18 Lecture 18: Central forces and angular momentum

In the previous lecture we learnt about conservative forces in three dimensions. We identified two constants of motion: energy and linear momentum. This allowed us in particular, to reduce the two-body problem in a general potential to an effective problem of one body, with a reduced mass.

We are now going to specialise our discussion of conservative forces further, and consider potentials that depend only on the distance between the two bodies. That is, instead of considering general potentials $V(r_1 - r_2)$ depending on three independent coordinates $x = x_1 - x_2$, $y = y_1 - y_2$ and $z = z_1 - z_2$ we are going to assume now that $V = V(r) = V(\sqrt{x^2 + y^2 + z^2})$, the potential only depends on the magnitude (absolute value) of $r = r_1 - r_2$. In other words the potential will not depend on the angles. We will see that as a consequence, there will be an additional constant of motion, the angular momentum.

Central forces

Having reduced the two-body problem to that of the relative motion we can consider directly the force $\vec{F}(r)$ as a function of the relative coordinate $r$, or equivalently, for a conservative force, $V(r)$. We now define a central force as a force that depends only on the distance $r = |\vec{r}|$, not on the angles, that is:

$$\vec{F}(\vec{r}) = F_r(\vec{r}) \hat{\vec{r}}$$

where $\hat{\vec{r}} = \vec{r}/r$. A conservative central force can be obtained from differentiating a central potential $V(r)$:

$$\vec{F}(r) = -\nabla V(r) = -\frac{\partial V(\sqrt{x^2 + y^2 + z^2})}{\partial x} \hat{x} - \frac{\partial V(\sqrt{x^2 + y^2 + z^2})}{\partial y} \hat{y} - \frac{\partial V(\sqrt{x^2 + y^2 + z^2})}{\partial z} \hat{z}$$

$$= -r \frac{dV(r)}{dr} \hat{\vec{r}}$$

(301)

where we used the chain rule. Thus the radial force $F_r$ (the only non-vanishing component of $\vec{F}(r)$) is simply given by the $r$ derivative of the potential:

$$F_r(r) = -\frac{dV(r)}{dr}$$

The inverse relating is of course an integral:

$$V(r) = -\int_r^\infty F_r(\hat{r})d\hat{r} + c$$

where the constant of integration (which is irrelevant for the motion and is purely a matter of convention) is usually fixed such that the potential vanishes at infinity, $r \to \infty$. With this convention we can write

$$V(r) = \int_r^\infty F_r(\hat{r})d\hat{r} = \int_r^\infty \vec{F}(\hat{r}) \cdot d\hat{r}$$

stating that the potential energy at distance $r$ equals to the total work done by the central force when a body moves from this position to infinity. Alternatively, it is minus the work done by an external force acting in the opposite direction, with the effect of moving the body from distance $r$ to infinity. Note that if the force is attractive then $F_r(\hat{r}) < 0$, and with our convention that the force vanishes at infinity, $V(r)$ is bound to be negative at any finite distance $r$.

Examples of central potentials: Newtonian gravity and electrostatics

The common two-body forces in nature are central forces. This include, in particular, Newtonian gravity where the potential is

$$V(r) = -G\frac{m_1 m_2}{r},$$

(302)

where $m_1$ and $m_2$ are the masses of the two bodies whose relative distance is $r = |r_1 - r_2|$ and $G = 6.67 \times 10^{-11} \text{Nm}^2\text{Kg}^{-2}$ is the universal Newton gravitational constant. The force is attractive; it acts to reduce $r$ and therefore it is negative, consistently with what one obtains by differentiation of the potential:

$$\vec{F}(r) = -\nabla V(r) = -\frac{dV(r)}{dr} \hat{\vec{r}} = -G\frac{m_1 m_2}{r^2} \hat{\vec{r}}.$$


In Newtonian gravity the force admits an inverse square law in the distance. In the following lectures we shall see that this particular power law behaviour is special.

In full analogy, the electrostatic Coulomb force also takes the form of an inverse square law. The electrostatic potential is

$$ V(r) = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r} , $$

where $q_1$ and $q_2$ are the charges of the two particle whose relative distance is $r = |\mathbf{r}_1 - \mathbf{r}_2|$. Here the force can be attractive, if the charges have opposite signs, and repulsive, if they have the same sign.

As an exercise, let us see how one recovers Galilean gravity from the more general case of Newtonian gravity when expanding at small distances around $z$ around the much larger earth radius $R$. We assume that the earth mass is $M$, while the body located at distance $r = R + z$ from the centre of the earth has mass $m_1$. Substituting $r = R + z$ in (302) we get:

$$ V(r) = -G\frac{m_1 M}{R + z} = -G\frac{m_1 M}{R(1 + z/R)} \approx -G\frac{m_1 M}{R} \left(1 - \frac{z}{R} + O((z/R)^2)\right) $$

$$ \approx -G\frac{m_1 M}{R} + G\frac{m_1 M}{R^2} z = \text{const} + g m_1 z $$

where we identified the earth gravitational acceleration, $g = GM/R^2$. Note that higher order terms in the expansion give corrections to the Galilean formula.

### Angular momentum

**Definition:** the angular momentum of a particle of mass $m$, linear momentum $\mathbf{p}$, and position $\mathbf{r}$ is

$$ \mathbf{L} \equiv \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}} . \quad (303) $$

Consider now the rate of change of the angular momentum (for a particle of a fixed mass $m$):

$$ \frac{d}{dt}\mathbf{L} = \dot{\mathbf{L}} = m\dot{\mathbf{r}} \times \dot{\mathbf{r}} + m\mathbf{r} \times \ddot{\mathbf{r}} = m\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \mathbf{F} = \mathbf{\tau} \equiv \mathbf{\tau} . \quad (304) $$

where we first used the differentiation of a product, where we recognized that one of the terms vanishes identically, and then, in the remaining term we substituted the second Newton law, $m\ddot{\mathbf{r}} = -\mathbf{F}$. Finally, the cross vector product of the position with the force as the **torque**: $\mathbf{r} \times \mathbf{F} \equiv \mathbf{\tau}$.

We conclude, in particular, that in **central forces**, where

$$ \mathbf{F}(\mathbf{r}) = F_r(r)\hat{r} $$

there torque vanishes identically:

$$ \mathbf{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times F_r(r)\hat{r} = 0 , $$

and therefore, eq. (304) implies that

$$ \frac{d}{dt}\mathbf{L} = 0 , $$

namely, that **for any central potential, angular momentum is a constant of motion**. Note that the origin of this conservation law is the fact that the problem has spherical symmetry. Rotation around the origin leaves the potential invariant, implying the conservation of angular momentum.

In particular, angular momentum conservation implies that motion under central forces will always be confined to a plane. To see this observe that the angular momentum vector as defined in (303) is perpendicular to both the momentum and the position vectors. The momentum $\mathbf{p}(t)$ and position $\mathbf{r}(t)$ of the particle at a given time $t$ define a plane, and $\mathbf{L}(t)$ is perpendicular to this plane. Because the vector $\mathbf{L}(t)$ is conserved this plane will be the same at any time. Thus, while the momentum $\mathbf{p}(t)$ and position $\mathbf{r}(t)$ vectors vary as the particle moves, the plane which they define is fixed, the motion is confined to this plane for ever.

### The Equation of Motion in plane polar coordinates

Knowing that the motion in central potential is confined to a plane amounts to a major simplification: without loss of generality we can define our inertial-frame coordinate system such that the motion is confined to the $x - y$ plane, always at $z = 0$. We can then describe the motion in terms of plane polar coordinates $r$ and $\theta$ such that:

$$ \mathbf{r} = x\hat{x} + y\hat{y} = r \left[ \cos(\theta)\hat{x} + \sin(\theta)\hat{y} \right] = r\hat{r} . $$
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with the two perpendicular directions:
\[ \dot{r} = \cos(\theta) \dot{x} + \sin(\theta) \dot{y}, \quad \dot{\theta} = -\sin(\theta) \dot{x} + \cos(\theta) \dot{y}. \]

Note that \( \dot{r} \cdot \dot{r} = 1, \dot{\theta} \cdot \dot{\theta} = 1 \) and \( \dot{r} \cdot \dot{\theta} = 0 \). As already seen in Lecture 6 when analysing the pendulum in the same plane polar coordinate system, the unit vectors \( \hat{r} \) and \( \hat{\theta} \) are not constants, but rather depend on time:
\[ \dot{\hat{r}} = \hat{\theta} \left[ -\sin(\theta) \dot{x} + \cos(\theta) \dot{y} \right] = \ddot{\theta} \hat{r}, \quad \dot{\hat{\theta}} = \hat{\theta} \left[ -\cos(\theta) \dot{x} - \sin(\theta) \dot{y} \right] = -\theta \dot{r}. \]

Thus when differentiating the position vector \( \mathbf{r} = r\hat{r} \) with respect to time we have to take a derivative of a product:
\[ \ddot{\mathbf{r}} = \frac{d}{dt} \left( r\ddot{r} \right) = \dddot{r} \hat{r} + r\dddot{\theta} \hat{\theta} \]
(305)

Similarly, for the acceleration we get:
\[ \dddot{\mathbf{r}} = \frac{d}{dt} \left( \dddot{r} \hat{r} + r\dddot{\theta} \hat{\theta} \right) = \dddot{r} \hat{r} + r\dddot{\theta} \hat{\theta} + r\dddot{\theta} \hat{\theta} - r\dddot{r} \hat{r} \]
(306)

For a central force \( \mathbf{F} = F_r(r) \hat{r} \) so the equation of motion in a vector form, \( m\ddot{\mathbf{r}} = F_r(r) \hat{r} \) becomes:
\[ m \left( \dddot{r} - r\dddot{\theta}^2 \right) \hat{r} + m \left( 2\dddot{r} \hat{\theta} + r\dddot{\theta} \right) \hat{\theta} = F_r(r) \hat{r}. \]
(307)

This vector equation can be readily split into two equations for the radial and angular acceleration components:
\[ \dddot{r} - r\dddot{\theta}^2 = F_r(r)/m \]
(308)
\[ 2\dddot{r} \hat{\theta} + r\dddot{\theta} = 0. \]
(309)

These equations determine the motion under the action of a central force.

The second equation can further be written as:
\[ \frac{1}{r} \frac{d}{dt} \left( r^2 \dot{\theta} \right) = 0 \quad \implies \quad r^2 \ddot{\theta} \equiv h = \text{const} \]
(310)

We therefore conclude that \( r^2 \dot{\theta} \) is a constant of motion. Indeed, based on the discussion above, we expect a constant of motion to appear: the angular momentum. Recall that we only used so far the fact that the direction of the angular momentum is fixed, restricting the motion to a plane. We have not used yet the fact that the magnitude of the angular momentum is fixed. Indeed, computing the angular momentum in terms of \( r \) and \( \theta \) using (305) above, we get:
\[ \mathbf{L} = \mathbf{r} \times \mathbf{p} = mr \dot{r} \times \dot{\mathbf{r}} = mr \dot{r} \times \left( \dot{r} \hat{r} + r\dot{\theta} \hat{\theta} \right) = mr \dot{r} \hat{r} \times r \dot{\theta} \hat{\theta} = mr^2 \dot{\theta} \hat{n} = mh \hat{n} \]
(311)
where we defined a unit vector in the direction perpendicular to both the velocity and the position by \( \hat{n} \equiv \dot{r} \times \dot{\theta} \).

We therefore deduce that the constant of motion emerging out of the angular equation (309) is nothing but the magnitude of the angular momentum, divided by the mass:
\[ r^2 \dot{\theta} \equiv h = \frac{|\mathbf{L}|}{m}. \]
(312)

Now, that we have identified the angular momentum as a constant of motion, we can use it to solve the remaining, radial equation of motion. Substituting \( \dot{\theta} = h/r^2 \) into (308) we get:
\[ \dddot{r} - \frac{h^2}{r^3} = \frac{F_r(r)}{m}. \]
(313)

This is a non-linear second order differential equation. Solving it would yield \( r(t) \), which in turn can be used in (312) to determine \( \dot{\theta}(t) \), and therefore \( \theta(t) \). In the next lecture we will see an alternative route to determining the motion.