5.6 Example of Golden rule - beta decay

A nucleus decays via the reaction n $\rightarrow$ p e$^-$ $\overline{\nu}$. to form a electron and antineutrino, releasing energy E_0.

The simplest form for the matrix element describing nuclear β-decay is given by the so-called Fermi ansatz $V_{mk} = G_F M/\Omega$ where Ω is the normalisation volume for the wavefunctions, $|M|^2 \approx 1$ is the wavefunction overlap between initial and final nuclear states and G_F is a constant.

We can work in the COM reference frame, so the kinetic energy of the nucleus is zero. Momentum is conserved, so the final state has nuclear, electron and neutrino momentum $\mathbf{P} + \mathbf{p} + \mathbf{q}$=0 while the energy released goes into the electron and neutrino, which for simplicity we treat as massless: $E_0 = E_e + qc$ The proton and neutron are heavy compared with the electron and neutrino. Given that momentum must be conserved, the kinetic energy must be concentrated in the lighter particles.

The density of final states for the electron is given by the phase space volume

$$dn = \frac{d^3\mathbf{p}d^3\mathbf{r}}{(2\pi\hbar)^3}$$

with a similar expression for the neutrino. Number of states in a volume of phase space is given by the number of electron states, times the number of neutrino states, provided energy is conserved:

$$dn = \frac{d^3\mathbf{p}d^3\mathbf{r}}{(2\pi\hbar)^3}\frac{d^3\mathbf{q}d^3\mathbf{r}}{(2\pi\hbar)^3}\delta(E_e + qc - E_0)$$

Using the relativistic relation $E^2 = p^2c^2 + m^2c^4$ implies $\quad \dfrac{dp}{dE} = \dfrac{E}{pc^2}$

the normalisation volume is just $\int d^3\mathbf{r} = \Omega$, and rotational invariance gives $d^3\mathbf{p} = 4\pi p^2 dp$.

All of which which simplifies the integral to

$$dn = \frac{\Omega^2}{4\pi^4\hbar^6c^6}E_e\sqrt{E_e^2 - m_e^2c^4}E_\nu^2\delta(E_e + E_\nu - E_0)dE_e dE_\nu$$

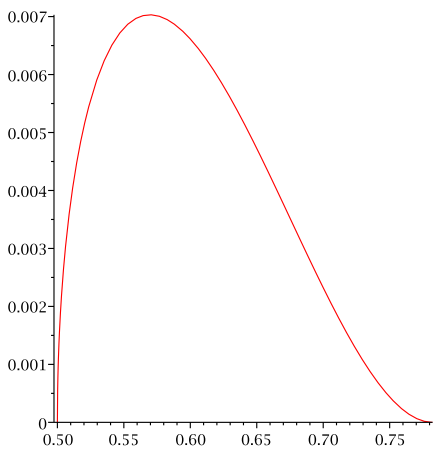
where E_e is the electron energy and E_ν is the neutrino energy. What can actually be measured is the electron energy, so we integrate over the neutrino energies,

$$\frac{dn}{dE_e} = \frac{\Omega^2}{4\pi^4\hbar^6c^6}E_e\sqrt{E_e^2 - m_e^2c^4}(E_0 - E_e)^2$$

This is the distribution of electron energies from beta decay: the rate fo emission of electrons at a particular energy is given by the Golden Rule

$$R = \frac{2\pi}{\hbar}\frac{G_F^2M^2}{4\pi^4\hbar^6c^6}E_e\sqrt{E_e^2 - m_e^2c^4}(E_0 - E_e)^2$$

Figure shows the simplest case of beta-emission: neutron decay. Conservation laws tell us that the electron energy must lie between its rest mass (0.51MeV) and the total energy available (0.7823MeV). But the entire shape can be deduced from geometry.



6 Two state systems

6.1 Time Dependence

The exact expression for the time dependence of a system with N states required a set of N simultaneous differential equations. One case where we can solve this problem exactly is when we have a small number of states. Consider a system which requires only two basis states. Say we prepare it in initial state $|1\rangle$ and we want to know how long it will take to go to the other state $|2\rangle$. From section 5, we have two coupled equations in the time dependent c_1 and c_2:

$$\begin{aligned} i\hbar\dot{c}_1 &= V_{11}c_1 + V_{12}c_2e^{i\omega_{12}t} \\ i\hbar\dot{c}_2 &= V_{22}c_2 + V_{21}c_1e^{i\omega_{21}t} \end{aligned}$$

where $c_1(0) = 1$ and $c_2(0) = 0$.

If the change is slow, we can use first order time-dependent perturbation theory. We thus replace the $c_n(t)$ by $c_n(0)$, and integrate whence:

$$\begin{aligned} c_1 &\approx \exp(iV_{11}t/\hbar) \\ |c_1|^2 &\approx 1 \\ c_2 &\approx \frac{-i}{\hbar}\int_0^t V_{21}e^{i\omega_{21}t}dt \end{aligned}$$

Including the constant of integration for $c_1(0) = 1$.

6.2 Notes

- The 'Matrix element' V_{21} determines whether there is a transition from an initial state 1 to a final state 2 even if $\hat{V}$ is *independent* of time. It also determines the rate of the transition.
- If the states $|1\rangle$ and $|2\rangle$ are eigenstates of the perturbation $\hat{V}$ then $V_{21} = V_{12} = 0$ and no transition occurs.
- Over a long period of time, the system will oscillate between the two states.
- Perturbation theory, in essence, ignores the third-order possibility of ending up in state 2 via $|1\rangle \to |2\rangle \to |1\rangle \to |2\rangle$
- The mathematics is the same as for two coupled pendula, where the energy moves back and forth between the two bobs.
- The states can represent *anything*, and oscillation will occur whenever there are off diagonal terms in the matrix.
- Examples: (see Feynman III Ch.9-11) Nitrogen atom in ammonia, electron in H_2^+, pion exchange, benzene, electron spins, photon polarisation, neutrino oscillations, neutral kaons.

6.3 Example: Oscillation in a fully mixing two state system

Consider the expectation value of a quantity S in a system which has two non-degenerate energy eigenstates $|1\rangle$ and $|2\rangle$, and where the Hermitian operator $\hat{S}$ is defined by $\hat{S}|1\rangle = |2\rangle$, $\hat{S}|2\rangle = |1\rangle$.

The general state can be written:

$$|\phi\rangle = c_1 \exp(-iE_1 t/\hbar)|1\rangle + c_2 \exp(-iE_2 t/\hbar)|2\rangle$$

if we assume real c_1, c_2 it follows that the expectation value $\langle \hat{S} \rangle$ will be:

$$\begin{aligned}
\langle \hat{S} \rangle &= \langle \phi|\hat{S}|\phi\rangle \\
&= \left[c_1 e^{iE_1 t/\hbar}\langle 1| + c_2 e^{iE_2 t/\hbar}\langle 2|\right]\left[c_1^* e^{-iE_1 t/\hbar}|2\rangle + c_2^* e^{-iE_2 t/\hbar}|1\rangle\right] \\
&= c_1 c_2 [e^{i\omega_{21} t} + e^{-i\omega_{21} t}] \\
&= 2c_1 c_2 \cos(\omega_{21} t)
\end{aligned}$$

Thus the expectation value of $\hat{S}$ oscillates in time at frequency $\omega_{21} = (E_2 - E_1)/\hbar$. This arises because $\hat{S}$ is not compatible with the hamiltonian, and hence does not define a constant of the motion.

6.4 Neutrino Oscillations

Neutrino oscillation is a phenomenon where a specific flavour of neutrino (electron, muon or tau) is later measured to have different flavour. The probability of measuring a particular flavour varies periodically. The three neutrino states are created by a radioactive decay in a flavour eigenstate as $|f_1\rangle$, $|f_2\rangle$, $|f_3\rangle$ (electron, muon, tauon). However, these are not eigenstates of energy with a definite mass $|m_1\rangle$, $|m_2\rangle$, $|m_3\rangle$. We can expand the flavour eigenstate using the energy eigenstates as a basis:

$$|f_i\rangle = \sum_j \langle m_j|f_i\rangle|m_j\rangle$$

the energy eigenstates show how the wavefunctions behave in time, $m_j(t) = m_j(0)\exp(i\omega_j t)$, where $\omega_j = m_j c^2/\hbar$. $\omega_{ij} = (m_i - m_j)c^2/\hbar$. Consider an electron neutrino produced by a fusion reaction in the sun, $\Phi(t = 0) = |f_1\rangle$, its wavefunction then varies as:

$$\Phi(t) = \sum_j |m_j\rangle\langle m_j|f_1\rangle \exp(i\omega_j t)$$

For real neutrinos, the $\langle m_j|f_1\rangle$ matrix has non-zero, possibly even complex elements everywhere, but here for simplicity we suppose that

$$\langle m_j|f_i\rangle = \begin{pmatrix} a & c & 0 \\ -c & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with a and c real, time independent and $a^2 + c^2 = 1$ for normalisation. Our electron neutrono then evolves as $\Phi(t) = a\exp(i\omega_1 t)|m_1(t)\rangle + c\exp(i\omega_2 t)|m_2(t)\rangle$, so the probability that some time later it is still an electron neutrino is

$$\begin{aligned}
|\langle f_1|\Phi(t)\rangle|^2 &= |a^2\exp(i\omega_1 t) + c^2\exp(i\omega_2 t)|^2 \\
&= a^4 + c^4 + a^2c^2(\exp(i\omega_{21} t) + a^2c^2\exp(-i\omega_{21} t) \\
&= 1 - 4a^2c^2\sin^2(\omega_{21} t/2)
\end{aligned}$$

which is less than 1: it can somehow "turn into" a muon neutrino. Often, one writes $a = \sin\theta$ in which case $4a^2c^2 = \sin^2\theta$. θ is refered to as a "mixing angle".

If $a = c = \sqrt{\frac{1}{2}}$, then with a frequency governed by the difference in masses, the electron neutrino turns completely into a muon neutrino, then back again. With smaller c, there's always some chance that it will still be an electron neutrino. In reality, it is also possible to oscillate into a tau neutrino. This underlies the "solar neutrino problem". Detection of solar neutrinos was the subject of the 2002 Nobel prize. Similar oscillation occurs in the kaon system due to a symmetry-breaking effect called "CP violation" subject of the 2008 Nobel prize. Here one of the states is subject to radioactive decay, so a particle not only "turns into" something else, it also disappears when it does so!

6.5 Strong force - Two state system or degenerate perturbation

The fundamental forces can be thought of as manifestations of two state systems. Consider a system comprising a proton and a neutron. The proton can decay into a neutron plus a pion, while the neutron can absorb the pion and become a proton. We can think of the system as two neutrons and a pion: the pion having two degenerate states $|a\rangle$ and $|b\rangle$ depending on which neutron it is located. The off-diagonal terms are now $\langle a|\hat{V}_a|b\rangle$, where V_a is the potential energy of the pion due to neutron a. The two state analysis shows that we can think of the pion hopping back and forth between the neutrons (the pion exchange mechanism). Or we can treat the system by degenerate perturbation theory and diagonalise the 2x2 matrix to find energies: $V_{aa} \pm V_{ab}$. The ground state has a binding energy of $|V_{ab}|$

Note that V_{ab} involves the overlap between the state with the pion on one site and the state with the pion on the other site. Obviously this depends on the separation (R), and so there is a force between the neutrons dE_g/dR. As the nucleons move apart, the force depends on the tails of the wavefunctions, which in turn are exponentially dependent on the pion mass. Thus the strength of the strong force falls off exponentially with distance.

Note also that we have described the basis states of our two state system as 'a proton and a neutron', but the actual ground state is a mixture of the two. When interacting via the strong force, the nucleons lose their well-defined identity.

This picture of forces arising from exchange of 'virtual' particles (the pion is not observed as a free particle here) is the standard way of thinking about fundamental forces - the electromagnetic force involves 'exchange of virtual photons', the gravitational force 'gravitons' etc. These forces are long ranged (not exponentially decaying) because the particles involved have zero mass.

All of this is analogous to covalent bonding: 'exchange of electrons': and in each case there is still another level of understanding lurking beneath to define the potential V: QED (photons) for electron-ion bonding and QCD (gluons and quarks) for nucleon binding.