17 Dirac field Feynman rules and Yukawa theory

17.1 Path integral over fermion fields

In the previous lecture we deduced, based on analogy with the complex scalar case, that the free-fermion generating functional is:

\[
Z_0(\eta, \bar{\eta}) \equiv \int D\psi D\bar{\psi} e^{i \int d^4x \left( \bar{\psi} \left( i \slashed{D} - m \right) \psi + \tau(x) \bar{\psi}(x) \psi(x) + \bar{\psi}(x) \eta(x) \right)},
\]

\[
= \exp \left[ - \int d^4x \int d^4y \, \bar{\eta}(x) S(x-y) \eta(y) \right].
\]

(301)

where \( S \) is the Dirac propagator:

\[
S_{\alpha\beta}(x-y) = \langle 0 | T \left( \psi_\alpha(x) \bar{\psi}_\beta(y) \right) | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} \frac{i(p_\mu + m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon}
\]

(302)

Performing the path integral over the fermion field in (301) requires defining integrals over Grassmann variables. We postpone the detailed discussion of this technique to the tutorial. Defining \( d^p \chi^a = d\chi_1^a \cdots d\chi_4^a \) we get

\[
\int d^p \chi^a d^p \bar{\chi}^a e^{i \mathcal{M} \chi + \bar{\chi} \gamma^0 \eta} = (\det \mathcal{M}) e^{-\eta \mathcal{M}^{-1} \eta}
\]

(303)

where the determinant of the Dirac operator (which is independent of the sources) is absorbed in the definition of the functional integral, while remaining exponent gives the result of (301) where the Dirac propagator is the inverse of the Dirac operator (see exercise 4 in the previous lecture):

\[
(i\slashed{D} - m)^{\alpha\beta} S_{\alpha\beta}(x-y) = i\delta^4(x-y) \delta_\eta.
\]

It is straightforward to verify that the propagator can be computed by taking the appropriate functional derivatives of the generating functional in the free theory, eq. (301). We have:

\[
\langle 0 | T \left( \psi_\alpha(x) \bar{\psi}_\beta(y) \right) | 0 \rangle = \left( \frac{1}{i} \delta \frac{\delta}{\delta \eta_\beta(x)} \right) \left( \frac{1}{i} \delta \frac{\delta}{\delta \eta_\alpha(y)} \right) \exp \left[ - \int d^4z_1 \int d^4z_2 \bar{\eta}_\alpha(z_1) S_{\gamma\delta}(z_1-z_2) \eta_\beta(z_2) \right] = S_{\alpha\beta}(x-y),
\]

where the extra minus sign in the exponent is compensated by the minus sign associated with taking the functional derivative with respect to \( \eta_\beta(z_1) \) first, while \( \eta_\alpha(z_2) \) is placed to the right of \( \eta_\beta(z_1) \).

17.2 Feynman rules for fermions: Yukawa theory

Let us consider now an interacting theory involving a Dirac fermion \( \mathcal{L}_D = i\bar{\psi} \slashed{D} \psi - m \bar{\psi} \psi \), and a real scalar \( \mathcal{L}_\varphi = \frac{1}{2} \partial^\mu \varphi \partial^\nu \varphi - \frac{1}{2} M^2 \varphi^2 \). The interaction Lagrangian is

\[
\mathcal{L}_I = -\lambda \varphi \bar{\psi} \psi
\]

Note that in four space-time dimensions \(|\psi| = \frac{1}{2}\) and \(|\varphi| = 1\) so \( \lambda \) is dimensionless. Thus we expect the theory to be renormalizable.

\[ \text{U(1) Symmetry} \]

The next observation is that the theory is invariant under the transformation \( \psi \rightarrow e^{-i\alpha} \psi \). This is a global \( U(1) \) symmetry (global means it does not depend on space-time). We recall the relation between symmetries and conservation laws: if we have a set of fields \( \phi_a \) transforming \( \phi_a(x) \rightarrow \phi_a(x) + \delta \phi_a(x) \), such that the Lagrangian density is invariant:

\[
\delta \mathcal{L} = \partial \frac{\partial \mathcal{L}}{\partial \phi_a(x)} \delta \phi_a(x) + \partial \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a(x))} \partial_\mu \delta \phi_a(x)
\]

\[
= \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a(x))} \right) \delta \phi_a(x) + \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a(x))} \delta \phi_a(x)
\]

\[
= \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a(x))} \right) \delta \phi_a(x)
\]

(304)

U(1) Transformation

\[ \text{U(1) Symmetry} \]

The next observation is that the theory is invariant under the transformation \( \psi \rightarrow e^{-i\alpha} \psi \). This is a global \( U(1) \) symmetry (global means it does not depend on space-time). We recall the relation between symmetries and conservation laws: if we have a set of fields \( \phi_a \) transforming \( \phi_a(x) \rightarrow \phi_a(x) + \delta \phi_a(x) \), such that the Lagrangian density is invariant:

\[
\delta \mathcal{L} = \partial \frac{\partial \mathcal{L}}{\partial \phi_a(x)} \delta \phi_a(x) + \partial \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a(x))} \partial_\mu \delta \phi_a(x)
\]

\[
= \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a(x))} \right) \delta \phi_a(x) + \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a(x))} \delta \phi_a(x)
\]

\[
= \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a(x))} \right) \delta \phi_a(x)
\]

(304)
where in the second line the first term is obtained using the Euler-Lagrange equation. In the third line we identified the Noether current \( j^\mu(x) \), which admits \( \partial_\mu j^\mu(x) = 0 \).

Let us now compute the conserved Noether current associated with the \( U(1) \) symmetry of the Yukawa theory. The transformation we consider is
\[
\psi \rightarrow \psi - i\alpha \psi, \quad \bar{\psi} \rightarrow \bar{\psi} + i\alpha \bar{\psi}
\]
so
\[
\alpha j^\mu(x) = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_a)} \delta \psi_a = (i\bar{\psi}\gamma^\mu)_a(-i\alpha\psi)_a = \alpha(i\bar{\psi}\gamma^\mu\psi)
\]
The conserved charge is \( Q = \bar{\psi}\gamma^0\psi \) and it has the meaning of total electric charge: it is proportional to the number of positively charged particles minus the number of negatively charged antiparticles.

Generating functional for the Yukawa theory

The generating functional is defined by the following path integral
\[
\mathcal{Z}(J, \eta, \bar{\eta}) = \int D\bar{\psi} D\psi D\varphi \exp \left[ i \int d^4x \left( \mathcal{L}_\varphi + \mathcal{L}_D + \mathcal{L}_I + J\varphi + \bar{\eta}\psi + \eta\bar{\psi} \right) \right]
\]  
(305)

Before turning on the interaction \( \mathcal{L}_I \) we have two free fields: a fermion and a real scalar, so the functional integral above factorises and the (free theory) generating functional is just a product of the two generating functionals in the two free field theories respectively:
\[
\mathcal{Z}_0(J, \eta, \bar{\eta}) = \exp \left[ -\frac{1}{2} \int d^4x \int d^4x' J(x) D_F(x-x') J(x') \right] \times \exp \left[ -\int d^4x \int d^4y (\bar{\psi}(x) S(x-y) \eta(y) \right]
\]  
(306)

Now with the Yukawa interaction term \( \mathcal{L}_I = -\lambda \bar{\psi}\varphi\psi \) may be generated by applying functional derivatives, so the generating functional in the interacting theory (still without any counterterms) is:
\[
\mathcal{Z}_1(J, \eta, \bar{\eta}) = \exp \left[ -i\lambda \int d^4x \left( \frac{1}{i} \delta J(x) \right) \left( \frac{1}{i} \delta \bar{\eta}_a(x) \right) \left( \frac{1}{i} \delta \eta_a(x) \right) \right] \mathcal{Z}_0(J, \eta, \bar{\eta})
\]  
(307)

This implies that the Feynman rule for the vertex is \( i\lambda \). Note the extra minus sign compared to the sign of the interaction term in the Lagrangian. Note also that the spin indices in (307) are contracted, as in \( \mathcal{L}_I = -\lambda \bar{\psi}_\alpha \varphi \psi_\alpha \).

Following lectures 5-6, by writing
\[
\mathcal{Z}_1(J, \eta, \bar{\eta}) = \exp(iW_1(J, \eta, \bar{\eta}))
\]
where \( iW_1 \) is the sum of all connected diagrams we can implement the normalization condition \( \mathcal{Z}_1(J, \eta, \bar{\eta}) = 1 \) by simply eliminating all vacuum diagrams (ones without sources).

At this point we can already compute tree-level diagrams (loop diagrams require introducing counter terms and will be discussed below). A correlation function involving external scalars and fermions can be obtained by applying the corresponding functional derivatives to (307). As usual, we focus on connected diagrams. An example is:
\[
\langle 0 \left| T(\varphi(z_1)\varphi(z_2)\psi_\alpha(x)\bar{\psi}_\beta(y)) \right| 0 \rangle = \left( \frac{1}{i} \frac{\delta}{\delta J(z_1)} \right) \left( \frac{1}{i} \frac{\delta}{\delta J(z_2)} \right) \left( \frac{1}{i} \frac{\delta}{\delta \bar{\eta}_a(x)} \right) \left( \frac{1}{i} \frac{\delta}{\delta \eta_a(y)} \right) \mathcal{Z}_1(J, \eta, \bar{\eta}) \bigg| _{J=0, \eta=0, \bar{\eta}=0}
\]  
(308)

Figure 18: Feynman diagram corresponding to the correlator in eq. (308). Here there are three types of sources: \( i\eta \) at the end of the fermion line where the arrow is pointing away from the source, \( \bar{\eta} \) at the end of the fermion line into which the arrow comes, and \( iJ \) at the end of a scalar line.

Note that this correlation function involves one fermion and one antifermion. This is consistent with charge conservation and thus does not vanish when the sources are put to zero (compare this to a correlation function with
two fermions!). Since the sources \( \eta \) and \( \bar{\eta} \) are distinct, in the corresponding Feynman diagram we may denote this by an arrow on the fermion line: the arrow point away from the \( \eta \) source and towards the \( \bar{\eta} \) one.

\[
\langle 0 | T \left( \varphi(z_1) \varphi(z_2) \psi_\alpha(x) \overline{\psi}_\beta(y) \right) | 0 \rangle = (i\lambda)^2 \int d^4w_1d^4w_2 S_{\alpha\gamma}(x-w_1) S_{\beta\delta}(w_1-w_2) S_{\gamma\delta}(w_2-y) \times \\
\left[ D_F(w_1-z_1) D_F(w_2-z_2) + D_F(w_1-z_2) D_F(w_2-z_1) \right]
\]

(309)

Having determined the correlation functions we may proceed and compute tree-level scattering amplitudes by applying the LSZ reduction formulae (33) and (283). For example, for fermion-antifermion annihilation this yields:

\[
i T \left( q_a(p_1) \overline{\eta}(p_2) \rightarrow \varphi(k_1) \varphi(k_2) \right) = (i\lambda)^2(\pi_\alpha(p_2)) \left[ \frac{i(p_1-k_1) - m}{(p_1-k_1)^2 - m^2 + i\delta} + \frac{i(p_1-k_2) + m}{(p_1-k_2)^2 - m^2 + i\delta} \right] (u_a(p_1))_\beta.
\]

17.3 Exercises

1. Prove eq. (303).

Guidance:

The first observation is that a general function \( f(\theta) \) of a Grassmann variable \( \theta \) has an expansion of the form

\[
f(\theta) = A + B\theta
\]

Higher order terms must be zero since \( \theta \) is anticommuting, \( \theta \theta = -\theta \). The integral over this function (the Grassmann equivalent to definite integration from \( -\infty \) to \( \infty \)) is then:

\[
\int d\theta f(\theta) = \int d\theta (A + B\theta) \equiv B
\]

where the last step is a definition.

Let us now consider now the equivalent of Gaussian integration over a complex pair of Grassmann variables:

\[
\int d\theta^* d\theta \ e^{-\theta^* \theta} = \int d\theta^* d\theta \ (1 - \theta^* \theta) = b
\]

(310)

where the sign flip is due to the order of integration compared with the order in the integrand (the convention is that the innermost integral is performed first and \( \int d\theta \int d\eta \theta = +1 \)). We also note that the result in (310) is very different from the Gaussian integral over commuting variables: \( \int dx^* dx \ e^{-x^* x} = \frac{2\pi}{b} \). The generalization of (310) to functional integration requires to introduce many Grassmann variables \( \theta_i \) where the index \( i \) represents space-time as well as spin degrees of freedom. We then obtain:

\[
\prod_{i=1}^n \int d\theta_i^* d\theta_i \ e^{-\theta_i^* B_i \theta_i} = det(B)
\]

(311)

We note again that this is quite different from the Gaussian integral over commuting variables:

\[
\prod_{i=1}^n \int dx_i^* dx_i \ e^{-x_i^* B_i x_i} = (2\pi)^n / det(B)
\]

(311)

Finally we derive (303) using eq. (311) we first shift the integration variables \( \chi = \psi - M^{-1} \eta \) and \( \chi^\dagger = \psi^\dagger - \eta^\dagger M^{-1} \).

2. Starting with the conserved current \( Q = \overline{\psi} \gamma^0 \psi \) and the expansion of \( \psi \) in terms of creation and annihilation operators in (278), show that it is proportional to the number of positively charged particles minus the number of negatively charged antiparticles. Compare your result to the expression for the Hamiltonian in (17).

3. Draw the diagrams and write down the corresponding tree-level expression for the following correlation functions by taking functional derivatives with respect to the sources.

(a) \( \langle 0 | T \left( \varphi(z_1) \varphi(z_2) \psi_\alpha(x) \overline{\psi}_\beta(y) \right) | 0 \rangle \).

(b) \( \langle 0 | T \left( \psi_{\alpha_1}(x_1) \overline{\psi}_{\beta_1}(y_1) \psi_{\alpha_2}(x_2) \overline{\psi}_{\beta_2}(y_2) \right) | 0 \rangle \).
Guidance: note that each of the above expressions receives contributions from two distinct diagrams. Interpret the relative sign between the two.

4. Use the LSZ reduction formulae (33) and (283) to compute the tree-level scattering amplitude

\[ iT(q_s(p_i)\varphi(k_i) \rightarrow q_a(p_f)\varphi(k_f)) \]

where \( q_s \) stands for a fermion with spin \( s \) and \( \varphi \) a real scalar, and the momenta are indicated in parenthesis. Write down the expression in the case the fermion is replaced by an antifermion.
18 Path integral for Photons

Maxwell’s equations in the vacuum

\[ \partial_\mu F^{\mu\nu} = 0 \]
\[ \partial^\mu F_{\mu\nu} + \partial^\nu F_{\mu\mu} + \partial^\delta F^{\delta\nu} = 0 \]  \hspace{1cm} (312)

correspond to the following Lagrangian density

\[ \mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]  \hspace{1cm} (313)

Let us formulate the quantum theory using a path integral. The action is:

\[
S_{\text{Maxwell}}(A) = \int d^4x \left[ -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu) \right]
\]
\[
= \frac{1}{2} \int d^4x \left[ (\partial_\mu A_\nu) (\partial^\mu A^\nu - \partial^\nu A^\mu) \right]
\]
\[
= \frac{1}{2} \int d^4x A_\nu (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\mu - \frac{1}{2} \int d^4x \partial_\nu [A_\nu (\partial^\mu A^\mu - \partial^\mu A^\nu)]
\]
\[
= \frac{1}{2} \int \left\{ \frac{d^4k}{(2\pi)^4} \tilde{A}_\nu(k) \left( -k^2 g^{\mu\nu} + k^\mu k^\nu \right) \tilde{A}_\mu(-k) \right\} \]  \hspace{1cm} (314)

where in the third line we performed integration by parts, and upon ignoring the surface term we changed into a Fourier representation.

The natural – although naive as we show below – starting point is to define a generating functional by

\[ Z(J) = \int DA \exp \{iS(A, J)\} = \int DA \exp \left\{ i \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu(x) \right] \right\} \]
\[ = \int DA \exp \left\{ \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\nu(k) \left( -k^2 g^{\mu\nu} + k^\mu k^\nu \right) \tilde{A}_\mu(-k) + \tilde{J}^\mu(k) \tilde{A}_\mu(-k) + \tilde{J}^\mu(-k) \tilde{J}_\mu(k) \right\} \]  \hspace{1cm} (315)

where \( DA = DA^0 DA^1 DA^2 DA^3 \). As in the scalar case the action is a quadratic form in \( A_\mu \), so naively one would expect to be able to perform the \( DA \) Gaussian integral getting

\[ Z(J) = \exp \left\{ \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{T}_\nu(k) D_{\mu\nu}^{\mu\nu}(k) \tilde{T}_\mu(-k) \right\} \]  \hspace{1cm} (316)

where the propagator \( D_{\mu\nu}^{\mu\nu}(k) \) would be the inverse of the matrix \( \left( -k^2 g^{\mu\nu} + k^\mu k^\nu \right) \), a solution to

\[ \left( -k^2 g^{\mu\nu} + k^\mu k^\nu \right) D_{\mu\nu}^{\mu\nu}(k) = i\delta_\mu^\nu \]  \hspace{1cm} (317)

One would discover, however, that such an inverse does not exist. In fact the matrix has a zero eigenvalue corresponding to solutions proportional to \( k_\mu \). Indeed, any longitudinal component of the gauge field, namely \( \tilde{A}_\mu(k) = k_\mu f(k) \), does not contribute to the action in (314). This redundancy is a consequence of gauge invariance: only transverse polarization states are physical.

The Maxwell operator we were trying to invert in (317) is proportional to

\[ T_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \]  \hspace{1cm} (318)

which is a projection operator selecting the transverse components of the gauge field (see exercise). Trying to perform a functional integral over all four gauge field components, \( DA = DA^0 DA^1 DA^2 DA^3 \), is bound to fail, because the integration over the subspace of gauge field of the form \( \tilde{A}_\mu(k) = k_\mu f(k) \) yields zero action and therefore exponent of the action reduces to unity, providing no suppression at large fields. This inevitably leads to an ill-defined generating functional.

The redundancy in the gauge field action which underlies this problem is associated with gauge invariance. The field strength, which is physical, is invariant under local gauge transformations of the form:

\[ A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x) \]  \hspace{1cm} (319)

We will see that the resolution to the problem above, will be based on a choice of gauge. It is essential to separate the space of gauge fields integrated over to ones that are mutually related by gauge transformations, and thus redundant, and physical ones. This is achieved is by the Faddeev-Popov procedure.
18.1 The Faddeev-Popov procedure

The troublesome modes, which we need not integrated over, are the pure gauge ones, where $A_\mu = \partial_\mu \alpha(x)$. To remove them we must fix the gauge. Let us express the gauge condition by the equation $G(A) = 0$, and integrate over the relevant subspace of gauge fields by inserting the delta function $\delta(G(A))$ into the functional integral. This can be done consistently by means of inserting 1 into the functional integral:

$$1 = \int D\alpha(x) \delta(G(A^\alpha)) \det \left( \frac{\delta G(A^\alpha)}{\delta \alpha} \right)$$

where $A_\mu^\alpha(x)$ denotes the gauge-transformed $A_\mu(x)$ with a particular $\alpha(x)$:

$$A_\mu^\alpha(x) = A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x).$$

Thus the unit is obtained by summing over all possible gauge transformations of the field $A_\mu(x)$, all redundant configurations. Inserting this into the path integral in (321) and swapping the order of integration between the $D\alpha$ and $DA$ integrals, we get:

$$Z_{I}(J) = \int D\alpha(x) \int DA \delta(G(A^\alpha)) \det \left( \frac{\delta G(A^\alpha)}{\delta \alpha} \right) \exp \left\{ i \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu(x) A_\mu(x) \right] \right\}$$

(320)

where the external source couples only to the modes which are subject to the constraint $G(A^\alpha) = 0$. Now, by replacing the integration variable $DA_\mu(x)$ by $DA_\mu^\alpha(x)$ we get:

$$Z_{II}(J) = \int D\alpha(x) \int DA^\alpha \delta(G(A^\alpha)) \det \left( \frac{\delta G(A^\alpha)}{\delta \alpha} \right) \exp \left\{ i \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu(x) A_\mu^\alpha(x) \right] \right\}$$

(321)

where in the second line we renamed the dummy integration variable $A_\mu^\alpha(x)$ as $A_\mu(x)$, and made the crucial observation that the integral over the gauge motions $DA$ factorizes: nothing depends on $\alpha$. This accomplishes the task of factorizing the irrelevant degrees of freedom. Clearly the integral ($\int D\alpha(x)$) is infinite, but it is also completely decoupled and we may well define our functional integral without it:

$$Z(J) = \int DA \delta(G(A)) \det \left( \frac{\delta G(A)}{\delta \alpha} \right) \exp \left\{ i \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu(x) A_\mu(x) \right] \right\}$$

(323)

In the Abelian theory discussed here (and for linear gauge choices) the determinant $\det \left( \frac{\delta G(A)}{\delta \alpha} \right)$ is itself gauge-field independent, and therefore reduces to an overall constant out of the functional integral. This is no more the case in a non-Abelian gauge theory (see exercise 3 below).

To proceed and compute the photon propagator implied by (323) we need to specify the gauge fixing function $G(A)$. We choose:

$$G(A) = \partial^\mu A_\mu - w(x)$$

corresponding to a family of covariant gauges. We then find that:

$$G(A^\alpha) = \partial^\mu A_\mu - \frac{\partial^2}{e} \alpha - w(x) \quad \Rightarrow \quad \frac{\delta G(A)}{\delta \alpha} = \frac{\partial^2}{e}$$

and then

$$Z(J) = \det \left( \frac{\partial^2}{e} \right) \int DA \delta(\partial^\mu A_\mu - w(x)) \exp \left\{ i \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu(x) A_\mu(x) \right] \right\}$$

(324)
This equality holds for any $w(x)$. We may just as well then integrate over all $w(x)$ with some weight of our choice:

\[
Z(J) = \det \left( -\frac{\partial^2}{\partial J - w} \right) \int DA \delta(\partial^a A_\mu - w(x)) \exp \left\{ i \int d^4x \left[ -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + J^\mu(x) A_\mu(x) - \frac{w^2}{2 \xi} \right] \right\}
\]

\[
= \det \left( -\frac{\partial^2}{\partial J - w} \right) \int DA \exp \left\{ i \int d^4x \left[ -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + J^\mu(x) A_\mu(x) - \frac{(\partial^a A_\mu)^2}{2 \xi} \right] \right\}
\]

(325)

where in the second line we used the $\delta$ function to integrate over $w(x)$. Expressing now the Maxwell Lagrangian as in (314) we get:

\[
Z(J) = \det \left( -\frac{\partial^2}{\partial J - w} \right) \int DA \exp \left\{ i \int d^4x \left[ \frac{1}{2} A_\nu \left( \partial^2 g^{\mu \nu} - \partial^\nu \partial^\rho \right) A^\rho + J^\mu(x) A_\mu(x) - (\partial^a A_\mu)^2 \right] \right\}
\]

(326)

where is the second line we integrated by part the new $\xi$-dependent term, discarded the surface term, and combined it with the Maxwell term. For a given $\xi$ the quadratic form we need to invert is now:

\[
\left( -k^2 g_{\mu \nu} + \left( 1 - \frac{1}{\xi} \right) k_\mu k_\nu \right) D^\sigma_{\mu \nu}(k) = i \delta_\mu^\sigma
\]

(327)

and in contrast with (317) this can be done. The inverse is:

\[
D^{\sigma \sigma}_{\mu \nu}(k) = \frac{-1}{k^2 + i\delta} \left( g^{\sigma \rho} - \left( 1 - \frac{1}{\xi} \right) \frac{k^\rho k^\sigma}{k^2} \right).
\]

(328)

Therefore now the path integral over $A_\mu$ may be performed as a Gaussian integral yielding

\[
Z(J) = \exp \left\{ \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} T_\mu(k) D^\mu_{\sigma \nu}(k) T_\nu(-k) \right\}
\]

(329)

where $D^\sigma_{\mu \nu}(k)$ computed in (328) is the gauge-dependent photon propagator.

18.2 Exercises

1. Prove that $T^{\mu \nu}$ defined in (318) acts as a transverse projection operator, i.e. that $k_\mu T^{\mu \nu} = 0$ and that $T^{\mu \nu} T^\nu_\mu = T^{\mu \mu}$.

2. Show that (328) solves (327).

3. Work though the Non-Abelian generalization of the Faddeev Popov procedure based on Section 16.2 in Peskin and Schroeder, and obtain the gauge field propagator as well as the propagator and interaction vertex Feynman rules for the ghost.

Guidance: The starting point is the Yang-Mills action:

\[
S = -\frac{1}{4} (F_{\mu \nu}) a_{\mu \nu}^a - \partial_\mu A^a_\mu - \partial_\nu A^a_\nu + g f^{abc} A^b_\mu A^c_\nu
\]

It is invariant under the following gauge transformation:

\[
A^a_\mu \rightarrow A^a_\mu + \frac{1}{g} \partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c
\]

The crucial difference with the abelian case is that the functional determinant depends on the gauge field:

\[
G(A) = \partial^\mu A^a_\mu - w^a(x) \quad \Rightarrow \quad G(A^a) = \partial^\mu \left( A^a_\mu + \frac{1}{g} \partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c \right) - w^a(x)
\]

so

\[
\delta G(A) = \partial^\mu \left( \frac{1}{g} \partial_\mu \delta^a + f^{abc} A^b_\mu \right)
\]

The final step is to represent this determinant as the integral over anticommuting “ghost” fields $c(x)$, which belong to the Adjoint representation:

\[
\det \left( \partial^\mu \left( \frac{1}{g} \partial_\mu \delta^a + f^{abc} A^b_\mu \right) \right) = \int Dc D\overline{c} \exp \left\{ -i \int d^4 x \overline{c} (x) \left( \partial^2 \delta^a + g f^{abc} \partial^\mu A^b_\mu \right) c(x) \right\}
\]

The Feynman rules for the ghost can be read off this expression.