}\cdot(\underline{\nabla}\times\underline{a})\;\mathrm{d}S$$

Mathematical note: For those worried about the 'error term', note that, for finite N, we can establish an upper bound

$$\left|\sum_{i=1}^{N}\epsilon^{(i)}\,\delta S^{(i)}\right| \;\le\; S\,\max_i\left\{\left|\epsilon^{(i)}\right|\right\}$$

The RHS tends to zero in the limit $N\to\infty$, because S is finite and $\epsilon^{(i)}\to 0,\ \forall i$. A similar analysis works in the proof of the divergence theorem.[15]

[15]The case of an infinite surface S (or infinite V in the case of the divergence theorem) requires more effort.

11.4 Examples of the use of Stokes' theorem

Hemisphere: Given the vector field $\underline{a} = 4y\,\underline{e}_x + x\,\underline{e}_y + 2z\,\underline{e}_z$, verify Stokes' theorem for the (open) hemispherical surface $x^2 + y^2 + z^2 = R^2$ with $z > 0$.

In this case, we have $\underline{\nabla}\times\underline{a} = -3\,\underline{e}_z$, and we have shown previously that $\mathrm{d}\underline{S} = R^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\underline{e}_r$ on the surface of a (hemi)sphere of radius R. Direct integration then gives

$$\begin{aligned}\int_{S_C} \underline{\nabla}\times\underline{a}\cdot \mathrm{d}\underline{S} &= \int_{S_C} (-3\underline{e}_z)\cdot R^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\underline{e}_r \\ &= -3R^2 \int_0^{\pi/2} \sin\theta\cos\theta\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \;=\; -3\pi R^2\end{aligned}$$

We can check our result using Stokes' theorem. The closed curve C bounding the hemisphere is a circle of radius R in the $x-y$ plane. Parameterising this by $x = R\cos\phi$, $y = R\sin\phi$, $z = 0$, gives $\mathrm{d}x = -R\sin\phi\,\mathrm{d}\phi$, $\mathrm{d}y = R\cos\phi\,\mathrm{d}\phi$, and $a_x = 4y = 4R\sin\phi$, $a_y = x = R\cos\phi$, $a_z = 2z = 0$. Hence

$$\begin{aligned}\oint_C \underline{a}\cdot\mathrm{d}\underline{r} &= \oint_C (4y\,\mathrm{d}x + x\,\mathrm{d}y) \\ &= \int_0^{2\pi} \left(-4R^2\sin^2\phi + R^2\cos^2\phi\right)\mathrm{d}\phi \;=\; -3\pi R^2\end{aligned}$$

Planar areas: Consider a planar surface S parallel to the $x-y$ plane, bounded by a closed curve C, and let the vector field $\underline{a}(\underline{r})$ be

$$\underline{a} = \frac{1}{2}\left[-y\,\underline{e}_x + x\,\underline{e}_y\right]$$

In this case $\underline{\nabla}\times\underline{a} = \underline{e}_z$, and the vector element of area normal to the $x-y$ plane is $\mathrm{d}\underline{S} = \mathrm{d}S\,\underline{e}_z$. Hence

$$\int_S \underline{\nabla}\times\underline{a}\cdot\mathrm{d}\underline{S} = \int_S \underline{e}_z\cdot\mathrm{d}\underline{S} = \int_S \mathrm{d}S = S$$

We can then use Stokes' theorem to find the area of the surface

$$S = \oint_C \underline{a}\cdot\mathrm{d}\underline{r} = \frac{1}{2}\oint_C (-y\,\underline{e}_x + x\underline{e}_y)\cdot(\mathrm{d}x\,\underline{e}_x + \mathrm{d}y\,\underline{e}_y)$$

which gives

$$S = \frac{1}{2}\oint_C (x\,\mathrm{d}y - y\,\mathrm{d}x)$$

Example: Find the area inside the curve

$$x^{2/3} + y^{2/3} = 1\,.$$

The curve can be parameterised by $x = \cos^3\phi$, $y = \sin^3\phi$, for $0 \le \phi \le 2\pi$, so that

$$\frac{\mathrm{d}x}{\mathrm{d}\phi} = -3\cos^2\phi\,\sin\phi\,, \quad \frac{\mathrm{d}y}{\mathrm{d}\phi} = 3\sin^2\phi\,\cos\phi$$

which gives

$$\begin{aligned}S &= \frac{1}{2}\oint_C (x\,\mathrm{d}y - y\,\mathrm{d}x) \;=\; \frac{1}{2}\oint_C \left(x\frac{\mathrm{d}y}{\mathrm{d}\phi} - y\frac{\mathrm{d}x}{\mathrm{d}\phi}\right)\mathrm{d}\phi \\ &= \frac{1}{2}\int_0^{2\pi}\left(3\cos^4\phi\,\sin^2\phi + 3\sin^4\phi\,\cos^2\phi\right)\mathrm{d}\phi \\ &= \frac{3}{2}\int_0^{2\pi}\sin^2\phi\,\cos^2\phi\,\mathrm{d}\phi \;=\; \frac{3}{8}\int_0^{2\pi}\sin^2 2\phi\,\mathrm{d}\phi \;=\; \frac{3\pi}{8}\end{aligned}$$

11.5 Corollaries of Stokes' theorem

We may deduce several immediate consequences of Stokes' theorem,

$$\int_S (\underline{\nabla}\times\underline{a})\cdot\mathrm{d}\underline{S} = \oint_C \underline{a}\cdot\mathrm{d}\underline{r}$$

where C is the boundary (traversed in the anticlockwise direction) of the open surface S.

(i) If $\underline{a} = \underline{c}$, where $\underline{c}$ is a constant vector, then $\underline{\nabla}\times\underline{a} = 0$. Therefore $\underline{c}\cdot\oint_C \mathrm{d}\underline{r} = 0$, and because $\underline{c}$ is arbitrary, we have

$$\oint_C \mathrm{d}\underline{r} = 0$$

(ii) Take $\underline{a} = -Q(x,y)\,\underline{e}_x + P(x,y)\,\underline{e}_y$, and S to lie in the $x-y$ plane with area A. Then

$$\begin{aligned}\underline{\nabla}\times\underline{a} &= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\underline{e}_z \quad \text{and} \quad \mathrm{d}\underline{S} = \mathrm{d}x\,\mathrm{d}y\,\underline{e}_z \\ \underline{a}\cdot\mathrm{d}\underline{r} &= -Q\,\mathrm{d}x + P\,\mathrm{d}y\end{aligned}$$

so

$$\int_A \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = \oint_C (-Q\,\mathrm{d}x + P\,\mathrm{d}y)$$

which is again Green's theorem in the plane, sometimes known as Stokes' theorem in the plane or the two-dimensional divergence theorem.

Taking $P = x/2$ and $Q = y/2$ and gives the planar area result of the previous section. Indeed, Green's theorem is a generalisation of this result.

(iii) Applying Stokes' theorem to $\underline{a} = \phi\,\underline{c}$ where $\underline{c}$ is a constant vector, we have

$$\underline{\nabla}\times(\phi\,\underline{c}) = (\underline{\nabla}\phi)\times\underline{c} + \phi\,(\underline{\nabla}\times\underline{c}) = (\underline{\nabla}\phi)\times\underline{c} + 0$$

Hence

$$(\underline{\nabla}\times(\phi\,\underline{c}))\cdot\mathrm{d}\underline{S} = ((\underline{\nabla}\phi)\times\underline{c})\cdot\mathrm{d}\underline{S} = \underline{c}\cdot(\mathrm{d}\underline{S}\times\underline{\nabla}\phi)$$

which gives

$$\int_S (\underline{\nabla}\times(\phi\,\underline{c}))\cdot\mathrm{d}\underline{S} = \underline{c}\cdot\int_S \mathrm{d}\underline{S}\times\underline{\nabla}\phi = \underline{c}\cdot\oint_C \phi\,\mathrm{d}\underline{r}$$

This holds for all constant vectors $\underline{c}$, so

$$\oint_C \phi\,\mathrm{d}\underline{r} = \int_S \mathrm{d}\underline{S}\times\underline{\nabla}\phi = -\int_S \underline{\nabla}\phi\times\mathrm{d}\underline{S}$$

Such results are hard to remember, but as we have seen, they can be derived quite easily.

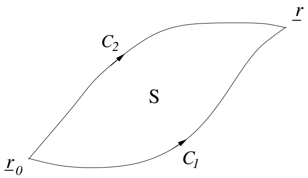
12 The scalar potential

A vector field $\underline{a}(\underline{r})$ is defined to be irrotational or *conservative* if its curl vanishes, *i.e.* if

$$\underline{\nabla} \times \underline{a} = 0$$

12.1 Path independence of line integrals for conservative fields

Let $\underline{\nabla} \times \underline{a} = 0$ everywhere in some region, and consider two (different) paths C_1 and C_2 from point $\underline{r}_0$ to point $\underline{r}$, say. Applying Stokes' theorem to the *open* surface S bounded by the *closed* path $C_1 - C_2$ gives



$$\int_S (\underline{\nabla} \times \underline{a}) \cdot d\underline{S} \;=\; 0 \;=\; \int_{C_1} \underline{a}(\underline{r}') \cdot d\underline{r}' \;-\; \int_{C_2} \underline{a}(\underline{r}') \cdot d\underline{r}'$$

where the $-$ve sign occurs in the second integral on the RHS because both paths are defined to go from $\underline{r}_0$ to $\underline{r}$. We use $\underline{r}'$ as integration variable to distinguish it from the integration *limits* $\underline{r}_0$ and $\underline{r}$.

Therefore, when $\underline{\nabla} \times \underline{a} = 0$ everywhere in S, we have

$$\int_{C_1} \underline{a}(\underline{r}') \cdot \mathrm{d}\underline{r}' \;=\; \int_{C_2} \underline{a}(\underline{r}') \cdot \mathrm{d}\underline{r}'$$

This is true for *any* S, and therefore for *any* paths C_1 and C_2 from $\underline{r}_0$ to $\underline{r}$.

Clearly, the converse is also true: if the line integral between two points is path independent, then the line integral around any closed curve (connecting the two points) is zero, and Stokes' theorem then gives $\underline{\nabla} \times \underline{a} = 0$. We just reverse the steps of the argument above.

Therefore

$$\boxed{\underline{\nabla} \times \underline{a} = 0 \quad \Leftrightarrow \quad \int_{\underline{r}_0}^{\underline{r}} \underline{a}(\underline{r}') \cdot \mathrm{d}\underline{r}' \text{ is path independent}}$$

12.2 Scalar potential for conservative vector fields

Since the line integral of a conservative vector field between two fixed points $\underline{r}_0$ and $\underline{r}$ is *path independent*, it can be a function *only* of the *end points* of the path. Hence there must exist a function $\phi(\underline{r})$ such that

$$\phi(\underline{r}) - \phi(\underline{r}_0) \;=\; \int_{\underline{r}_0}^{\underline{r}} \underline{a}(\underline{r}') \cdot \mathrm{d}\underline{r}' \,. \tag{25}$$

The scalar field $\phi(\underline{r})$ is called the *scalar potential* of the *vector field* $\underline{a}(\underline{r})$.

It is useful to invert this equation (and to give a more conventional result) by considering two neighbouring points $\underline{r}$ and $\underline{r} + \mathrm{d}\underline{r}$, for which

$$\begin{aligned}
\mathrm{d}\phi &= \phi(\underline{r} + \mathrm{d}\underline{r}) - \phi(\underline{r}) \\
&= \Big[\phi(\underline{r} + \mathrm{d}\underline{r}) - \phi(\underline{r}_0)\Big] - \Big[\phi(\underline{r}) - \phi(\underline{r}_0)\Big]
\end{aligned}$$

$$\begin{aligned}
&= \int_{\underline{r}_0}^{\underline{r}+\mathrm{d}\underline{r}} \underline{a}(\underline{r}') \cdot \mathrm{d}\underline{r}' - \int_{\underline{r}_0}^{\underline{r}} \underline{a}(\underline{r}') \cdot \mathrm{d}\underline{r}' && \text{(using equation (25))} \\
&= \int_{\underline{r}}^{\underline{r}+\mathrm{d}\underline{r}} \underline{a}(\underline{r}') \cdot \mathrm{d}\underline{r}' && \text{(along } \textit{any} \text{ path from } \underline{r} \text{ to } \underline{r} + \mathrm{d}\underline{r}) \\
&= \underline{a}(\underline{r}) \cdot \int_{\underline{r}}^{\underline{r}+\mathrm{d}\underline{r}} \mathrm{d}\underline{r}' \;+\; O(|\mathrm{d}\underline{r}|^2) && (\textit{choosing} \text{ the straight line from } \underline{r} \text{ to } \underline{r} + \mathrm{d}\underline{r}) \\
&= \underline{a}(\underline{r}) \cdot [(\underline{r} + \mathrm{d}\underline{r}) - \underline{r}] \;=\; \underline{a}(\underline{r}) \cdot \mathrm{d}\underline{r}
\end{aligned}$$

because $\underline{a}(\underline{r})$ is approximately constant between $\underline{r}$ and $\underline{r}+\mathrm{d}\underline{r}$, and the correction term $O(|\mathrm{d}\underline{r}|^2)$ can be ignored as $\mathrm{d}\underline{r} \to 0$.

But $\mathrm{d}\phi = \underline{\nabla}\,\phi \cdot \mathrm{d}\underline{r}$ (by definition), and so, since $\mathrm{d}\underline{r}$ is *arbitrary*, we must have

$$\underline{a}(\underline{r}) \;=\; \underline{\nabla}\,\phi(\underline{r})$$

The converse is much easier to prove. If $\underline{a} = \underline{\nabla}\phi$, then $\underline{\nabla} \times \underline{a} = \underline{\nabla} \times (\underline{\nabla}\phi) \equiv 0$.

Therefore

$$\boxed{\underline{\nabla} \times \underline{a} = 0 \quad \Leftrightarrow \quad \underline{a} = \underline{\nabla}\phi}$$

To determine whether a vector field is conservative, one simply checks whether $\underline{\nabla} \times \underline{a} = 0$ (in the region of interest).

NB: The scalar potential $\phi(\underline{r})$ is only determined up to a *constant.* If $\psi = \phi + \textit{constant}$ then $\underline{\nabla}\,\psi = \underline{\nabla}\,\phi$, so ψ is an equally good potential. The freedom in the constant corresponds to the freedom in choosing $\underline{r}_0$ when calculating the potential. So $\phi(\underline{r}_0)$ in equation (25) is just an irrelevant constant. Equivalently, the absolute value of a scalar potential has no meaning, only *potential differences* are significant.

12.3 Finding scalar potentials

Method (1): Integration along a straight line

We have shown that the scalar potential $\phi(\underline{r})$ for a *conservative* vector field $\underline{a}(\underline{r})$ can be constructed from a line integral which is *independent* of the path of integration between the endpoints. A convenient way of evaluating such integrals is to integrate along a *straight line* from $\underline{r}_0$ to $\underline{r}$. Depending on the convergence of the integral, there are two standard choices:

(i) $\underline{r}_0 = 0$: If $\phi(\underline{r})$ is *finite* or *zero* at $\underline{r} = 0$, we parameterise the straight line by $\underline{r}' = \lambda\,\underline{r}$ with $0 \le \lambda \le 1$. Thus $\mathrm{d}\underline{r}' = \mathrm{d}\lambda\,\underline{r}$, and hence

$$\phi(\underline{r}) \;=\; \int_0^{\underline{r}} \underline{a}(\underline{r}') \cdot \mathrm{d}\underline{r}' \;=\; \int_{\lambda=0}^{\lambda=1} \underline{a}(\lambda\underline{r}) \cdot \underline{r}\,\mathrm{d}\lambda\,,$$

(ii) $|\underline{r}_0| = \infty$: If $\phi(\underline{r})$ is *finite* or *zero* as $|\underline{r}| \to \infty$, we again parameterise the straight line by $\underline{r}' = \lambda\,\underline{r}$, but this time with $1 \le \lambda < \infty$. Again, we have $\mathrm{d}\underline{r}' = \mathrm{d}\lambda\,\underline{r}$, and hence

$$\phi(\underline{r}) \;=\; \int_\infty^{\underline{r}} \underline{a}(\underline{r}') \cdot \mathrm{d}\underline{r}' \;=\; \int_{\lambda=\infty}^{\lambda=1} \underline{a}(\lambda\underline{r}) \cdot \underline{r}\,\mathrm{d}\lambda\,,$$

Example 1: Let $\underline{a}(\underline{r}) = (2xy + z^3)\underline{e}_x + x^2\underline{e}_y + 3xz^2\underline{e}_z$.

First check that $\underline{\nabla} \times \underline{a} = 0$, so the field is conservative (exercise). Then

$$\begin{aligned}
\phi(\underline{r}) &= \int_0^{\underline{r}} \underline{a}(\underline{r}')\cdot \mathrm{d}\underline{r}' \;=\; \int_0^1 \underline{a}(\lambda\underline{r})\cdot\underline{r}\,\mathrm{d}\lambda \\
&= \int_0^1 \left[\left(2\lambda^2 xy + \lambda^3 z^3\right)x + \left(\lambda^2 x^2\right)y + \left(\lambda^3\, 3xz^2\right)z\right]\mathrm{d}\lambda \\
&= \frac{2}{3}x^2y + \frac{1}{4}xz^3 + \frac{1}{3}x^2y + \frac{3}{4}xz^3 \\
&= x^2y + xz^3
\end{aligned}$$

NB: Always check that your potential $\phi(\underline{r})$ satisfies $\underline{a}(\underline{r}) = \underline{\nabla}\,\phi(\underline{r})$. (Exercise)

Example 2: Let $\underline{a}(\underline{r}) = 2\,(\underline{c}\cdot\underline{r})\,\underline{r} + r^2\,\underline{c}$ where $\underline{c}$ is a constant vector.

$$\underline{\nabla}\times\underline{a} \;=\; 2\,\left[\underline{\nabla}\,(\underline{c}\cdot\underline{r})\times\underline{r} + (\underline{c}\cdot\underline{r})\,\underline{\nabla}\times\underline{r}\right] + \left(\underline{\nabla}\,r^2\right)\times\underline{c} \;=\; 2\left[\underline{c}\times\underline{r} + 0\right] + 2\,\underline{r}\times\underline{c} \;=\; 0$$

Then

$$\begin{aligned}
\phi(\underline{r}) &= \int_0^{\underline{r}} \underline{a}(\underline{r}')\cdot\mathrm{d}\underline{r}' \;=\; \int_0^1 \underline{a}\,(\lambda\,\underline{r})\cdot(\mathrm{d}\lambda\,\underline{r}) \\
&= \int_0^1 \left(2\,(\underline{c}\cdot\lambda\,\underline{r})\,\lambda\,\underline{r} \;+\; \lambda^2 r^2\,\underline{c}\right)\cdot\underline{r}\,\mathrm{d}\lambda \;=\; \left(2\,(\underline{c}\cdot\underline{r})\,\underline{r}\cdot\underline{r} \;+\; r^2\,(\underline{c}\cdot\underline{r})\right)\int_0^1 \lambda^2\,\mathrm{d}\lambda \\
&= r^2\,(\underline{c}\cdot\underline{r})
\end{aligned}$$

Integration along a straight line is a straightforward and fairly elegant method, and it's generally applicable.

Method (2): Direct integration

Since $\underline{a} = \underline{\nabla}\phi$, we have

$$\frac{\partial\phi}{\partial x} = a_x(x,y,z) \qquad \frac{\partial\phi}{\partial y} = a_y(x,y,z) \qquad \frac{\partial\phi}{\partial z} = a_z(x,y,z)$$

We can integrate these equations separately to give

$$\begin{aligned}
\phi(x,y,z) &= \int^x a_x(x',y,z)\,dx' + f(y,z) \\
\phi(x,y,z) &= \int^y a_y(x,y',z)\,dy' + g(x,z) \\
\phi(x,y,z) &= \int^z a_z(x,y,z')\,dz' + h(x,y)
\end{aligned}$$

and then determine the "constants" of integration $f(y,z)$, $g(x,z)$ and $h(x,y)$ by consistency.

Example 1 (revisited): Let $\underline{a} = (2xy + z^3)\underline{e}_x + x^2\underline{e}_y + 3xz^2\underline{e}_z$. Then

$$\begin{aligned}
\phi &= x^2y + xz^3 + f(y,z) \\
\phi &= x^2y + g(x,z) \\
\phi &= xz^3 + h(x,y)
\end{aligned}$$

These agree if we choose $f(y,z) = 0$, $g(x,z) = xz^3$ and $h(x,y) = x^2y$, hence

$$\phi(\underline{r}) \;=\; x^2y + xz^3$$

as before. This method is straightforward but it's rather clumsy for problems such as Example 2, which is typical of many Physics applications.

Method (3): Direct integration "by inspection" (guessing)

Sometimes the result can be spotted directly.

For example, if $\underline{a}(\underline{r}) = (\underline{c}\cdot\underline{r})\underline{c}$ where $\underline{c}$ is a constant vector, then

$$\underline{a}(\underline{r}) \;=\; (\underline{c}\cdot\underline{r})\,\underline{c} \;=\; (\underline{c}\cdot\underline{r})\,\underline{\nabla}\,(\underline{c}\cdot\underline{r}) \;=\; \underline{\nabla}\left(\frac{1}{2}\,(\underline{c}\cdot\underline{r})^2 \;+\; \text{constant}\right)$$

Example 2 (revisited)

$$\underline{a}(\underline{r}) \;=\; 2\,(\underline{c}\cdot\underline{r})\,\underline{r} \;+\; r^2\,\underline{c} \;=\; (\underline{c}\cdot\underline{r})\underline{\nabla}\,r^2 \;+\; r^2\,\underline{\nabla}\,(\underline{c}\cdot\underline{r}) \;=\; \underline{\nabla}\left((\underline{c}\cdot\underline{r})\,r^2 \;+\; \text{constant}\right)$$

in agreement with what we had before if we choose the integration constant to be zero.

12.4 Conservative forces: conservation of energy

We now show how the name *conservative field* arises in Physics. Let the vector field $\underline{F}(\underline{r})$ (assumed time-independent) be the total force acting on a particle of mass m at position $\underline{r}$. We will show that for a conservative/irrotational force, where we can write

$$\underline{F}(\underline{r}) = -\underline{\nabla}\,V(\underline{r})\,,$$

the total energy is *constant* in time. Note that the *force* is *minus* the gradient of the (scalar) potential. The minus sign is conventional.

Proof: Let $\underline{r}(t)$ be the position vector of a particle at time t. Denote the first and second derivatives of $\underline{r}$ with respect to time by $\dot{\underline{r}}$ (velocity) and $\ddot{\underline{r}}$ (acceleration) respectively.

The particle moves under the influence of Newton's second law (N2):

$$m\ddot{\underline{r}} = \underline{F}(\underline{r})$$

In time $\mathrm{d}t$ the particle moves from $\underline{r}$ to $\underline{r} + \mathrm{d}\underline{r}$. From N2, we get

$$m\,\ddot{\underline{r}}\cdot\mathrm{d}\underline{r} \;=\; \underline{F}(\underline{r})\cdot\mathrm{d}\underline{r} \;=\; -\,\underline{\nabla}\,V(\underline{r})\cdot\mathrm{d}\underline{r}$$

Integrating this expression along the path of the particle starting from $\underline{r}_A$ at time t_A, to $\underline{r}_B$ at time t_B, gives

$$m \int_{\underline{r}_A}^{\underline{r}_B} \ddot{\underline{r}} \cdot \mathrm{d}\underline{r} \;=\; -\int_{\underline{r}_A}^{\underline{r}_B} \underline{\nabla}\, V(\underline{r}) \cdot \mathrm{d}\underline{r} \tag{26}$$

We can evaluate the left-hand side of equation (26)

$$m \int_{\underline{r}_A}^{\underline{r}_B} \ddot{\underline{r}} \cdot \mathrm{d}\underline{r} \;=\; m \int_{t_A}^{t_B} \ddot{\underline{r}} \cdot \frac{\mathrm{d}\underline{r}}{dt}\,\mathrm{d}t \;=\; m \int_{t_A}^{t_B} \frac{1}{2}\frac{d}{dt}\left(\dot{\underline{r}} \cdot \dot{\underline{r}}\right)\mathrm{d}t \;=\; \frac{1}{2} m \Big[\, |\dot{\underline{r}}|^2 \Big]_{t_A}^{t_B} \;=\; \frac{1}{2} m \left(v_B^2 - v_A^2\right),$$

where v_A and v_B are the magnitudes of the particle's velocity at points A and B respectively.

The right-hand side of equation (26) gives

$$-\int_{\underline{r}_A}^{\underline{r}_B} \underline{\nabla}\, V(\underline{r}) \cdot \mathrm{d}\underline{r} \;=\; -\int_{\underline{r}_A}^{\underline{r}_B} dV = V_A - V_B$$

where V_A and V_B are the values of the potential V at $\underline{r}_A$ and $\underline{r}_B$, respectively. Therefore

$$\frac{1}{2} m \left(v_B^2 - v_A^2\right) \;=\; V_A - V_B$$

Rearranging, we get

$$\frac{1}{2} m v_A^2 + V_A \;=\; \frac{1}{2} m v_B^2 + V_B$$

Hence the *total energy*, defined as $E \equiv \frac{1}{2} m v^2 + V$, is *conserved* – it's *constant in time.*

(Choosing $\underline{F} = +\underline{\nabla}\, V$ would lead to $E \equiv \frac{1}{2} m v^2 - V$, a less desirable convention.)

Examples: *Newtonian gravity* and the *electrostatic force* are both conservative. *Frictional forces* are not conservative: energy is dissipated and work is done in traversing a closed path. In general, time-dependent forces are not conservative.

We now return to where we started in section (1.3).

12.5 Gravitation and Electrostatics (revisited)

The foundation of Newtonian Gravity is *Newton's Law of Gravitation.* The force $\underline{F}(\underline{r})$ on a particle of mass m_1 at $\underline{r}$ due to a particle of mass m situated at the origin is given (in SI units) by

$$\underline{F}(\underline{r}) \;=\; -\,G m m_1 \frac{\underline{r}}{r^3}$$

where $G = 6.672\,59(85) \times 10^{-11}\, Nm^2kg^2$ is Newton's Gravitational Constant.

The *gravitational field* $\underline{G}(\underline{r})$ due to the mass at the origin is *defined* by

$$\underline{F}(\underline{r}) \;\equiv\; m_1\, \underline{G}(\underline{r}) \qquad \text{or} \qquad \underline{G}(\underline{r}) \;=\; -\,G\, m\, \frac{\underline{r}}{r^3} \tag{27}$$

where the test mass m_1 is so small that its gravitational field can be ignored. The gravitational field is conservative because

$$\underline{\nabla} \times \left(\frac{\underline{r}}{r^3}\right) \;=\; \underline{\nabla}\left(\frac{1}{r^3}\right) \times \underline{r} \;+\; \frac{1}{r^3}\left(\underline{\nabla} \times \underline{r}\right) \;=\; \left(-\frac{3\underline{r}}{r^5}\right) \times \underline{r} \;+\; 0 \;=\; 0$$

The *gravitational potential* defined by

$$\underline{G} = -\underline{\nabla}\, \phi$$

can be obtained from equation (27) by spotting the direct integration, $\underline{\nabla}\,(1/r) = -\underline{r}/r^3$, giving

$$\phi = -\frac{Gm}{r}$$

Alternatively, we may evaluate it explicitly by a line integral. Choosing $\underline{r}_0$ at infinity gives

$$\begin{aligned} \phi(\underline{r}) \;&=\; -\int_{\underline{r}_0}^{\underline{r}} \underline{G}(\underline{r}') \cdot \mathrm{d}\underline{r}' \;=\; -\int_{\infty}^{1} \underline{G}(\lambda \underline{r}) \cdot \mathrm{d}\lambda \underline{r} \\ &=\; (-)^2 \int_{\infty}^{1} \frac{Gm\,(\underline{r} \cdot \underline{r})}{r^3} \frac{\mathrm{d}\lambda}{\lambda^2} \;=\; -\,\frac{Gm}{r} \end{aligned}$$

Note: In this example, the vector field $\underline{G}$ is *singular* at the origin $\underline{r} = 0$. This implies that we have to exclude the origin, so it's not possible to obtain the scalar potential at $\underline{r}$ by integration along a path from the origin. Instead we integrate from infinity, which in turn means that the gravitational potential at infinity is zero.

Note: Since $\underline{F} = m_1\, \underline{G} = -\underline{\nabla}\,(m_1\, \phi)$, the *potential energy* of the mass m_1 is $V(\underline{r}) = m_1 \phi(\underline{r})$. The distinction (a convention) between potential and potential energy is a common source of confusion.

Electrostatics: Coulomb's Law states that the force $\underline{F}(\underline{r})$ on a particle of charge q_1 situated at $\underline{r}$ in the electric field $\underline{E}(\underline{r})$ due to a particle of charge q situated at the origin is given (in SI units) by

$$\underline{F} \;=\; q_1\, \underline{E} \;=\; \frac{q_1\, q}{4\pi\epsilon_0}\, \frac{\underline{r}}{r^3}\,,$$

where $\epsilon_0 = 10^7/(4\pi c^2) = 8.854\,187\,817\ldots \times 10^{-12}\, C^2N^{-1}m^{-2}$ is called the *permittivity of free space.* Again the test charge q_1 is taken as small, so as not to disturb the electric field.

The *electrostatic potential* may be obtained by inspection, or by integrating $E = -\underline{\nabla}\,\phi$ from infinity to $\underline{r}$,

$$\phi(\underline{r}) \;=\; \frac{q}{4\pi\epsilon_0 r} \tag{28}$$

The *potential energy* of a charge q_1 in the electric field is $V(\underline{r}) = q_1\phi(\underline{r})$.

Note that electrostatics and gravitation are very similar mathematically, the only real difference being that the gravitational force between two masses is always attractive, whereas like charges repel.

12.6 The equations of Poisson and Laplace

In section (10.6), we derived Gauss' Law and Maxwell's first equation (ME1) for the electrostatic field

$$\int_S \underline{E}\cdot d\underline{S} \;=\; \frac{Q}{\epsilon_0} \qquad \text{and} \qquad \underline{\nabla}\cdot\underline{E}(\underline{r}) \;=\; \frac{\rho(\underline{r})}{\epsilon_0}$$

where $\rho(\underline{r})$ is the charge density at $\underline{r}$, and $Q = \int_V \rho(\underline{r})\,dV$ is the total charge in volume V.

Writing $\underline{E}(\underline{r}) = -\underline{\nabla}\,\phi(\underline{r})$ and using Maxwell's first equation gives *Poisson's equation*

$$\nabla^2\phi \;=\; -\frac{\rho}{\epsilon_0}$$

If $\rho(\underline{r}) = 0$ everywhere in some region, we have

$$\nabla^2\phi = 0$$

which is *Laplace's equation.*

These *partial differential equations* are important in many branches of Physics and Mathematics. You will study (and solve) them next year.

13 The vector potential

We have shown that an *irrotational* vector field $\underline{a}(\underline{r})$, *i.e.* one that satisfies $\underline{\nabla}\times\underline{a} = 0$, can be written as the *gradient* of a *scalar* field, $\underline{a} = \underline{\nabla}\,\phi$.

Under what conditions can we write a vector field $\underline{B}(\underline{r})$ as the *curl* of a *vector* field $\underline{A}(\underline{r})$?

(i) If $\underline{\nabla}\cdot\underline{B} = 0$, it can be shown that a vector field $\underline{A}$ can be found such that $\underline{B} = \underline{\nabla}\times\underline{A}$.

(ii) The converse is easy to prove. If the field $\underline{B}$ can be written as $\underline{B} = \underline{\nabla}\times\underline{A}$, then $\underline{\nabla}\cdot\underline{B} = \underline{\nabla}\cdot(\underline{\nabla}\times\underline{A}) = 0$ because 'div curl' is always zero.

Hence

$$\underline{\nabla}\cdot\underline{B}(\underline{r}) \;=\; 0 \qquad \Leftrightarrow \qquad \text{There exists a field } \underline{A}(\underline{r}) \text{ such that } \underline{B}(\underline{r}) \;=\; \underline{\nabla}\times\underline{A}(\underline{r})$$

The field $\underline{A}$ is called the *vector potential* for the *solenoidal* field $\underline{B}$.

For such a field $\underline{B}$, which is finite or zero at the origin, it can be shown that

$$\underline{A}(\underline{r}) \;=\; -\,\underline{r}\times\int_0^1 \underline{B}(\lambda\underline{r})\,\lambda\,d\lambda \tag{29}$$

is a vector potential for $\underline{B}(\underline{r})$.

Example: Find a vector potential $\underline{A}$ for the field $\underline{B} = \underline{c}\times\underline{r}$, where $\underline{c}$ is a constant vector.

It is easy to show that $\underline{\nabla}\cdot\underline{B} = 0$ (exercise). Equation (29) then gives

$$\begin{aligned} \underline{A}(\underline{r}) \;&=\; -\underline{r}\times\int_0^1 \underline{B}(\lambda\underline{r})\,\lambda\,d\lambda \;=\; -\,\underline{r}\times\int_0^1 (\underline{c}\times\lambda\underline{r})\,\lambda\,d\lambda \\ &=\; -\Big(r^2\,\underline{c} - (\underline{r}\cdot\underline{c})\,\underline{r}\Big)\int_0^1 \lambda^2\,d\lambda \;=\; \frac{1}{3}\Big((\underline{r}\cdot\underline{c})\,\underline{r} - r^2\,\underline{c}\Big) \end{aligned}$$

You should always check at the end that $\underline{A}$ satisfies $\underline{B} = \underline{\nabla}\times\underline{A}$ (exercise).

Gauge invariance: We can always add the *gradient* of an *arbitrary* scalar field $f(\underline{r})$ to the vector potential

$$\underline{A}(\underline{r}) \;\to\; \underline{A}'(\underline{r}) \;=\; \underline{A}(\underline{r}) + \underline{\nabla}\,f(\underline{r})$$

without changing $\underline{B}$.

We have $\underline{\nabla}\times\underline{A}' = \underline{\nabla}\times\underline{A} + 0$ because $\underline{\nabla}\times\underline{\nabla}\,f = 0$ for *any* scalar field ('curl grad' is always zero). This is called *gauge invariance* in electromagnetism, and is one of the most important symmetries in Physics. (*Electroweak gauge invariance* is (partly) broken by the Higgs field.)

13.1 Physical examples of vector potentials

Magnetism: Magnetic field lines do not have sources or sinks – the lines of a magnetic field are continuous. For example, the magnetic field lines around a straight current-carrying wire are *circles.* It can be shown that the magnetic field $\underline{B}$ satisfies $\underline{\nabla}\cdot\underline{B} = 0$ everywhere, so we can write $\underline{B} = \underline{\nabla}\times\underline{A}$, where $\underline{A}$ is the *magnetic vector potential* (tutorial).

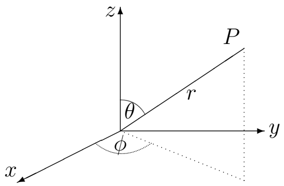
Fluid mechanics: For incompressible fluids (which have constant mass density, ρ) with no sources or sinks, we showed in section (10.5) that the *velocity field* $\underline{v}(\underline{r})$ satisfies $\underline{\nabla}\cdot\underline{v} = 0$. In this case, there exists a velocity potential $\underline{\Psi}(\underline{r})$ such that $\underline{v} = \underline{\nabla}\times\underline{\Psi}$.

14 Orthogonal curvilinear coordinates

As we have seen, it is often convenient to work with coordinate systems other than Cartesian coordinates $\{x_i\}$, *i.e.* $(x_1,\, x_2,\, x_3)$ or $(x,\, y,\, z)$.

For example, spherical polar coordinates $(r,\, \theta,\, \phi)$ are defined by:

$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
$$



We shall set up a formalism to deal with rather general coordinate systems, of which spherical polars are a very important example.

Suppose we make a transformation from the Cartesian coordinates (x_1, x_2, x_3) to the variables $(u_1,\, u_2,\, u_3)$, which are functions of the $\{x_i\}$

$$
\begin{aligned}
u_1 &= u_1(x_1,\, x_2,\, x_3) \\
u_2 &= u_2(x_1,\, x_2,\, x_3) \\
u_3 &= u_3(x_1,\, x_2,\, x_3)
\end{aligned}
$$

If the variables $\{u_i\}$ are single-valued functions of the variables $\{x_i\}$, then we can make the inverse transformations,

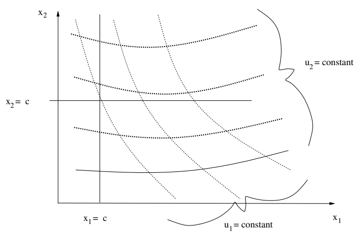
$$x_i \;=\; x_i(u_1,\, u_2,\, u_3) \quad \text{for } i = 1, 2, 3,$$

except possibly at certain points.

A point may be specified by its Cartesian coordinates $\{x_i\}$, or its *curvilinear coordinates* $\{u_i\}$.

We may define the the curvilinear coordinates by equations giving $\{x_i\}$ as functions of $\{u_i\}$, or vice-versa.[16]

- For Cartesian coordinates, the surfaces 'x_i = constant' $(i = 1, 2, 3)$ are *planes*, with (constant) normal vectors $\underline{e}_i$ (the Cartesian basis vectors) *intersecting at right angles.*
- For curvilinear coordinates, the surfaces 'u_i = constant' do *not*, in general, have constant normal vectors, nor do they intersect at right angles. For example, in 2-D, we might have



[16]Sometimes the curvilinear coordinates are called (u, v, w), just as Cartesians are called (x, y, z).

From the definition of spherical polar coordinates $(r,\, \theta,\, \phi)$, we have

$$r \;=\; \sqrt{x^2+y^2+z^2} \qquad \theta \;=\; \cos^{-1}\left\{\frac{z}{\sqrt{x^2+y^2+z^2}}\right\} \qquad \phi \;=\; \tan^{-1}\left(\frac{y}{x}\right) .$$

The surfaces of constant r, θ, and ϕ are

$$
\begin{aligned}
r &= \text{constant} \;\Rightarrow\; \text{spheres centred at the origin} \\
\theta &= \text{constant} \;\Rightarrow\; \text{cones of semi-angle } \theta \text{ and axis along the } z\text{-axis} \\
\phi &= \text{constant} \;\Rightarrow\; \text{planes passing through the } z\text{-axis}
\end{aligned}
$$

Not all of these surfaces are planes, but they *do* intersect at right angles.

14.1 Orthogonal curvilinear coordinates

If the coordinate surfaces (surfaces of constant u_i), intersect at right angles, as in the above example of spherical polars, the curvilinear coordinates are said to be *orthogonal*.

Scale factors and basis vectors: Suppose the point P has position vector $\underline{r} = \underline{r}(u_1, u_2, u_3)$.

If we change curvilinear coordinate u_1 by $\mathrm{d}u_1$ (with u_2 and u_3 fixed), then $\underline{r} \to \underline{r} + \mathrm{d}\underline{r}$, with

$$\mathrm{d}\underline{r} \;=\; \frac{\partial \underline{r}}{\partial u_1}\,\mathrm{d}u_1 \;\equiv\; h_1\, \underline{e}_1\, \mathrm{d}u_1$$

where we have defined the *scale factor* h_1 and the *unit vector* $\underline{e}_1$ by

$$h_1 \;=\; \left|\frac{\partial \underline{r}}{\partial u_1}\right| \quad \text{and} \quad \underline{e}_1 \;=\; \frac{1}{h_1}\frac{\partial \underline{r}}{\partial u_1}$$

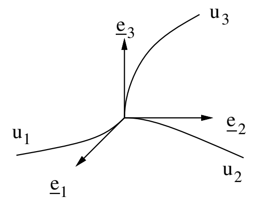
- The *scale factor* h_1 gives the *length* $h_1\mathrm{d}u_1$ of $\mathrm{d}\underline{r}$ when we change $u_1 \to u_1 + \mathrm{d}u_1$.
- $\underline{e}_1$ is a *unit vector* in the direction of increasing u_1 (with *fixed* u_2 and u_3.)

Similarly, we can define h_i and $\underline{e}_i$ for $i = 2$ and 3.

In general, if we change a single u_i, keeping the other two fixed, we have

$$\frac{\partial \underline{r}}{\partial u_i} \;=\; h_i\, \underline{e}_i \qquad i = 1, 2, 3$$

- The unit vectors $\{\underline{e}_i\}$ are in general *not* constant vectors – their directions depend on the position vector $\underline{r}$, and hence on the curvilinear coordinates $\{u_i\}$. [They should perhaps be called $\{\underline{e}_{u_i}\}$ or $\{\underline{e}_u, \underline{e}_v, \underline{e}_w\}$ to avoid confusion with Cartesian basis vectors.]
- If the curvilinear unit vectors satisfy $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$, the $\{u_i\}$ are said to be *orthogonal curvilinear coordinates*, and the three unit vectors $\{\underline{e}_i\}$ form an orthonormal basis.



14.1.1 Examples of orthogonal curvilinear coordinates (OCCs)

Cartesian coordinates:

$$\underline{r} \; = \; x\,\underline{e}_x + y\,\underline{e}_y + z\,\underline{e}_z \qquad \Rightarrow \qquad h_x\,\underline{e}_x \; = \; \frac{\partial \underline{r}}{\partial x} \; = \; \underline{e}_x\,, \; etc.$$

The scale factors are all *unity*, and the unit vectors point in the *same* direction everywhere.

Spherical polar coordinates: $u_1 = r\,,\; u_2 = \theta\,,\; u_3 = \phi$ (in that order)

$$\underline{r} \; = \; r\,\sin\theta\,\cos\phi\,\underline{e}_x + r\,\sin\theta\,\sin\phi\,\underline{e}_y + r\,\cos\theta\,\underline{e}_z$$

$$\frac{\partial \underline{r}}{\partial r} \; = \; \sin\theta\,\cos\phi\,\underline{e}_x + \sin\theta\,\sin\phi\,\underline{e}_y + \cos\theta\,\underline{e}_z \qquad \Rightarrow \quad h_r \; = \; \left|\frac{\partial \underline{r}}{\partial r}\right| \; = \; 1$$

$$\frac{\partial \underline{r}}{\partial \theta} \; = \; r\,\cos\theta\,\cos\phi\,\underline{e}_x + r\,\cos\theta\,\sin\phi\,\underline{e}_y - r\,\sin\theta\,\underline{e}_z \qquad \Rightarrow \quad h_\theta \; = \; \left|\frac{\partial \underline{r}}{\partial \theta}\right| \; = \; r$$

$$\frac{\partial \underline{r}}{\partial \phi} \; = \; -r\,\sin\theta\,\sin\phi\,\underline{e}_x + r\,\sin\theta\,\cos\phi\,\underline{e}_y \qquad \Rightarrow \quad h_\phi \; = \; \left|\frac{\partial \underline{r}}{\partial \phi}\right| \; = \; r\,\sin\theta$$

Hence the unit vectors for spherical polars are

$$\begin{aligned} \underline{e}_r &= \sin\theta\cos\phi\,\underline{e}_x + \sin\theta\sin\phi\,\underline{e}_y + \cos\theta\,\underline{e}_z \; = \; \underline{r}\,/\,r \\ \underline{e}_\theta &= \cos\theta\cos\phi\,\underline{e}_x + \cos\theta\sin\phi\,\underline{e}_y - \sin\theta\,\underline{e}_z \\ \underline{e}_\phi &= -\sin\phi\,\underline{e}_x + \cos\phi\,\underline{e}_y \end{aligned}$$

These unit vectors are normal to the surfaces described above (spheres, cones and planes).

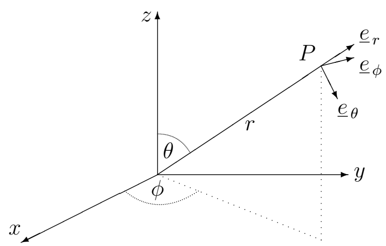
They are orthogonal:

$$\underline{e}_r \cdot \underline{e}_\theta \; = \; \underline{e}_r \cdot \underline{e}_\phi \; = \; \underline{e}_\theta \cdot \underline{e}_\phi \; = \; 0$$

And they form a right-handed orthonormal basis:

$$\underline{e}_r \times \underline{e}_\theta = \underline{e}_\phi\,, \quad \underline{e}_\theta \times \underline{e}_\phi = \underline{e}_r\,, \quad \underline{e}_\phi \times \underline{e}_r = \underline{e}_\theta\,.$$

See also tutorial question (7.6).



Cylindrical coordinates: $u_1 = \rho\,,\; u_2 = \phi\,,\; u_3 = z$ (in that order)

$$\underline{r} \; = \; \rho\,\cos\phi\,\underline{e}_x + \rho\,\sin\phi\,\underline{e}_y + z\,\underline{e}_z$$

$$\Rightarrow \quad \frac{\partial \underline{r}}{\partial \rho} \; = \; \cos\phi\,\underline{e}_x + \sin\phi\,\underline{e}_y \qquad \frac{\partial \underline{r}}{\partial \phi} \; = \; -\,\rho\,\sin\phi\,\underline{e}_x + \rho\,\cos\phi\,\underline{e}_y \qquad \frac{\partial \underline{r}}{\partial z} \; = \; \underline{e}_z$$

The scale factors are then (tutorial) $h_\rho = 1\,,\; h_\phi = \rho\,,\; h_z = 1\,,$ and the basis vectors are

$$\underline{e}_\rho \; = \; \cos\phi\,\underline{e}_x + \sin\phi\,\underline{e}_y \qquad \underline{e}_\phi \; = \; -\sin\phi\,\underline{e}_x + \cos\phi\,\underline{e}_y \qquad \underline{e}_z \; = \; \underline{e}_z$$

These unit vectors are normal to surfaces which are (respectively): cylinders centred on the z-axis (ρ = constant), planes through the z-axis (ϕ = constant), planes perpendicular to the z axis (z = constant), and they are clearly orthonormal.

14.2 Elements of length, area and volume in OCCs

Length: If we change $u_1 \to u_1 + \mathrm{d}u_1$, keeping u_2 and u_3 *fixed*, then $\underline{r} \to \underline{r} + \mathrm{d}\underline{r}_1$ where $\mathrm{d}\underline{r}_1 = h_1\,\underline{e}_1\,\mathrm{d}u_1$. The infinitesimal element of length along $\underline{e}_1$ is $h_1\,\mathrm{d}u_1$.

Clearly, the infinitesimal elements of length along the three curvilinear basis vectors $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$, respectively, are

$$h_1\,\mathrm{d}u_1 \qquad h_2\,\mathrm{d}u_2 \qquad h_3\,\mathrm{d}u_3$$

If we change all three of the $\{u_i\}$, then

$$\mathrm{d}\underline{r} \; = \; h_1\,\mathrm{d}u_1\,\underline{e}_1 + h_2\,\mathrm{d}u_2\,\underline{e}_2 + h_3\,\mathrm{d}u_3\,\underline{e}_3$$

Arc length: If $\mathrm{d}s$ is the length of the infinitesimal vector $\mathrm{d}\underline{r}$, then $(\mathrm{d}s)^2 = \mathrm{d}\underline{r} \cdot \mathrm{d}\underline{r}$.

In Cartesian coordinates

$$(\mathrm{d}s)^2 \; = \; (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2$$

In curvilinear coordinates, if we change the i^{th} coordinate u_i by $\mathrm{d}u_i$, then[17]

$$(\mathrm{d}s)^2 \; = \; \mathrm{d}\underline{r} \cdot \mathrm{d}\underline{r} \; = \; \left(\sum_i h_i\,\underline{e}_i\,\mathrm{d}u_i\right) \cdot \left(\sum_j h_j\,\underline{e}_j\,\mathrm{d}u_j\right) \; = \; \sum_{ij} h_i\,h_j\,\left(\underline{e}_i \cdot \underline{e}_j\right)\,\mathrm{d}u_i\,\mathrm{d}u_j$$

For *orthogonal* curvilinear coordinates, we have $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$, which tells us to set $j = i$, and we can perform the sum over j. Hence

$$(\mathrm{d}s)^2 \; = \; \sum_i h_i^2\,(\mathrm{d}u_i)^2 \; = \; h_1^2\,\mathrm{d}u_1^2 + h_2^2\,\mathrm{d}u_2^2 + h_3^2\,\mathrm{d}u_3^2$$

For spherical polars, $h_r = 1\,,\; h_\theta = r\,,\; h_\phi = r\sin\theta\,,$ therefore

$$(\mathrm{d}s)^2 \; = \; (\mathrm{d}r)^2 + r^2(\mathrm{d}\theta)^2 + r^2\sin^2\theta\,(\mathrm{d}\phi)^2$$

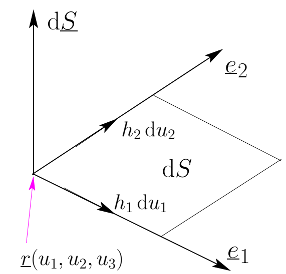
Vector Area:

If we let $u_1 \to u_1 + \mathrm{d}u_1$, then

$$\underline{r} \to \underline{r} + \mathrm{d}\underline{r}_1 \text{ with } \mathrm{d}\underline{r}_1 = h_1\,\underline{e}_1\,\mathrm{d}u_1$$

If we let $u_2 \to u_2 + \mathrm{d}u_2$, then

$$\underline{r} \to \underline{r} + \mathrm{d}\underline{r}_2 \text{ with } \mathrm{d}\underline{r}_2 = h_2\,\underline{e}_2\,\mathrm{d}u_2$$



The *vector area* of the infinitesimal parallelogram (actually a rectangle for OCCs) whose sides are the vectors $\mathrm{d}\underline{r}_1$ and $\mathrm{d}\underline{r}_2$ is

$$\mathrm{d}\underline{S}_3 \; = \; (\mathrm{d}\underline{r}_1) \times (\mathrm{d}\underline{r}_2) \; = \; (h_1\,\mathrm{d}u_1\,\underline{e}_1) \times (h_2\,\mathrm{d}u_2\,\underline{e}_2) \; = \; h_1\,h_2\,\mathrm{d}u_1\,\mathrm{d}u_2\,\underline{e}_3\,,$$

because $\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$ for orthogonal systems. Clearly, $\mathrm{d}\underline{S}_3$ points in the direction of $\underline{e}_3$ which is *normal* to the surfaces u_3 =constant.

The vector areas $\mathrm{d}\underline{S}_1$ and $\mathrm{d}\underline{S}_2$ are defined similarly.

[17] The shorthand $\sum_i$ means $\sum_{i=1}^{3}$ etc.

Example: For *spherical polars*, if we vary θ and ϕ, keeping r fixed, we obtain very easily the familiar result

$$\mathrm{d}\underline{S}_r \;=\; (h_\theta\,\mathrm{d}\theta\,\underline{e}_\theta)\times\left(h_\phi\,\mathrm{d}\phi\,\underline{e}_\phi\right) \;=\; h_\theta\,h_\phi\,\mathrm{d}\theta\,\mathrm{d}\phi\,\underline{e}_r \;=\; r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\underline{e}_r$$

Similarly, if we vary ϕ and r, keeping θ fixed, we obtain the vector element of area on the cone of semi-angle θ, with its axis along the z axis

$$\mathrm{d}\underline{S}_\theta \;=\; \left(h_\phi\,\mathrm{d}\phi\,\underline{e}_\phi\right)\times(h_r\,\mathrm{d}r\,\underline{e}_r) \;=\; h_\phi\,h_r\,\mathrm{d}\phi\,\mathrm{d}r\,\underline{e}_\theta \;=\; r\sin\theta\,\mathrm{d}r\,\mathrm{d}\phi\,\underline{e}_\theta$$

and similarly for $\mathrm{d}\underline{S}_\phi$. See also tutorial question (7.6).

Volume: The volume of the infinitesimal parallelepiped (actually a cuboid for OCCs) with edges $\mathrm{d}\underline{r}_1$, $\mathrm{d}\underline{r}_2$ and $\mathrm{d}\underline{r}_3$ is

$$\begin{aligned}\mathrm{d}V \;&=\; \left(\mathrm{d}\underline{r}_1\times\mathrm{d}\underline{r}_2\right)\cdot\mathrm{d}\underline{r}_3 \;=\; (h_1\,\mathrm{d}u_1\,\underline{e}_1)\times(h_2\,\mathrm{d}u_2\,\underline{e}_2)\cdot(h_3\,\mathrm{d}u_3\,\underline{e}_3)\\ &=\; h_1\,h_2\,h_3\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3\end{aligned}$$

because $(\underline{e}_1\times\underline{e}_2)\cdot\underline{e}_3 = 1$ for orthogonal curvilinear coordinates.

For *spherical polars*, we have $\mathrm{d}V \;=\; h_r\,h_\theta\,h_\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \;=\; r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$

14.3 Components of a vector field in curvilinear coordinates

A vector field $\underline{a}(\underline{r})$ can be expressed in terms of *curvilinear components* a_i, defined by

$$\underline{a}(\underline{r}) \;=\; \sum_{i=1}^{3} a_i(u_1,u_2,u_3)\,\underline{e}_i$$

where $\underline{e}_i$ is the i^{th} *curvilinear* basis vector (which again should really be called $\underline{e}_{u_i}$ to avoid confusion with the Cartesian basis vectors.)

For *orthogonal* curvilinear coordinates, the component a_i can be obtained by taking the scalar product of $\underline{a}$ with the i^{th} curvilinear basis vector $\underline{e}_i$

$$a_i \;=\; \underline{a}(\underline{r})\cdot\underline{e}_i$$

NB a_i must be expressed in terms of u_1, u_2, u_3 (not x, y, z) when working in the $\{u_i\}$ basis.

Example: If $\underline{a} = a\,\underline{e}_x$ in Cartesians, then in spherical polars

$$a_r \;=\; \underline{a}\cdot\underline{e}_r \;=\; (a\,\underline{e}_x)\cdot\left(\sin\theta\cos\phi\,\underline{e}_x + \sin\theta\sin\phi\,\underline{e}_y + \cos\theta\,\underline{e}_z\right) \;=\; a\,\sin\theta\cos\phi$$

Similarly, $a_\theta = \underline{a}\cdot\underline{e}_\theta$ and $a_\phi = \underline{a}\cdot\underline{e}_\phi$, and we obtain $\underline{a}$ in the spherical-polar basis (exercise)

$$\underline{a}(r,\theta,\phi) \;=\; a\left(\sin\theta\cos\phi\,\underline{e}_r + \cos\theta\cos\phi\,\underline{e}_\theta - \sin\phi\,\underline{e}_\phi\right)$$

- You can often spot the curvilinear components "by inspection". See tutorial question (5.6) for an example of this.
- In general, one chooses the set of coordinates which matches most closely the *symmetry* of the problem.

14.4 Div, grad, curl and the Laplacian in orthogonal curvilinears

14.4.1 Gradient

In section (2) we defined the gradient in terms of the change in a scalar field[18] $f(\underline{r})$ when we let $\underline{r}\to\underline{r}+\mathrm{d}\underline{r}$

$$\mathrm{d}f(\underline{r}) \;=\; \underline{\nabla}\,f(\underline{r})\cdot\mathrm{d}\underline{r} \tag{30}$$

Now consider writing $f(\underline{r})$ in terms of orthogonal curvilinear coordinates, $f(\underline{r}) = f(u_1,\,u_2,\,u_3)$.

As usual, we denote the *curvilinear* basis vectors by $\{\underline{e}_1,\,\underline{e}_2,\,\underline{e}_3\}$.

Let $u_1\to u_1+\mathrm{d}u_1$, $u_2\to u_2+\mathrm{d}u_2$, and $u_3\to u_3+\mathrm{d}u_3$.

Using Taylor's theorem, we have

$$\mathrm{d}f \;=\; \frac{\partial f}{\partial u_1}\,\mathrm{d}u_1 + \frac{\partial f}{\partial u_2}\,\mathrm{d}u_2 + \frac{\partial f}{\partial u_3}\,\mathrm{d}u_3 \tag{31}$$

We can manipulate the RHS of this equation into the form of equation (30). Start with

$$\mathrm{d}\underline{r} \;=\; h_1\,\mathrm{d}u_1\,\underline{e}_1 + h_2\,\mathrm{d}u_2\,\underline{e}_2 + h_3\,\mathrm{d}u_3\,\underline{e}_3$$

Now use orthogonality of the *curvilinear* basis vectors, $\underline{e}_i\cdot\underline{e}_j = \delta_{ij}$, to rewrite equation (31) as

$$\begin{aligned}\mathrm{d}f \;&=\; \frac{\partial f}{\partial u_1}\,\mathrm{d}u_1 + \frac{\partial f}{\partial u_2}\,\mathrm{d}u_2 + \frac{\partial f}{\partial u_3}\,\mathrm{d}u_3\\ &=\; \left(\frac{\partial f}{\partial u_1}\,\underline{e}_1 + \frac{\partial f}{\partial u_2}\,\underline{e}_2 + \frac{\partial f}{\partial u_3}\,\underline{e}_3\right)\cdot(\underline{e}_1\,\mathrm{d}u_1 + \underline{e}_2\,\mathrm{d}u_2 + \underline{e}_3\,\mathrm{d}u_3)\\ &=\; \left(\frac{1}{h_1}\frac{\partial f}{\partial u_1}\,\underline{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\,\underline{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\,\underline{e}_3\right)\cdot(h_1\,\underline{e}_1\,\mathrm{d}u_1 + h_2\,\underline{e}_2\,\mathrm{d}u_2 + h_3\,\underline{e}_3\,\mathrm{d}u_3)\\ &=\; \left(\frac{1}{h_1}\frac{\partial f}{\partial u_1}\,\underline{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\,\underline{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\,\underline{e}_3\right)\cdot\mathrm{d}\underline{r}\end{aligned}$$

Comparing this result with equation (30), which holds for all $\mathrm{d}\underline{r}$, we obtain $\underline{\nabla}\,f$ in orthogonal curvilinear coordinates

$$\boxed{\underline{\nabla}\,f \;=\; \frac{1}{h_1}\frac{\partial f}{\partial u_1}\,\underline{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\,\underline{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\,\underline{e}_3 \;=\; \sum_{i=1}^{3}\frac{1}{h_i}\frac{\partial f}{\partial u_i}\,\underline{e}_i}$$

For *spherical polars*, $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$, and we have

$$\underline{\nabla}\,f(r,\theta,\phi) \;=\; \frac{\partial f}{\partial r}\,\underline{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\,\underline{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\,\underline{e}_\phi$$

[18] We use $f(\underline{r})$ rather than $\phi(\underline{r})$ here in order to avoid confusion with the angle ϕ in spherical polars.

14.4.2 Divergence

Let $\underline{a}(\underline{r})$ be a vector field[19], which we write in orthogonal curvilinear coordinates as

$$\underline{a}(\underline{r}) \;=\; \sum_{i=1}^{3} a_i(u_1, u_2, u_3)\; \underline{e}_i$$

where a_i are the components of $\underline{a}$ in the curvilinear basis, and $\underline{e}_i$ is the i^{th} curvilinear basis vector.

We obtain $\underline{\nabla} \cdot \underline{a}$ in orthogonal curvilinears using the integral definition of divergence

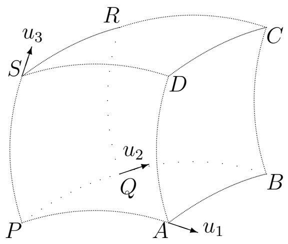
$$\underline{\nabla} \cdot \underline{a} \;=\; \lim_{\delta V \to 0} \frac{1}{\delta V} \int_{\delta S} \underline{a} \cdot \mathrm{d}\underline{S}\,,$$

where δS is the closed surface bounding δV.

Let the point P have curvilinear coordinates (u_1, u_2, u_3).

Choose δV to be a small "cuboid" with its three edges $\{\delta \underline{r}_i\}$ along the basis vectors $\{\underline{e}_i\}$ at P:

$$\begin{aligned} \delta \underline{r}_1 &= h_1\, \delta u_1\, \underline{e}_1 \\ \delta \underline{r}_2 &= h_2\, \delta u_2\, \underline{e}_2 \\ \delta \underline{r}_3 &= h_3\, \delta u_3\, \underline{e}_3 \end{aligned}$$



The outward element of area on the face $ABCD$ is $\mathrm{d}\underline{S} = +h_2\, h_3\, \mathrm{d}u_2\, \mathrm{d}u_3\, \underline{e}_1$

The outward element of area on the face $PQRS$ is $\mathrm{d}\underline{S} = -h_2\, h_3\, \mathrm{d}u_2\, \mathrm{d}u_3\, \underline{e}_1$

The contributions to the surface integral from the faces $ABCD$ and $PQRS$ are then

$$\begin{aligned} &\int_{u_3}^{u_3+\delta u_3} \int_{u_2}^{u_2+\delta u_2} \Big\{ [a_1\, h_2\, h_3]_{ABCD} \;-\; [a_1\, h_2\, h_3]_{PQRS} \Big\}\; \mathrm{d}u_2\, \mathrm{d}u_3 \\ =\;& \iint \Big\{ [a_1\, h_2\, h_3]_{(u_1+\delta u_1, u_2, u_3)} \;-\; [a_1\, h_2\, h_3]_{(u_1, u_2, u_3)} \Big\}\; \mathrm{d}u_2\, \mathrm{d}u_3 \\ =\;& \iint \left\{ \delta u_1 \left[\frac{\partial}{\partial u_1} (a_1\, h_2\, h_3) \right]_{(u_1, u_2, u_3)} \right\} \mathrm{d}u_2\, \mathrm{d}u_3 \qquad \text{(by Taylor's theorem)} \\ =\;& \delta u_1\, \delta u_2\, \delta u_3 \left[\frac{\partial}{\partial u_1} (a_1\, h_2\, h_3) \right]_{(u_1, u_2, u_3)} \end{aligned} \tag{32}$$

In the last step, we assumed that δV is small enough that the integrand is approximately constant over the range of integration. We then approximated the integrals over u_2 and u_3 by the integrand evaluated at the point P,

$$\delta u_1 \left[\frac{\partial}{\partial u_1} (a_1\, h_2\, h_3) \right]_{(u_1, u_2, u_3)}$$

multiplied by the ranges of integration $\delta u_2\, \delta u_3$.

[19] $\underline{a}$ must be continuously differentiable.

The contributions of the other four faces to the integral over δS can be obtained similarly, or by cyclic permutations of the indices $\{1, 2, 3\}$ in equation (32).

Dividing by the volume of the cuboid $\delta V = h_1\, h_2\, h_3\, \delta u_1\, \delta u_2\, \delta u_3$, we obtain our final expression for $\underline{\nabla} \cdot \underline{a}$ in orthogonal curvilinear coordinates

$$\boxed{\underline{\nabla} \cdot \underline{a} \;=\; \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1}(a_1 h_2 h_3) \;+\; \frac{\partial}{\partial u_2}(a_2 h_3 h_1) \;+\; \frac{\partial}{\partial u_3}(a_3 h_1 h_2) \right\}}$$

For Cartesian coordinates, the scale factors are all *unity*, and we recover the usual expression for $\underline{\nabla} \cdot \underline{a}$ in Cartesians.

For *spherical polars* we have

$$\begin{aligned} \underline{\nabla} \cdot \underline{a}(r, \theta, \phi) &= \frac{1}{r^2 \sin\theta} \left\{ \frac{\partial}{\partial r} \left(r^2 \sin\theta\, a_r \right) \;+\; \frac{\partial}{\partial \theta} \left(r \sin\theta\, a_\theta \right) \;+\; \frac{\partial}{\partial \phi} \left(r\, a_\phi \right) \right\} \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2\, a_r \right) \;+\; \frac{1}{r \sin\theta} \left\{ \frac{\partial}{\partial \theta} \left(\sin\theta\, a_\theta \right) \;+\; \frac{\partial}{\partial \phi} \left(a_\phi \right) \right\} \end{aligned}$$

where a_r, a_θ, and a_ϕ are the components of the vector field $\underline{a}$ in the basis $\{\underline{e}_r,\, \underline{e}_\theta,\, \underline{e}_\phi\}$.

14.4.3 Curl

We obtain $\underline{\nabla} \times \underline{a}$ in orthogonal curvilinear coordinates using the line integral definition of curl.

The component of $\underline{\nabla} \times \underline{a}$ in the direction of the unit vector $\underline{n}$ is

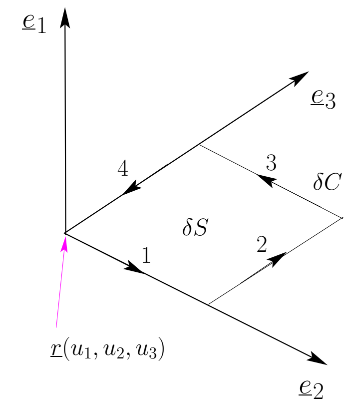
$$\underline{n} \cdot (\underline{\nabla} \times \underline{a}) \;=\; \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_{\delta C} \underline{a} \cdot \mathrm{d}\underline{r}$$

where δS is a small planar surface, with unit normal $\underline{n}$, bounded by the closed curve δC.

Let δS be a small rectangular surface parallel to the $\underline{e}_2 - \underline{e}_3$ plane with one corner at $\underline{r}(u_1, u_2, u_3)$, and with edges

$$\delta \underline{r}_2 \;=\; h_2\, \delta u_2\, \underline{e}_2 \quad \text{and} \quad \delta \underline{r}_3 \;=\; h_3\, \delta u_3\, \underline{e}_3$$

which lie along the basis vectors, so that $\underline{n} = \underline{e}_1$.



The line integral around the curve δC is the sum of the line integrals along the lines $1 \to 4$ respectively,

$$\begin{aligned} \oint_{\delta C} \underline{a} \cdot \mathrm{d}\underline{r} \;=\; &\int [a_2\, h_2]_{(u_1, u_2, u_3)}\; \mathrm{d}u_2 \quad + \int [a_3\, h_3]_{(u_1,\, u_2+\delta u_2, u_3)}\; \mathrm{d}u_3 \\ &- \int [a_2\, h_2]_{(u_1, u_2, u_3+\delta u_3)}\; \mathrm{d}u_2 \quad - \int [a_3\, h_3]_{(u_1, u_2, u_3)}\; \mathrm{d}u_3 \end{aligned}$$

Using Taylor's theorem, we can write this as

$$\oint_{\delta C} \underline{a} \cdot \mathrm{d}\underline{r} \;=\; \int \left\{ \delta u_2 \left[\frac{\partial}{\partial u_2}(a_3\, h_3) \right]_{(u_1,\, u_2, u_3)} \right\} \mathrm{d}u_3 \;-\; \int \left\{ \delta u_3 \left[\frac{\partial}{\partial u_3}(a_2\, h_2) \right]_{(u_1,\, u_2, u_3)} \right\} \mathrm{d}u_2$$

In each case, we approximate the integrals over u_3 and u_2 by the product of the integrand and the integration ranges δu_3 and δu_2, respectively. Hence

$$\oint_{\delta C} \underline{a} \cdot \mathrm{d}\underline{r} \;=\; \frac{\partial}{\partial u_2}\,(a_3\, h_3)\; \delta u_2\, \delta u_3 \;-\; \frac{\partial}{\partial u_3}\,(a_2\, h_2)\; \delta u_3\, \delta u_2$$

where all the $\{a_i\}$ and $\{h_i\}$ are evaluated at $\underline{r}(u_1, u_2, u_3)$.

Finally, we divide by the area of the rectangle $\delta S = h_2 h_3\, \delta u_2 \delta u_3$, to obtain

$$\underline{e}_1 \cdot (\underline{\nabla} \times \underline{a}) \;=\; (\underline{\nabla} \times \underline{a})_1 \;=\; \frac{1}{h_2\, h_3} \left\{ \frac{\partial}{\partial u_2}(a_3 h_3) \;-\; \frac{\partial}{\partial u_3}(a_2 h_2) \right\}$$

The components of $\underline{\nabla} \times \underline{a}$ in the directions of the curvilinear basis vectors $\underline{e}_2$ and $\underline{e}_3$ may be obtained similarly, or by cyclic permutations of the indices.

It is convenient to write the final result in the form

$$\boxed{\underline{\nabla} \times \underline{a} \;=\; \frac{1}{h_1\, h_2\, h_3} \begin{vmatrix} h_1\, \underline{e}_1 & h_2\, \underline{e}_2 & h_3\, \underline{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1\, a_1 & h_2\, a_2 & h_3\, a_3 \end{vmatrix}}$$

For *spherical polars* we have

$$\underline{\nabla} \times \underline{a} \;=\; \frac{1}{r^2\, \sin\theta} \begin{vmatrix} \underline{e}_r & r\, \underline{e}_\theta & r \sin\theta\, \underline{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ a_r & r a_\theta & r \sin\theta a_\phi \end{vmatrix}$$

14.4.4 Laplacian of a scalar field

The action of the Laplacian operator on a *scalar field* $f(\underline{r})$ is defined by $\nabla^2 f = \underline{\nabla} \cdot (\underline{\nabla}\, f)$.

Using the expression for $\underline{\nabla} \cdot \underline{a}$, with $\underline{a} = \underline{\nabla}\, f$, derived above, we find

$$\boxed{\nabla^2 f \;=\; \frac{1}{h_1\, h_2\, h_3} \left\{ \frac{\partial}{\partial u_1} \left(\frac{h_2\, h_3}{h_1} \, \frac{\partial f}{\partial u_1} \right) \;+\; \frac{\partial}{\partial u_2} \left(\frac{h_3\, h_1}{h_2} \, \frac{\partial f}{\partial u_2} \right) \;+\; \frac{\partial}{\partial u_3} \left(\frac{h_1\, h_2}{h_3} \, \frac{\partial f}{\partial u_3} \right) \right\}}$$

In *spherical polars*, we have

$$\begin{aligned} \nabla^2 f(r, \theta, \phi) \;&=\; \frac{1}{r^2 \sin\theta} \left\{ \frac{\partial}{\partial r} \left(r^2 \sin\theta\, \frac{\partial f}{\partial r} \right) \;+\; \frac{\partial}{\partial \theta} \left(\sin\theta\, \frac{\partial f}{\partial \theta} \right) \;+\; \frac{\partial}{\partial \phi} \left(\frac{1}{\sin\theta}\, \frac{\partial f}{\partial \phi} \right) \right\} \\ &=\; \frac{1}{r^2}\, \frac{\partial}{\partial r} \left(r^2\, \frac{\partial f}{\partial r} \right) \;+\; \frac{1}{r^2 \sin^2\theta} \left\{ \sin\theta\, \frac{\partial}{\partial \theta} \left(\sin\theta\, \frac{\partial f}{\partial \theta} \right) \;+\; \frac{\partial^2 f}{\partial \phi^2} \right\} \end{aligned}$$

14.4.5 Laplacian of a vector field

The Laplacian of a *vector field* $\underline{a}(\underline{r})$ in curvilinear coordinates is *defined* by means of the identity

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{a}) \;=\; \underline{\nabla}\, (\underline{\nabla} \cdot \underline{a}) \;-\; \nabla^2\, \underline{a}$$

in the form[20]

$$\boxed{\nabla^2\, \underline{a} \;=\; \underline{\nabla}\; (\underline{\nabla} \cdot \underline{a}) \;-\; \underline{\nabla} \times (\underline{\nabla} \times \underline{a})}$$

The expressions on the right hand side are evaluated using the expressions for grad, div and curl derived above.

The expression for the Laplacian of a scalar field in spherical polars is one of the most important results in the course, with applications in Quantum Mechanics, Electromagnetism, Optics, Meteorology, Fluid/Solid Mechanics, Cosmology, ...

[The End]

[20] Remember the mnemonic 'grad-div-minus-curl-curl' or GDMCC in Glaswegian.