15. The Fokker-Plank equation: equivalence with the Langevin equation

In the preceding Lecture, we have seen that a popular way to characterise a non-equilibrium statistical mechanics system, either on its way to thermodynamic equilibrium or not, is to write down a stochastic differential equation, e.g. to describe the coarse grained dynamics of the order parameter, or of other quantities which characterise this system.

However, it is uncommon to come across a Langevin equation which can be solved *exactly*, i.e. explicitly as a parametric function of the stochastic forcing. Most often we are forced to a numerical solution, and this can be computationally costly as we are bound to simulate several trajectories in order to obtain accurate averages. Another option is to switch to a different, deterministic, description, in which we trade the stochastic differential equation for a random variable with a partial differential equation (PDE) which governs the dynamical evolution of the probability distribution function (pdf) of such a random variable, i.e. which tells us at a given time t what the probability that the random variable $x(t)$ attains the value X, for all physically admissible values of X.

In this lecture we will construct the Fokker-Planck equation equivalent to a given (1-dimensional) Langevin equation. In the next lecture we will discuss specific examples in 1d of Langevin/Fokker-Planck equations and will see how to solve these in practice.

15.1 Taylor expanding a stochastic function

Consider the following Langevin equation:

$$\frac{dx(t)}{dt} = f(x(t),t) + g(x(t),t)\eta(t), \tag{1}$$

where $\eta(t)$ is a Gaussian (or white) noise with unit variance, viz:

$$\begin{aligned} \langle \eta(t) \rangle &= 0 & (2)\\ \langle \eta(t)\eta(t') \rangle &= \delta(t-t'). & (3) \end{aligned}$$

Let us start by considering a constant $g(x) \equiv g$ – we will generalise to space dependent functions later on in this section.

Discretizing Eq. 1, we obtain:

$$x(t+dt) - x(t) = f(x(t),t)dt + g\sqrt{dt}\tilde{\eta}(t), \tag{4}$$

where $\tilde{\eta}$ is a random variable with zero mean and variance equal to 1.

Let us now consider a generic function $\phi(x(t))$, and compute its derivative *with respect to time*, $\frac{d\phi(x(t))}{dt}$. To do so, we need to compute $\phi(x(t+dt)) - \phi(x(t))$:

$$\begin{aligned} \phi(x(t+dt)) - \phi(x(t)) &= \phi\left(x(t) + f(x)dt + g\sqrt{dt}\tilde{\eta}\right) - \phi(x(t)) & (5)\\ &\simeq \phi'(x)f(x)dt + \phi'(x)g\tilde{\eta}\sqrt{dt} & (6)\\ &+ \frac{\phi''(x)}{2}g^2\tilde{\eta}^2 dt + O\left((dt)^{3/2}\right), & (7) \end{aligned}$$

where we have Taylor expanded ϕ close to time t by using Eq. 4, and where a dash denotes a derivative with respect to x.

If we now divide by dt and take the average over noise realisations, we obtain that the second term in Eq. 6 cancels as $\tilde{\eta}$ has zero mean, and we are left with:

$$\langle\frac{d\phi(x(t))}{dt}\rangle = \langle\phi'(x)f(x) + \frac{\phi''(x)}{2}g^2\rangle, \tag{8}$$

where we have used the fact that $\tilde{\eta}$ has unit variance.

Note: If g were not constant but a function of x, $g(x)$, there is a subtle point which may invalidate our derivation of Eq. 8. In order to get this equation, we need to prove that the second term in Eq. 6 vanishes when averaged. This is only true if we can decouple the averages of $\phi'(x(t))g(x(t))$ and of $\tilde{\eta}$ (the latter is computed at time t). In order for this to be true for any $g(x)$, we can prescribe that when discretizing the Langevin equation we are to compute $g(x(t))$ at the *beginning* of the time step, i.e. using the value t, and not for instance $t + dt/2$. In ordinary calculus this would not make any difference of course, but in stochastic calculus it does because of the presence of the noise term which only scales as $(dt)^{1/2}$. In practice therefore we may still use Eq. 8, and its consequences which we will see later on, also for non-constant $g(x)$, provided that we compute g at the "beginning" of a time step – this is also known as Ito's prescriptions (other alternatives exist, but they would lead to different equivalent Fokker-Planck equations !).

15.2. Deriving an equivalent Fokker-Planck equation

To compute the averages in Eq.8, we may proceed in two ways. We may, as we have already done, average over several noise realisations. Alternatively, we may use the probability distribution $P(x, t)$, which measures the probability that the system (our particle moving about in 1 dimension) is in the position x at time t. In practice, we can write, for any function $F(x(t))$,

$$\langle F(x(t))\rangle = \int dz P(z, t)F(z) \tag{9}$$

The left hand side of Eq. 8 may therefore be written as:

$$\langle\frac{d\phi(x(t))}{dt}\rangle = \frac{\partial}{\partial t}\int dz P(z, t)\phi(z), \tag{10}$$

whereas for the right hand side we may observe that:

$$\langle\phi'(x)f(x)\rangle = \int dz P(z, t)f(z)\phi'(z) \tag{11}$$

$$\langle\frac{\phi''(x)}{2}g(x)^2\rangle = \int dz P(z, t)g(z)\frac{\phi''(z)}{2}. \tag{12}$$

Let us now specialise these formulas for the case of

$$\phi(x(t)) = \delta(x(t) - X). \tag{13}$$

The left hand side becomes:

$$\begin{aligned} \langle \frac{d\phi(x(t))}{dt} \rangle &= \frac{\partial}{\partial t} \int dz P(z,t) \delta(z-X) &(14)\\ &= \frac{\partial}{\partial t} P(X,t), &(15) \end{aligned}$$

To simplify the other terms, we need to recall the properties of Dirac's delta function, which "transfers" derivatives applied to itself on the "test function". The first term on the right hand side then becomes

$$\begin{aligned} \langle \phi'(x) f(x) \rangle &= \int dz P(z,t) f(z) \frac{d}{dz} [\delta(z-X)] &(16)\\ &= -\int dz \frac{d}{dz} [P(z,t) f(z)] \delta(z-X) &(17)\\ &= -\frac{\partial}{\partial X} (P(X,t) f(X)), &(18) \end{aligned}$$

while the second one may be written as:

$$\begin{aligned} \langle \frac{\phi''(x)}{2} g^2(x) \rangle &= \int dz P(z,t) \frac{g^2(z)}{2} \frac{d^2}{dz^2} [\delta(z-X)] &(19)\\ &= \int dz \frac{d^2}{dz^2} \left[P(z,t) \frac{g^2(z)}{2} \right] [\delta(z-X)] &(20)\\ &= \frac{\partial^2}{\partial X^2} \left(\frac{g^2(X)}{2} P(X,t) \right). &(21) \end{aligned}$$

In conclusion we find that the Langevin equation 1 (with the Ito prescription, in case the g term depends on x) is equivalent to the following Fokker-Planck equation for the probability distribution $P(x,t)$

$$\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x} [f(x) P(x,t)] + \frac{\partial^2}{\partial x^2} \left[\frac{g^2(x)}{2} P(x,t) \right]. \quad (22)$$

This is also known as the *forward* Fokker-Planck equation. The equation for P is deterministic and can be solved either numerically or analytically with standard methods for partial differential equations (PDEs).

15.3 Examples: the random walk again

As a simple example let us consider the case of a random walk,

$$\frac{dx}{dt} = \sqrt{2D}\eta(t) \quad (23)$$

with $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(t') \rangle = \delta(t-t')$. The corresponding Fokker-Planck equation is the diffusion equation, as we should be expected from the lecture on Random Walks.

$$\frac{\partial}{\partial t} P(x,t) = D \frac{\partial^2}{\partial x^2} P(x,t) \quad (24)$$

If the initial condition is, e.g., $P(x,0) = \delta(x)$, this is solved by:

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right). \tag{25}$$

In the presence of a constant biasing force f, the Langevin equation for the random walk becomes:

$$\frac{dx}{dt} = f + \sqrt{2D}\eta(t), \tag{26}$$

so that if, e.g., $f > 0$, the random walk is biased towards the positive x direction. The corresponding Fokker-Planck equation now contains a "drift term" (the one proportional to $\frac{\partial P(x,t)}{\partial x}$):

$$\frac{\partial}{\partial t}P(x,t) = f\frac{\partial}{\partial x}P(x,t) + D\frac{\partial^2}{\partial x^2}P(x,t). \tag{27}$$