Quantum Physics 2006/07

Lecture 17: More on the Dirac Equation

In the last lecture we showed that the Dirac equation for a free particle

$$i\hbar \frac{\partial}{\partial t} \psi(\underline{r}, t) \;=\; \left(-i\hbar c\, \underline{\alpha} \cdot \underline{\nabla} \;+\; \beta\, mc^2\right) \psi(\underline{r}, t) \tag{1}$$

has plane wave solutions $\psi(\underline{r}, t) = \exp(-ip \cdot x/\hbar)\, w(p)$ if the 4-component column vector $w(p)$, or 4-component *Dirac spinor*, satisfies the momentum- or p-space Dirac equation

$$p^0\, w(p) \;=\; \left(\underline{\alpha} \cdot \underline{p} \;+\; \beta mc\right)\, w(p) \tag{2}$$

and $p^0 = \pm\left\{(m^2c^2 \;+\; |\underline{p}|^2\right\}^{1/2}$.

Two-by-two block form of the Dirac equation

The p-space Dirac equation can be solved for $w(p)$ by brute force, but we shall introduce a more elegant formalism by writing the *four-component spinor* $w(p)$ in terms of two *two-component spinors* ϕ and χ:

$$w \;=\; \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

The p-space Dirac equation (2), for plane wave solutions becomes

$$p^0 \begin{pmatrix} \phi \\ \chi \end{pmatrix} \;=\; \begin{pmatrix} mc & \underline{\sigma} \cdot \underline{p} \\ \underline{\sigma} \cdot \underline{p} & -mc \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

Note that the elements of the 2×2 *block matrix* are themselves 2×2 matrices. More explicitly

$$p^0\, \phi \;=\; mc\, \phi \;+\; \underline{\sigma} \cdot \underline{p}\, \chi \tag{3}$$

$$p^0\, \chi \;=\; \underline{\sigma} \cdot \underline{p}\, \phi \;-\; mc\, \chi \tag{4}$$

Positive energy solutions: Let us first choose $p^0 > 0$, ie

$$p^0 \;=\; +\,(m^2c^2 + |\underline{p}|^2)^{1/2} \equiv p^0_+ \;=\; \frac{E}{c} \quad (E > 0)$$

Equation (4) can be used to solve for χ in terms of ϕ

$$\chi \;=\; \frac{\underline{\sigma} \cdot \underline{p}}{p^0 + mc}\, \phi \tag{5}$$

which we can substitute back into equation (3) to obtain

$$p^0\, \phi \;=\; \left\{ mc \;+\; \frac{(\underline{\sigma} \cdot \underline{p})^2}{p^0 + mc} \right\} \phi$$

but $(\underline{\sigma} \cdot \underline{p})^2 \;=\; |\underline{p}|^2$ (exercise), therefore $w(p)$ is a free particle solution of the Dirac equation for all two-component spinors ϕ if

$$(p^0)^2 \;+\; p^0 mc \;=\; p^0 mc \;+\; (mc)^2 \;+\; |\underline{p}|^2$$

which reduces to the usual relation between energy & momentum.

Therefore, the two positive-energy plane-wave solutions can be written as

$$w^{(1),(2)}(p) \;=\; \begin{pmatrix} \phi^{(1),(2)} \\ \left(\dfrac{c\,\underline{\sigma}\cdot\underline{p}}{E+mc^2}\right)\phi^{(1),(2)} \end{pmatrix}$$

It is conventional to choose the two linearly-independent two-spinors

$$\phi^{(1)} \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \phi^{(2)} \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Negative energy solutions: Now choose $p^0 < 0$, ie

$$p^0 \;=\; -\,(m^2c^2 + |\underline{p}|^2)^{1/2} \equiv p^0_- \;=\; -\,\frac{E}{c} \quad (E > 0)$$

It is conventional to write down the two negative energy solutions with spatial momenta $-\underline{p}$, *ie* with $p^\mu_- = (p^0_-, -\underline{p})$. By solving equation (3) for ϕ in terms of χ we obtain (exercise):

$$w^{(3),(4)}(-p) \;=\; \begin{pmatrix} \left(\dfrac{c\,\underline{\sigma}\cdot\underline{p}}{E+mc^2}\right)\chi^{(1),(2)} \\ \chi^{(1),(2)} \end{pmatrix}$$

For the negative energy solutions, it is conventional to choose the two linearly-independent two-spinors

$$\chi^{(1)} \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \chi^{(2)} \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The reason for the (apparently perverse) choice of negative momenta and two-spinors should become clearer when we interpret the negative energy states.

Summary: With the above conventions, the two positive energy solutions with four momenta $p^\mu_+ = (E/c,\ \underline{p})$ have components

$$w^{(1)}(p) \;=\; \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp^3}{E+mc^2} \\ \dfrac{c(p^1+ip^2)}{E+mc^2} \end{pmatrix} \qquad \text{and} \qquad w^{(2)}(p) \;=\; \begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p^1-ip^2)}{E+mc^2} \\ \dfrac{-cp^3}{E+mc^2} \end{pmatrix}$$

The two negative energy solutions with spatial momenta $-\underline{p}$, ie with $p^\mu_- = (-E/c,\ -\underline{p})$, have components

$$w^{(3)}(-p) \;=\; \begin{pmatrix} \dfrac{c(p^1-ip^2)}{E+mc^2} \\ \dfrac{-cp^3}{E+mc^2} \\ 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad w^{(4)}(-p) \;=\; \begin{pmatrix} \dfrac{cp^3}{E+mc^2} \\ \dfrac{c(p^1+ip^2)}{E+mc^2} \\ 1 \\ 0 \end{pmatrix}$$

Note: The quantity $E > 0$ in all equations above. Unfortunately, conventions in labelling the $w^{(i)}$ differ widely.

Rest-frame solutions, spin and angular momentum: When $\underline{p} = 0$ we have

$$w^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad w^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad w^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad w^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and the positive-energy solutions are simply

$$\psi^{(1)} = \exp(-imc^2t/\hbar) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \psi^{(2)} = \exp(-imc^2t/\hbar) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

These are degenerate in energy. Therefore, by the compatibility theorem, there must be another operator which commutes with the Hamiltonian (for $\underline{p} = 0$) and whose eigenvalues label the states. One such operator is

$$\Sigma^3 \equiv \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

The rest-frame *four-component spinors* $w^{(i)}(0)$ are eigenvectors of Σ^3 with eigenvalues ± 1.

The appearance of the Pauli spin matrix σ^3 suggests that we interpret the Dirac equation as describing a spin 1/2 particle. If we introduce the three 4×4 matrices

$$\Sigma^i \equiv \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \qquad \text{or, in vector notation} \qquad \underline{\Sigma} \equiv \begin{pmatrix} \underline{\sigma} & 0 \\ 0 & \underline{\sigma} \end{pmatrix}$$

$$\left.\begin{array}{lll} \text{Then} & \left(\frac{1}{2}\hbar\,\underline{\Sigma}\right) \cdot \left(\frac{1}{2}\hbar\,\underline{\Sigma}\right) = \frac{3}{4}\hbar^2\,\hat{1} = s(s+1)\,\hbar^2\,\hat{1} \ \text{with} \ s = \frac{1}{2} \\ \text{and} & \frac{1}{2}\hbar\,\Sigma^3 \qquad \text{has eigenvalues} \ \pm\frac{1}{2}\hbar \end{array}\right\} \Rightarrow \ \hat{\underline{s}} = \frac{1}{2}\hbar\,\underline{\Sigma}$$

ie we interpret $\frac{1}{2}\hbar\underline{\Sigma}$ as the *spin operator* for the Dirac theory – the Dirac particle *necessarily* has an *intrinsic spin* which is not related to ordinary orbital angular momentum.

However, $\underline{\Sigma}$ does *not* commute with the hamiltonian $c\underline{\alpha} \cdot \hat{\underline{p}} + \beta mc^2$ in any frame other than the rest frame, $\underline{p} = 0$, so the expectation value of $\underline{\Sigma}$ is not a conserved quanity.

Also, the operator $\hat{\underline{L}} = \hat{\underline{r}} \times \hat{\underline{p}}$ does *not* commute with the Hamiltonian in any frame other than the rest frame, so orbital angular momentum isn't a conserved quantity either.

However, the operator

$$\hat{\underline{J}} = \hat{\underline{L}} + \frac{1}{2}\hbar\,\underline{\Sigma}$$

commutes with the Hamiltonian in all frames, suggesting that it may be interpreted as the operator for the *total* angular momentum, and this ***is** conserved* (tutorial).

Helicity: As we have seen, there are two independent states for any given four-momentum. A different (p-space) operator which commutes with $c\underline{\alpha} \cdot \underline{p} + \beta mc^2$, and which can be used to label the states is the *helicity operator*

$$\hat{h}(\underline{p}) \;=\; \begin{pmatrix} \dfrac{\underline{\sigma}\cdot\underline{p}}{|\underline{p}|} & 0 \\ 0 & \dfrac{\underline{\sigma}\cdot\underline{p}}{|\underline{p}|} \end{pmatrix}$$

which has eigenvalues ± 1. Therefore, general plane wave states with $\underline{p} \neq 0$ can be chosen to be helicity eigenstates. (Of course, it still remains to *interpret* the negative energy states!)

Interpreting the negative energy solutions

As we saw, the plane wave solutions of the Dirac equation satisfy the energy-momentum relation

$$E \;=\; \pm\left(|\underline{p}|^2 c^2 \;+\; m^2 c^4\right)$$

therefore

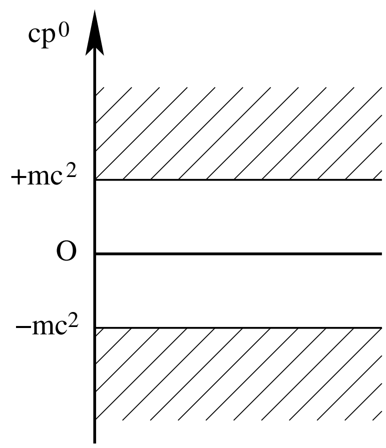
$$\textbf{either} \quad E \;\geq\; mc^2 \qquad \textbf{or} \quad E \;\leq -mc^2$$

ie there is a continuum of positive energy states starting at $E = mc^2$ and a continuum of negative energy states going down from $E = -mc^2$.

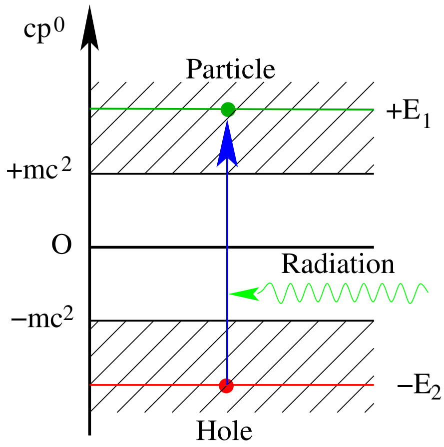
Since the Dirac equation appears to describe spin-half particles, let's assume these particles are electrons, and let's take the negative energy solutions seriously. The problem we must address is the following:

> What is to prevent a positive energy electron from making transitions under the influence of a perturbation to negative energy states?

A solution to this problem was suggested by Dirac in 1930.



The Dirac Sea: Taking the negative energy states seriously, Dirac proposed that all negative energy states are filled, each energy level holding two electrons with opposite spins. Since electrons are fermions, he then evoked the Pauli exclusion principle to prevent any transition of a positive energy electron to a negative energy state! In this picture the 'vacuum' is an infinite sea of negative energy electrons – the Dirac sea. He then argued that the infinite negative energy and infinite negative charge of this 'vacuum', are unobservable – we only measure finite changes of charge and energy *relative* to this vacuum.



Pair Production: One important consequence of this picture is that we can excite a negative energy particle from the 'sea' into a positive energy state.

Suppose an electron in the 'sea' absorbs photons[1] with sufficient energy ($> 2mc^2$) to make a transition to a state in the positive energy continuum. What we will observe is an electron of charge $-e$ and energy $+E_1$, say, together with a 'hole' in the sea. The 'hole' which is the *absence* of an electron with charge $-e$ and energy $-E_2$ would be interpreted by an observer as a *particle* of charge $+e$ and *energy* $+E_2$, in other words as a *positive* energy *anti-particle* or *positron*. Furthermore, the threshold for this process is just $2mc^2$, the size of the gap in the energy eigenvalue spectrum, and we have arrived at a description of electron-positron *pair production*. Thus Dirac predicted the existence of antiparticles.

- Although we started with a single-particle wave equation, the Dirac theory forces us into a many-particle interpretation.
- The positron was discovered four years later, thus 'confirming' Dirac's prediction.
- The *absence* of a *negative energy* particle with spin *'up'* in its rest-frame is equivalent to the *presence* of a *positive-energy* particle with spin *'down'*. This 'explains' the 'apparently perverse' choice of negative-momentum solutions and two-component spinors we made on page 2.

  The full solution requires Relativistic Quantum Field Theory (RQFT) , the subject of a fifth-year course.
- The Dirac sea picture doesn't work for bosons. There is no Pauli principle, therefore nothing can seemingly stop the positive energy particles decaying into oblivion. This is one of the reasons that lead Dirac to discard the Klein-Gordon equation for spinless particles. However, it turns out that there is no such problem in RQFT.

Covariant form of the Dirac equation – non-examinable

In most advanced applications of the Dirac equation, a covariant notation is used. Anyone taking RQFT next year is advised to become familiar with it now.

Defining the 'natural' system of units, $\hbar = c = 1$, the Dirac equation for a free particle is

$$i\frac{\partial}{\partial t}\,\psi(\underline{r},t) \;=\; (\,-i\,\underline{\alpha}\cdot\underline{\nabla}\;+\;\beta\,m)\,\psi(\underline{r},t)$$

[1] At least two photons must be absorbed to conserve 4-momentum.

If we multiply by β:

$$i\beta\frac{\partial}{\partial t}\,\psi(\underline{r},t) \;=\; (\,-i\,\beta\,\underline{\alpha}\cdot\underline{\nabla}\;+\;m)\,\psi(\underline{r},t) \tag{6}$$

and introduce the matrices

$$\begin{aligned}\gamma^0 &\equiv \beta\\ \gamma^i &\equiv \beta\,\alpha^i\end{aligned}$$

then we may rewrite equation (6) as

$$\left\{i\left(\gamma^0\,\frac{\partial}{\partial x^0}\;+\;\gamma^i\,\frac{\partial}{\partial x^i}\right)\;-\;m\right\}\,\psi(x)\;=\;0$$

where $x = x^\mu$ $(\mu = 0,\,\cdots,\,3)$. More compactly,

$$\begin{aligned}(i\gamma^\mu\,\partial_\mu\;-\;m)\;\psi(x)\;&=\;0\\ \text{or}\qquad (i\not{\partial}\;-\;m)\;\psi(x)\;&=\;0\end{aligned}$$

where we have introduced the Feynman *slash* notation:

$$\not{a}\;\equiv\;\gamma^\mu\,a_\mu\;=\;\gamma_\mu\,a^\mu$$

pronounced 'a-slash'. Similarly, $\not{\partial}$ is 'd-slash'.

Positive energy plane-wave solutions of the type $\psi(x) = \exp(-ip\cdot x)\,u(p)$ thus satisfy

$$(\gamma^\mu p_\mu\;-\;m)\;u(p)\;\equiv\;(\not{p}\;-\;m)\;u(p)\;=\;0$$

whilst negative energy (negative four-momentum) solutions $\psi(x) = \exp(+ip{\cdot}x)\,v(-p)$ satisfy

$$(\gamma^\mu p_\mu\;+\;m)\;v(-p)\;\equiv\;(\not{p}\;+\;m)\;v(-p)\;=\;0$$

It is easy to verify that the gamma matrices satisfy the anticommutation relations known as the *Clifford algebra*

$$\gamma^\mu\,\gamma^\nu\;+\;\gamma^\nu\,\gamma^\mu\;\equiv\;\{\gamma^\mu,\,\gamma^\nu\}\;=\;2\,g^{\mu\nu}$$

In the standard representation of $\underline{\alpha}$ and β

$$\gamma^i\;=\;\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}\qquad\text{and}\qquad\gamma^0\;=\;\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

so that γ^0 is hermitian $\gamma^{0\dagger} = \gamma^0$, the γ^i are anti-hermitian $\gamma^{i\dagger} = -\gamma^i$, thus $(\gamma^0)^2 = 1$ and $(\gamma^i)^2 = -1$.

It is also convenient to work with

$$\begin{aligned}\overline{\psi}\;&\equiv\;\psi^\dagger\,\gamma^0\\ \text{and similarly}\qquad \overline{u}\;&\equiv\;u^\dagger\,\gamma^0\end{aligned}$$

where $\overline{\psi}$ is pronounced 'psi-bar'.

The new notation treats space and time on an (even more) equal basis, and is known as the *covariant formulation.* One can derive the properties of the Dirac wave-function $\psi(x)$ under Lorentz boosts and use them to prove explicitly that the conserved current

$$\begin{aligned}j^\mu\;&\equiv\;\left(\,\psi^\dagger\psi,\;\psi^\dagger\,\underline{\alpha}\,\psi\,\right)\\ &=\;\overline{\psi}\gamma^\mu\psi\end{aligned}$$

does indeed transform as a 4-vector under Lorentz transformations. Similarly, one can show that the quantity $\overline{\psi}\psi$ is invariant under Lorentz transformations, ie it's a Lorentz scalar.