Physics 4 - Quantum Physics 2000-1 - Answers.

These answers are intended to reassure that you have got the problems right. They are not model solutions, and would be unacceptably terse as examination solutions.

1 Perturbation and Matrix Elements

For integer n:

$$\frac{1}{\sqrt{a}}\cos\frac{(2n+1)\pi x}{2a}; \qquad \frac{1}{\sqrt{a}}\sin\frac{2n\pi x}{2a}$$

For perturbation theory va should be much smaller than the difference between energy levels of different n:

$$va << \frac{(2n+1)\pi^2\hbar^2}{8ma^2}$$

First order energy shift: $\langle n|vx|n\rangle = 0$.

Matrix elements, by symmetry, integer n,m: $\langle 2m|vx|2n\rangle = \langle 2m+1|vx|2n+1\rangle = 0$

For n odd, m even:

$$V_{nm} = \frac{4av}{\pi^2}\int_{-\pi/2}^{\pi/2} y\cos(ny)\sin(my)dy \qquad \text{e.g.} \qquad V_{12} = \frac{32va}{9\pi^2}$$

$$\langle 267|vx|387\rangle = \langle m|vx|m+738\rangle = 0$$

2 Completeness and Orthonormality in Vectors

Consider a unit vector $\mathbf{a} = (1,1,3)/\sqrt{11}$. What are its components (dot products) in the following 'basis sets' $\mathbf{u_i}$:

a) $1/\sqrt{11}$, $1/\sqrt{11}$, $3/\sqrt{11}$: complete, orthogonal, normalised

b) $(\sqrt{3}+1)/2\sqrt{11}$, $(1-\sqrt{3})/2\sqrt{11}$: incomplete, orthogonal, normalised

c) $(\sqrt{3}+3)/2\sqrt{11}$, $(\sqrt{3}+3)/2\sqrt{11}$, $1/\sqrt{11}$: complete, non-orthogonal, normalised

d) $(\sqrt{3}+3)/2\sqrt{11}$, $(1-3\sqrt{3})/2\sqrt{11}$, $1/\sqrt{11}$: complete, orthogonal, normalised

e) 0, $7/\sqrt{11}$, $-6/\sqrt{11}$: complete, orthogonal, unnormalised

f) $(3-\sqrt{3})/2\sqrt{11}$, $(1+3\sqrt{3})/2\sqrt{11}$, $1/\sqrt{11}$, 0. overcomplete, non-orthogonal, unnormalised

g) $\sqrt{2}/\sqrt{11}$, $3/\sqrt{11}$: incomplete, orthogonal, normalised

h) $(1+\sqrt{3})/2\sqrt{11}$, $(1-\sqrt{3})/2\sqrt{11}$, $\sqrt{2}/\sqrt{11}$: incomplete, non-orthogonal, normalised

Complete, orthogonal and normalised iff for any $\mathbf{a}$:

$$\sum_i |\mathbf{u_i}\cdot\mathbf{a}|^2 = 1 \qquad \text{and} \qquad \mathbf{u_i}\cdot\mathbf{u_j} = \delta_{ij}$$

The Fourier series is orthogonal and tends to completeness as $\mathbf{k}\to\infty$. Atomic orbitals centred on different sites are not orthogonal and may become overcomplete. Thus Fourier series have an advantage in completeness. However, the bonding orbitals in H_2 are more similar to the atomic ones, and a basis set of just two 1s wavefunctions gives as good a description of the bonding as could hundreds of Fourier components.

3 One electron atoms

a) $\langle\Phi_{nl0}|v\hat{l}_z|\Phi_{nl0}\rangle = 0$; ; $\langle\Phi_{211}|v\hat{l}_z|\Phi_{211}\rangle = v\hbar$; ; $\langle\Phi_{21-1}|v\hat{l}_z|\Phi_{21-1}\rangle = -v\hbar$;

b)$\langle\Phi_{100}|vr^2|\Phi_{100}\rangle = 3v(a_0/Z)^2$; $\langle\Phi_{200}|vr^2|\Phi_{200}\rangle = 42v(a_0/Z)^2$; $\langle\Phi_{210}|vr^2|\Phi_{210}\rangle = 30v(a_0/Z)^2$;

$\langle\Phi_{21\pm1}|vr^2|\Phi_{21\pm1}\rangle = 30v(a_0/Z)^2$

c) $\langle\Phi_{100}|vr|\Phi_{100}\rangle = 1\cdot 5v(a_0/Z)$; $\langle\Phi_{200}|vr|\Phi_{200}\rangle = 6v(a_0/Z)$; $\langle\Phi_{210}|vr|\Phi_{210}\rangle = 2\cdot 5v(a_0/Z)$;

$\langle\Phi_{21\pm1}|vr|\Phi_{21\pm1}\rangle = 2\cdot 5v(a_0/Z)$

d) $\langle\Phi_{100}|ve^{-2r/a_0}|\Phi_{100}\rangle = v(\frac{Z}{Z+1})^3$; $\langle\Phi_{200}|ve^{-2r/a_0}|\Phi_{200}\rangle = vZ^3[(\frac{1}{Z+2})^3 + (\frac{3Z^2}{(Z+2)^5} + \frac{3Z}{(Z+2)^4}]$;

$\langle\Phi_{21m}|ve^{-2r/a_0}|\Phi_{21m}\rangle = fracv2(\frac{Z}{Z+2})^5$;

e) $\langle\Phi_{100}|ve^{-Zr/a_0}|\Phi_{100}\rangle = \frac{8v}{27}$; $\langle\Phi_{200}|ve^{-Zr/a_0}|\Phi_{200}\rangle = \frac{v}{32}$; $\langle\Phi_{21m}|ve^{-Zr/a_0}|\Phi_{21m}\rangle = \frac{v}{64}$

Notice that all off-diagonal terms with the same n are zero: none of these perturbations mix the eigenstates. (a) actually commutes with the Hamiltonian, so doesn't mix n=1 with n=2 states, in this case the energy shifts are exact for any v. (b-e) mix states of n=1 and n=2, and so perturbation is only correct when the matrix element is much less than the energy difference between the n=1 and n=2 states $(0.75Z^2$ Ryd.).

4 Commutation

$-i\hbar$, 0, 0, $i\hbar z$, 0, $i\hbar\hat{p}_z$, $i\hbar\hat{l}_z$

5 Spin-Orbit Coupling

$$\hat{j}^2 = (\hat{l}+\hat{s})^2 = \hat{l}^2 + 2\hat{l}\cdot\hat{s} + \hat{s}^2 \Rightarrow \hat{l}\cdot\hat{s} = \frac{1}{2}[\hat{j}^2 - \hat{l}^2 - \hat{s}^2]$$

For 1s and 2s, $\Delta E = 0$

For 2p, fourfold degenerate, $J = \pm\frac{3}{2}$, $\Delta E = \eta/2$

For 2p, twofold degenerate, $J = \pm\frac{1}{2}$, $\Delta E = -\eta$

First order because we assume the perturbed wavefunctions are the same as the hydrogen ones. Degeneracy is only partly lifted because there is still symmetry with respect to rotating the atom. Note that four levels are raised by $\eta/2$ while two are lowered by η: the 'average energy' of a 2p state in unaffected, though in fact each atom will be in one state or the other, so this is not a conservation law.

6 Degenerate Perturbation

Other orthogonal state: $\cos(\theta + \frac{\pi}{2})|\alpha_1\rangle + \sin(\theta + \frac{\pi}{2})|\alpha_2\rangle$.

$\langle\alpha_2|\hat{V}|\alpha_1\rangle = V_1$ since operator must be Hermitian. Eigenvalues of the matrix are $V_0 \pm V_1$, so splitting between levels is $2V_1$. Eigenstates correspond to $\theta = \pm\pi/4$.

Initial state is $|\alpha_1\rangle$. When $\hat{V}$ is applied it collapses onto $|\alpha(\pm\pi/4)\rangle$. For both of these states $|\langle\alpha(\pm\pi/4)|\alpha_2\rangle|^2 = \frac{1}{2}$, thus the probability of measuring $|\alpha_2\rangle$ when the perturbation is removed is $\frac{1}{2}$.

Time dependent perturbation theory would be inappropriate because the perturbation is larger than the energy difference between the unperturbed states (which is zero).

7 Good quantum numbers in hydrogen

b), e) and j) are good sets. a) is not good since L_z and L_x do not commute. d) is not good since x does not commute with the other operators. c), g), h), i) and j) do not give different labels to $|200\uparrow\rangle$ and $|210\uparrow\rangle$. f) does not give different labels to $|211\downarrow\rangle$ and $|210\uparrow\rangle$.

8 Periodic perturbation

Energy of $\phi_\pm$ is $\hbar^2 k^2/2m$ and they are eigenfunctions of $\hat{H}_0$.

Momentum of $\phi_\pm$ is $\hbar k$ and they are eigenfunctions of $\hat{\mathbf{p}}$.

$$\langle \phi_+ | \hat{V}(x) | \phi_+ \rangle = \langle \phi_- | \hat{V}(x) | \phi_- \rangle = 0$$

$$\langle \phi_+ | \hat{V}(x) | \phi_- \rangle = \langle \phi_- | \hat{V}(x) | \phi_+ \rangle = 0 \quad \text{unless} \quad 2Lk = N\pi$$

Think about what this means - to first order, a free electron travelling through a periodic potential (e.g. a crystal) is *not* affected by that potential! This is why the free electron theory of metals works reasonably well.

$$\text{If} \quad 2Lk = N\pi \quad \text{then}: \qquad \langle \phi_+ | \hat{V}(x) | \phi_- \rangle = \langle \phi_- | \hat{V}(x) | \phi_+ \rangle = V_0/2$$

Appropriate states are $(\phi_+ \pm \phi_-)/\sqrt{2}$, which are still eigenfunctions of H_0 but not of $\hat{\mathbf{p}}$.

This analysis is similar to the opening of a band gap by a periodic potential in solid state band theory of a one dimensional crystal.

The non-zero matrix element from combining terms on either side of $|k| = N\pi/2L$ is

$$\int_0^L \exp[i(N\pi + \delta)x/2L] v_o \cos(2kx) \exp[i(N\pi - \delta)x/2L] dx = L/2$$

Thus the second order energy shift of the state at $|k| = N\pi/2L - \delta$ is:

$$\Delta E = -\frac{(v_o L/2)^2}{\hbar^2 2N\pi\delta/8mL^2} = -\frac{mv_0^2}{N\hbar^2\pi\delta}$$

i.e. its energy is reduced. Even second order perturbation breaks down as $\delta \to 0$ (because the unperturbed states are closer together in energy than the size of the perturbation), but we do know the exact limit to which the energy shift should tend from the degenerate result above.

9 Stark Effect

There are 81 elements, of which 73 are zero. Of the remaining eight, there are four equal pairs $\langle 300|z|310\rangle$, $\langle 320|z|310\rangle$ and $\langle 32 \pm 1|z|31 \pm 1\rangle$, and in fact only two are distinct since $\langle 320|z|310\rangle = \langle 32 \pm 1|z|31 \pm 1\rangle$.

Defining $a = \langle 300|z|310\rangle$ and $b = \langle 320|z|310\rangle$, we can block diagonalise the matrix and find the eigenvalues:

$-\sqrt{(a^2 + b^2)}$, 0 , $\sqrt{(a^2 + b^2)}$, b, $-b$, b, $-b$, 0 , 0.

10 Neutral kaons

$$CP|K^0\rangle = |\overline{K}^0\rangle; \qquad\qquad CP|\overline{K}^0\rangle = |K^0\rangle$$

Consider a general state $|\phi\rangle = a_0|K^0\rangle + b_0|\overline{K}^0\rangle$

The total intensity is proportional to $\langle \phi|\phi\rangle = |a_0|^2 + |b_0|^2$, of which $|a_0|^2$ are K^0s. And

$$\langle\phi|\frac{1}{2}(\hat{S}+1)|\phi\rangle = |a_0|^2$$

Similarly, the intensity of $\overline{K}^0$ is $|b_0|^2$ and comes from the expectation value of $\frac{1}{2}(1-\hat{S})$

If we assume K^0 intensity $I_{K^0}(t) = |a_0(t)|^2$, then.

$$|a_0(t)|^2 = \langle\frac{1}{2}(\hat{S}+1)\rangle = \frac{1}{4}|a_0(0)|^2 \left[e^{-t/\tau_1} + e^{-t/\tau_2} + 2e^{-t/2\tau_1}e^{-t/2\tau_2}\cos(m_{12}t)\right]$$

$$\langle\hat{S}\rangle = |a_0(0)|^2 e^{-t/2\tau_1}e^{-t/2\tau_2}\cos(m_{12}t)$$

$$\langle\hat{S}^2\rangle = \frac{1}{2}|a_0(0)|^2 \left(e^{-t/\tau_1} + e^{-t/\tau_2}\right)$$

$$\langle\frac{1}{2}(\hat{C}\hat{P}+1)\rangle = \frac{1}{2}|a_0(0)|^2 e^{-t/\tau_1}$$

$$\langle\hat{C}\hat{P}\rangle = \frac{1}{2}|a_0(0)|^2 \left(e^{-t/\tau_1} - e^{-t/\tau_2}\right)$$

Leaving the matter, the appropriate collapsed eigenstates are $|K^0\rangle$ and $|\overline{K}^0\rangle$

The final kaon intensity is a quarter of the initial intensity. Had the matter not intervened, the final kaon intensity would have been half of the initial intensity.

11 Variational principle

a) $\quad a^2 = \sqrt{\frac{\pi^2-6}{12}\frac{m\omega}{\hbar}};\qquad \langle\hat{H}_0\rangle = 0.568\hbar\omega$

b) $\quad a^2 = \sqrt{\frac{35}{2}\frac{\hbar}{m\omega}};\qquad \langle\hat{H}_0\rangle = 0.598\hbar\omega$

c) $\quad \alpha = \frac{m\omega}{2\hbar};\qquad \langle\hat{H}_0\rangle = 0.5\hbar\omega$

d) $\quad a^2 = \sqrt{15\pi}\frac{\hbar}{m\omega};\qquad \langle\hat{H}_0\rangle = 0.548\hbar\omega$

e) $\quad a^2 = \sqrt{\frac{2\pi^2-3}{6}\frac{m\omega}{\hbar}};\qquad \langle\hat{H}_0\rangle = 1.607\hbar\omega$

12 Degenerate Perturbation

Eigenstates: $[\sin\frac{\pi x}{L}\cos\frac{\pi y}{2L} \pm \sin\frac{\pi y}{L}\cos\frac{\pi x}{2L}]/L$

Energy shifts $= \pm\frac{1024}{81\pi^4}vL^2$

13 Properties of Legendre Polynomials

$$\int_0^\pi P_l(\cos\theta)P_m(\cos\theta)d\theta = \delta_{ml}\frac{2}{2l+1}$$

$P_l(1) = 1$ and $P_0(0) = 1$, $P_2(0) = 0$, $P_3(0) = \frac{1}{2}$, $P_4(0) = 0$.

For the dipole, $V = eEz$ etc. $\Delta l = \pm 1$ (θ integral), $\Delta m_s = 0$ (spin integral), $\Delta m_l = \pm 1, 0$ (ϕ integral)

For the quadrupole, $V = eExy$ etc. $\Delta l = \pm 2, 0$ (θ integral), $\Delta m_s = 0$ (spin integral), $\Delta m_l = \pm 2, \pm 1, 0$ (ϕ integral)

14 Abstract Operators

$$\hat{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The eigenvectors are just $|p\rangle = [1\,0]$ and $|n\rangle = [0\,1]$, the eigenvalues are 1 and -1

$$\hat{T}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

eigenstates of $\hat{T}_1$ are $[\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}]$ and $[\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}]$ both with charge expectation values of $\frac{1}{2}$

15 Density of states - Born Approximation

States between E and E+dE: $\frac{mkL^3}{2\hbar^2\pi^2}dE$

States between E and E+dE and angle between θ and $\theta + d\theta$: $\frac{mkL^3}{4\hbar^2\pi^2}\sin\theta d\theta dE$

16 Impact Parameters

Impact parameter for 1eV p-electron: $2.75\text{\AA}$

17 Partial Waves

According to Levinson's Theorem there are two bound states, both with $l = 0$.

There will be a minimum in the cross section when $\delta_0 = 2\pi$, E=1eV.

There are maxima in the cross section at $\delta_0 = 3\pi/2$: $E = 1.78eV$ and when $\delta_1 = \pi/2$: $E = 10eV$. However, the $\delta_1 = \pi/2$ maximum is coincident with the maxima from the other l-components, and thus this is the energy with strongest scattering. The ratio of scattering strengths is $35/10 : 1/1.78 \approx 6.2$

At high energy, the scattering falls as $\sigma_{max}/2E$, since the average value of $\sin^2\delta \approx 1/2$.

18 Atomic Scattering

$$\frac{d\sigma}{d\theta} = \frac{m^2e^4}{4\pi^2\epsilon_0^2\hbar^4}[2k\sin\frac{\theta}{2}]^{-4}[F(\chi) - Z]^2 = \frac{e^4}{64\pi^2\epsilon_0^2}\sin^{-4}\frac{\theta}{2}\frac{1}{m^2v^4}[F(\chi) - Z]^2$$

19 Coulomb Scattering

No Planck constant - the expression is the same as the classical Rutherford cross section.

20 Hydrogen Scattering

Redefine axes for integral over all space such that polar axis is along χ. Use standard formula for $\int r e^{-ar} dr$ with complex a.

21 Centre of mass coordinates

With $M = m_1 + m_2$ and $\mu = (m_1 m_2)/(m_1 + m_2)$ and total energy $E = E_r + E_R$, the equations are:

$$-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}\Phi + V(r)\Phi = E_r\Phi$$

$$-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial R^2}\Phi = E_R\Phi$$

22 Imaginary Potentials

$$\Phi = Ae^{V_1 t/\hbar}e^{ikx-iwt} + Be^{V_1 t/\hbar}e^{-ikx-iwt}$$

Is the general solution, so that $\langle\Phi|\Phi\rangle = (A_0^2 + B_0^2)e^{2V_1 t/\hbar}$. The probability current density is then :

$$(\frac{\hbar k}{m})e^{2V_1 t/\hbar}(B_0^2 - A_0^2)$$

Particles are being created. If V_1 were negative it would represent particles decaying.

23 Electron-electron scattering

$$(a) \qquad \frac{d\sigma}{d\theta} = (\frac{e^2}{8\pi\epsilon_0 2E})^2 \sin^{-4}\frac{\theta}{2}$$

$$(b) \qquad \frac{d\sigma}{d\theta} = (\frac{e^2}{8\pi\epsilon_0 2E})^2\frac{1}{2}(\sin^{-2}\frac{\theta}{2} - \cos^{-2}\frac{\theta}{2})^2$$

$$(c) \qquad \frac{d\sigma}{d\theta} = (\frac{e^2}{8\pi\epsilon_0 2E})^2\frac{1}{2}\left([\sin^{-2}\frac{\theta}{2} - \cos^{-2}\frac{\theta}{2}]^2 + \sin^{-4}\frac{\theta}{2}\right)$$

24 Bell's Theory

(a) $\frac{1}{4}$

(b) $\cos^2(\theta_A)\cos^2(\theta_B)$

(c) $\frac{1}{2}\cos^2(\theta_A - \theta_B)$

(d) $\frac{1}{4}$

(e) $\frac{1}{2}\cos^2\frac{1}{2}(\theta_A - \theta_B)$

The factor of two arises because opposite-polarised photons (x and y) are at $90°$ while opposite spins $S_z = \pm\frac{1}{2}$ are at $180°$ to one another.

Hidden variables methods would give the same result for (a), (b) and (d), but will incorrectly predict for (c) $\frac{1}{4}\cos^2(\theta_A - \theta_B)$ and for (e) $\frac{1}{4}\cos^2\frac{1}{2}(\theta_A - \theta_B)$

The probability averaged over all θ_A and θ_B is $\frac{1}{4}$ in every case.

25 Kronig-Penney

When $l = b$, the equation becomes simply $\cos k_1 b = \cos kb$, whence the energy is $E = \hbar^2 k^2/2m$.

When $b = 0$ we have $\cos k_2 l = \cos kl$. For $E > V_0$ this is just the free electron again. For $E < V_0$, k_2 is imaginary, $i\kappa_2$ say, the equation becomes $\cosh \kappa_2 l = \cos kl$ which has no solution except the trivial l=0.

The first two cases are simply free electrons, the final non solution shows that an electron must have at least the potential energy of the region where it is found.

26 Hydrogen Molecular Ion

The ground state is simply one hydrogen atom and one bare proton. Since there is a choice for which proton has the electron, it is doubly degenerate with energy E_0. The wavefunction is simply the 1s orbital in hydrogen. Refer to these states as $|1\rangle$ and $|2\rangle$.

The integrals are the electron wavefunction acted on by the electric field of the other proton, premultiplied either by itself or the equivalent wavefunction on the other ion. Each increases towards infinity at R=0 and drops to zero as R$\rightarrow \infty$. V_{12} is larger.

The energies of the perturbed states are:

$E_0 + V_{11} \pm V_{12}$

The electron will fall into the lower energy state $E_0 + V_{11} - V_{12}$. This will produce a force between the atoms of

$$\frac{dE}{dR} = \frac{dV_{11}}{dR} - \frac{dV_{12}}{dR}$$

since $\frac{dV_{12}}{dR} < \frac{dV_{11}}{dR}$, this force is attractive. (Check it using MAPLE)

The protons do not collide because of their mutual Coulombic repulsion.

The approximation comes in not including $n = 2$ basis functions in the wavefunction. It will break down when $V_{12} \approx \frac{3}{4}R_H$, where R_H is the Rydberg constant.

Some Integrals

$$\int_{-a}^{a} (\text{Any odd function}) \, dx = 0$$

$$\int_{0}^{a} x \cos^2\left(\frac{n\pi x}{2a}\right) dx = \frac{a^2}{4}\left[1 - \frac{4}{n^2\pi^2}\right] \quad \text{n odd} \quad ; \qquad \int_{0}^{a} x \sin^2\left(\frac{n\pi x}{2a}\right) dx = \frac{a^2}{4}\left[1 + \frac{4}{n^2\pi^2}\right] \quad \text{n odd}$$

$$\int_{0}^{a} x \sin^2\left(\frac{n\pi x}{2a}\right) dx = \int_{0}^{a} x \cos^2\left(\frac{n\pi x}{2a}\right) dx = \frac{a^2}{4} \quad \text{n even}$$

$$\int_{-\pi/2}^{\pi/2} x \cos^2 x \sin x \, dx = \frac{4}{9}$$

$$\int_{0}^{\infty} x^n \exp(-ax) \, dx = n! \, a^{-(n+1)}$$

$$\int_{0}^{L} \cos^2(\pi x/L) \, dx \int_{0}^{L} \sin^2(\pi x/L) \, dx = L/2$$

$$\int_{-\pi/a}^{\pi/a} x^2 \sin^2 nax \, dx = \frac{\pi(2n^2\pi^2 - 3)}{6n^2a^3} \quad ; \qquad \int_{-\pi/a}^{\pi/a} x^2 \cos^2 nax \, dx = \frac{\pi(2n^2\pi^2 + 3)}{6n^2a^3}$$

$$\int_{-a}^{a} a^4 - 2a^2x^2 + x^4 \, dx = \frac{16}{15}a^5$$

$$\int_{-\infty}^{\infty} x^2 \exp(-x^2/\sigma^2) \, dx = \frac{\sigma^3\sqrt{\pi}}{2}$$

$$\int_{-\infty}^{\infty} \exp(-x^2/\sigma^2) \, dx = \sigma\sqrt{\pi}$$

$$\int_{0}^{a} x^2(x-a)^2 \, dx = \frac{a^5}{30}$$

$$\int_{-\pi/2}^{\pi/2} \sum_{n=1}^{\infty} \frac{\cos^2 nx}{n^2} dx = \frac{\pi^3}{16}$$

$$\int_{-\infty}^{\infty} d^3\mathbf{x} \int_{-\infty}^{\infty} d^3\mathbf{y} \, \frac{\rho(x)\exp(-i\mathbf{k}.\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} = \int_{-\infty}^{\infty} d^3\mathbf{x} \int_{-\infty}^{\infty} d^3\mathbf{z} \, \frac{\rho(x)\exp -i\mathbf{k}.(\mathbf{x} + \mathbf{z})}{|\mathbf{z}|} = \frac{4\pi}{k^2} \int_{-\infty}^{\infty} d^3\mathbf{x} \, \rho(x)\exp -i\mathbf{k}.\mathbf{x}$$

$$\int_{0}^{\infty} x \exp(-2x/a) \cos(\chi x) = \frac{a^2(\chi^2 a^2 - 4)}{(\chi^2 a^2 + 4)^2}$$