Quantum Physics 2011/12

Solutions to Tutorial Sheet 2: Perturbations

A note about notation

In perturbation theory we need distinguish between different unperturbed systems (hydrogen, square well, SHO) perturbed and unperturbed systems, order of perturbation theory and the quantum numbers defining a particular state.

There is no unique notation in quantum mechanics: different authors use different symbols for the same thing. The best you can do is be sure that you properly define your notation in your answers, and understand what it means.

1. *A quantum dot is a self assembled nanoparticle in which a single electron state can be confined. A model for such an object is a particle moving in one dimension in the potential*

$$V(x) = \infty, \quad |x| > a, \qquad V(x) = V_0 \cos(\pi x/2a), \quad |x| \leq a$$

Identify an appropriate unperturbed system and perturbation term. The potential is an infinite square well with a bump at the bottom. The appropriate unperturbed system is the infinite square well without the bump. The energy scale of the bump is set by V_0, so perturbation theory will be valid provided this is less than the energy difference between square-well states, i.e.

$$V_0 << \frac{\hbar^2\pi^2}{8ma^2}(2n+1)$$

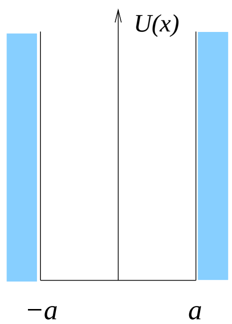
For sufficiently large n, this will always be true. But we're only asked about n=1 and n=2.

Calculate the energies of the two lowest states to first order in perturbation theory.

We write

$$\hat{H} = \hat{H}_0 + \Delta\hat{V}$$

where $\hat{H}_0$ is the Hamiltonian of the 1-dimensional infinite square well:



$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + U(x)$$

and

$$U(x) = \begin{cases} \infty & |x| > a \\ 0 & |x| \leq a \end{cases}$$

with known exact solution: the eigenvalues of $\hat{H}_0$ are given by

$$E_n^{(0)} = \frac{\hbar^2\pi^2n^2}{8ma^2}, \qquad n = 1, 2, 3, \cdots$$

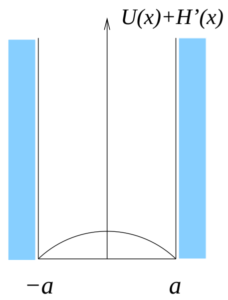
and the corresponding eigenfunctions are

$$u_n(x) = \frac{1}{\sqrt{a}} \left\{ \begin{array}{c} \cos \\ \sin \end{array} \right\} \left(\frac{n\pi x}{2a} \right) \qquad \left\{ \begin{array}{l} n \text{ odd} \\ n \text{ even} \end{array} \right.$$

The perturbation is

$$\Delta\hat{V} = V_0 \cos\left(\frac{\pi x}{2a}\right) \qquad |x| \le a$$

and to first order in perturbation theory, the energy shifts are given by



$$\Delta E_n = \langle u_n | \Delta\hat{V} | u_n \rangle = \int_{-a}^{a} u_n^*(x) \Delta\hat{V} u_n(x)\, \mathrm{d}x$$

For the ground state:

$$\begin{aligned}
\Delta E_1 &= \frac{V_0}{a} \int_{-a}^{a} \cos\left(\frac{\pi x}{2a}\right) \cos\left(\frac{\pi x}{2a}\right) \cos\left(\frac{\pi x}{2a}\right) \mathrm{d}x \;=\; \frac{V_0}{a} \int_{-a}^{a} \cos^3\left(\frac{\pi x}{2a}\right) \mathrm{d}x \\
&= \frac{2V_0}{\pi} \int_{-\pi/2}^{+\pi/2} \cos^3\theta \,\mathrm{d}\theta \qquad \text{where } \theta \equiv \frac{\pi x}{2a} \\
&= \frac{2V_0}{\pi} \int_{-1}^{+1} \left(1 - \sin^2\theta\right) \mathrm{d}(\sin\theta) \qquad \text{using the identity } \cos^2\theta \equiv 1 - \sin^2\theta \\
&= \frac{2V_0}{\pi} \left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_{-1}^{+1}
\end{aligned}$$

Thus

$$\boxed{\Delta E_1 = \frac{8V_0}{3\pi}}$$

Similarly for the first excited state ($n = 2$):

$$\begin{aligned}
\Delta E_2 &= \frac{V_0}{a} \int_{-a}^{a} \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi x}{2a}\right) \sin\left(\frac{\pi x}{a}\right) \mathrm{d}x \\
&= \frac{2V_0}{\pi} \int_{-\pi/2}^{+\pi/2} \sin^2 2\theta \, \cos\theta \,\mathrm{d}\theta \qquad \text{where } \theta \equiv \frac{\pi x}{2a} \\
&= \frac{8V_0}{\pi} \int_{-\pi/2}^{+\pi/2} \sin^2\theta \, \cos^3\theta \,\mathrm{d}\theta \qquad \text{using the identity } \sin 2\theta \equiv 2\sin\theta\cos\theta \\
&= \frac{8V_0}{\pi} \int_{-1}^{+1} \sin^2\theta \left(1 - \sin^2\theta\right) \mathrm{d}(\sin\theta) \\
&= \frac{8V_0}{\pi} \left[\frac{1}{3}\sin^3\theta - \frac{1}{5}\sin^5\theta\right]_{-1}^{+1}
\end{aligned}$$

Thus

$$\boxed{\Delta E_2 = \frac{32V_0}{15\pi}}$$

What is the sign of V_0?

Since the electron is attracted to the atoms comprising the dot, it is likely to be more strongly bound at the centre. Hence V_0 will be negative.

State two ways in which the colour of a material containing dots can be shifted towards the red.

To vary the colour, we need to change the difference between energy levels. The simplest way to do this is to use a larger dot, increasing a in the unperturbed energies and hence reducing (redshifting) their energy. Alternately, we just saw that the perturbation lowered the ground state by more than the excited state: thus a smaller V_0 (i.e. a material which less strongly bound the electron) would give a red shift.

2. *A particle moves in one dimension in the potential*

$$V(x) = \infty, \quad |x| > a, \qquad V(x) = V_0 \sin(\pi x/a), \quad |x| \le a$$

- *show that the first order energy shift is zero;*

Just as in the previous question, we write

$$\hat{H} = \hat{H}_0 + \Delta\hat{V}$$

where $\hat{H}_0$ is the Hamiltonian of the 1-dimensional infinite square well.

The first order formula for the energy shift is then

$$\Delta E_n = \langle n|\Delta\hat{V}|n\rangle = \int_{-a}^{a} u_n^*(x)\Delta\hat{V} u_n(x)\,\mathrm{d}x$$

Thus

$$\Delta E_n = \frac{V_0}{a}\int_{-a}^{a} \left\{ \begin{array}{c} \cos \\ \sin \end{array} \right\} \left(\frac{n\pi x}{2a}\right) \cdot \sin\left(\frac{\pi x}{a}\right) \cdot \left\{ \begin{array}{c} \cos \\ \sin \end{array} \right\} \left(\frac{n\pi x}{2a}\right) \mathrm{d}x \qquad \left\{ \begin{array}{l} n \text{ odd} \\ n \text{ even} \end{array} \right.$$

We observe that the integrand is always an odd function of x regardless of the value of n, and hence the integral from $-a$ to a must vanish. Hence

$$\boxed{\Delta E_n \equiv 0}$$

- * *obtain an expression for the second order correction to the energy of the ground state.*

At second order, the correction to the ground-state energy is given by

$$\Delta E_1^{(2)} = \sum_{m \ne 1} \frac{|\langle m|\Delta\hat{V}|1\rangle|^2}{\left(E_1^{(0)} - E_m^{(0)}\right)}$$

For m odd:

$$\langle m|\Delta\hat{V}|1\rangle = \frac{V_0}{a}\int_{-a}^{a} \cos\left(\frac{m\pi x}{2a}\right) \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi x}{2a}\right) \mathrm{d}x \equiv 0$$

because the integrand is an odd function of x.

For m even, no such argument applies and we have to work hard to evaluate the required integral!

$$
\begin{aligned}
\langle m|\Delta\hat{V}|1\rangle &= \frac{V_0}{a}\int_{-a}^{a} \sin\left(\frac{m\pi x}{2a}\right)\ \sin\left(\frac{\pi x}{a}\right)\ \cos\left(\frac{\pi x}{2a}\right)\ \mathrm{d}x \\
&= \frac{2V_0}{\pi}\int_{-\pi/2}^{+\pi/2} \sin m\theta\ \sin 2\theta\ \cos\theta\ \mathrm{d}\theta \qquad \text{where } \theta \equiv \frac{\pi x}{2a} \\
&= \frac{4V_0}{\pi}\int_{0}^{\pi/2} \sin m\theta\ \sin 2\theta\ \cos\theta\ \mathrm{d}\theta
\end{aligned}
$$

We now use the trigonometric identities (from $\cos A \cos B$ etc.) :

$$
\begin{aligned}
\sin m\theta\ \sin 2\theta &\equiv \frac{1}{2}\left[\cos(m-2)\theta - \cos(m+2)\theta\right] \\
\cos(m-2)\theta\ \cos\theta &\equiv \frac{1}{2}\left[\cos(m-1)\theta + \cos(m-3)\theta\right] \\
\cos(m+2)\theta\ \cos\theta &\equiv \frac{1}{2}\left[\cos(m+3)\theta + \cos(m+1)\theta\right]
\end{aligned}
$$

to write

$$
\begin{aligned}
\langle m|\Delta\hat{V}|1\rangle &= \frac{V_0}{\pi}\int_{0}^{\pi/2} \left[\cos(m-1)\theta + \cos(m-3)\theta - \cos(m+3)\theta - \cos(m+1)\theta\right]\ \mathrm{d}\theta \\
&= \frac{V_0}{\pi}\left[\frac{\sin(m-1)\pi/2}{(m-1)} + \frac{\sin(m-3)\pi/2}{(m-3)} - \frac{\sin(m+3)\pi/2}{(m+3)} - \frac{\sin(m+1)\pi/2}{(m+1)}\right]
\end{aligned}
$$

Recall that m is even and ≥ 2 so that

$$
\begin{aligned}
\sin(m-1)\pi/2 &= \sin(m+3)\pi/2 = (-1)^{\frac{m}{2}+1} \\
\sin(m-3)\pi/2 &= \sin(m+1)\pi/2 = (-1)^{\frac{m}{2}}
\end{aligned}
$$

Thus

$$
\langle m|\Delta\hat{V}|1\rangle = (-1)^{m/2}\ \frac{V_0}{\pi}\left[-\frac{1}{(m-1)} + \frac{1}{(m-3)} + \frac{1}{(m+3)} - \frac{1}{(m+1)}\right]
$$

Putting everything over a common denominator yields

$$
\boxed{\langle m|\Delta\hat{V}|1\rangle = (-1)^{m/2}\ \frac{V_0}{\pi}\frac{16m}{(m^2-9)(m^2-1)}}
$$

Thus

$$
\langle 2|\Delta\hat{V}|1\rangle = \frac{32V_0}{15\pi}; \qquad \langle 4|\Delta\hat{V}|1\rangle = \frac{64V_0}{105\pi} \quad \text{etc}
$$

Now for the unperturbed Hamiltonian

$$
(E_1 - E_2) = \frac{\pi^2\hbar^2}{8ma^2}(1-4) = -\frac{3\pi^2\hbar^2}{8ma^2}
$$

and so we find that the second order energy shift is an infinite sum of the even m terms, changing the index so that $n = 2m$ ...

$$\Delta E_1^{(2)} = \sum_n \left(\frac{V_0}{\pi} \frac{n}{2(n^2-36)(n^2-4)} \right)^2 \cdot \frac{8ma^2}{(1-4n^2)\pi^2\hbar^2}$$

Noting that this ultimately scales as n^{-9}, we can assume that the sum converges, and we already saw that the first order term is about four times the second.

$$\Delta E_1^{(2)} = -\left(\frac{32V_0}{15\pi}\right)^2 \frac{8ma^2}{3\pi^2\hbar^2} + \ldots$$

3. *The 1-d anharmonic oscillator: a particle of mass m is described by the Hamiltonian*

$$\hat{H} = \frac{\hat{p}^2}{2m} + \tfrac{1}{2}m\omega^2\,\hat{x}^2 + \gamma\hat{x}^4$$

- *Assuming that γ is small, use first-order perturbation theory to calculate the ground state energy;*
- **show more generally that the energy eigenvalues are approximately*

$$E_n \;\simeq\; (n+\tfrac{1}{2})\hbar\omega + 3\gamma\left(\frac{\hbar}{2m\omega}\right)^2(2n^2+2n+1)$$

Hint: to evaluate matrix elements of powers of $\hat{x}$, write $\hat{x}$ in terms of the harmonic oscillator raising and lowering operators $\hat{a}$ and $\hat{a}^\dagger$. Recall that the raising and lowering operators are defined by

$$\hat{a} \;\equiv\; \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2m\omega\hbar}}\,\hat{p} \qquad \text{and} \qquad \hat{a}^\dagger \;\equiv\; \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - \frac{i}{\sqrt{2m\omega\hbar}}\,\hat{p}$$

with the properties that

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \text{and} \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Since the unperturbed energy eigenstates are non-degenerate, we can use the standard result to calculate the first order energy level shifts:

$$\boxed{\Delta E_n = \langle n|\Delta\hat{V}|n\rangle = \gamma\langle n|\hat{x}^4|n\rangle}$$

The integral associated with the ground state is then

$$\gamma\sqrt{\frac{m\omega}{\pi\hbar}}\int x^4\exp\left(-m\omega x^2/\hbar\right)\mathrm{dx}$$

Which is deeply unpleasant. You can look it up:

$$\int_{-\infty}^{\infty} x^{2n}\exp\left(-ax^2\right)\mathrm{dx} = \frac{1}{2^{n+1}}\sqrt{\frac{\pi}{a^{2n+1}}}\prod_{j=1}^{2n-1}(2j-1)$$

so

$$\gamma\sqrt{\frac{m\omega}{\pi\hbar}}\int x^4 \exp\left(-m\omega x^2/\hbar\right) \mathrm{dx} = 3\gamma\left(\frac{\hbar}{2m\omega}\right)^2$$

But to do it yourself, its best to use raising and lowering operators.

The unperturbed energy eigenstates satisfy

$$\hat{H}_0|n\rangle = E_n|n\rangle \qquad \text{with} \qquad E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega, \quad n = 0, 1, 2, \ldots$$

We can write $\hat{x}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right)$$

Thus

$$\Delta E_n = \gamma\left(\frac{\hbar}{2m\omega}\right)^2 \langle n|\left(\hat{a}+\hat{a}^\dagger\right)^4|n\rangle$$

Expanding the bracket $\left(\hat{a}+\hat{a}^\dagger\right)^4$ looks pretty awful until you realise that, from the raising and lowering properties and the orthonormality of the energy eigenstates, *only terms which contain an equal number of raising and lowering operators will give a non-zero contribution to the diagonal matrix element* $\langle n|\Delta\hat{V}|n\rangle$. Thus

$$\Delta E_n = \gamma\left(\frac{\hbar}{2m\omega}\right)^2 \langle n|\left(\hat{a}^2\hat{a}^{\dagger 2}+\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger+\hat{a}\hat{a}^\dagger\hat{a}^\dagger\hat{a}+\hat{a}^\dagger\hat{a}\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger 2}\hat{a}^2\right)|n\rangle$$

For the ground state, $n = 0$, this simplifies even further because $\hat{a}|0\rangle = 0$, so the third, fifth and sixth terms all give zero. Evaluating the remaining three terms,

$$\begin{aligned}
\hat{a}^2\hat{a}^{\dagger 2}|0\rangle &= \hat{a}^2\hat{a}^\dagger|1\rangle = \sqrt{2}\,\hat{a}^2|2\rangle = \sqrt{2}\,\sqrt{2}\,\hat{a}|1\rangle = 2|0\rangle \\
\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger|0\rangle &= \hat{a}\hat{a}^\dagger\hat{a}|1\rangle = \hat{a}\hat{a}^\dagger|0\rangle = \hat{a}|1\rangle = |0\rangle \\
\hat{a}^\dagger\hat{a}\hat{a}\hat{a}^\dagger|0\rangle &= \hat{a}^\dagger\hat{a}\hat{a}|1\rangle = \hat{a}^\dagger\hat{a}|0\rangle = 0
\end{aligned}$$

Thus

$$\Delta E_0 = 3\gamma\left(\frac{\hbar}{2m\omega}\right)^2\langle 0|0\rangle = 3\gamma\left(\frac{\hbar}{2m\omega}\right)^2$$

For the general case, we have to work a little harder;

$$\begin{aligned}
\hat{a}^2\hat{a}^{\dagger 2}|n\rangle &= \sqrt{n+1}\,\hat{a}^2\hat{a}^\dagger|n+1\rangle = \sqrt{(n+1)(n+2)}\,\hat{a}^2|n+2\rangle \\
&= \sqrt{n+1}\,(n+2)\hat{a}|n+1\rangle = (n+1)(n+2)|n\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger|n\rangle &= \sqrt{n+1}\,\hat{a}\hat{a}^\dagger\hat{a}|n+1\rangle = (n+1)\,\hat{a}\hat{a}^\dagger|n\rangle = (n+1)\sqrt{n+1}\,\hat{a}|n+1\rangle \\
&= (n+1)^2|n\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{a}^\dagger\hat{a}\hat{a}\hat{a}^\dagger|n\rangle &= \sqrt{n+1}\,\hat{a}^\dagger\hat{a}\hat{a}|n+1\rangle = (n+1)\hat{a}^\dagger\hat{a}|n\rangle = (n+1)\sqrt{n}\,\hat{a}^\dagger|n\rangle \\
&= n(n+1)|n\rangle
\end{aligned}$$

but we also need to consider the previously neglected terms:

$$\begin{aligned}
\hat{a}\hat{a}^\dagger\hat{a}^\dagger\hat{a}|n\rangle &= \sqrt{n}\,\hat{a}\hat{a}^\dagger\hat{a}^\dagger|n-1\rangle \;=\; n\hat{a}\hat{a}^\dagger|n\rangle \;=\; n\sqrt{n+1}\,\hat{a}|n+1\rangle \\
&= n(n+1)|n\rangle \\
\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|n\rangle &= \sqrt{n}\,\hat{a}^\dagger\hat{a}\hat{a}^\dagger|n-1\rangle \;=\; n\hat{a}^\dagger\hat{a}|n\rangle \;=\; n\sqrt{n}\,\hat{a}^\dagger|n-1\rangle \\
&= n^2|n\rangle \\
\hat{a}^{\dagger 2}\hat{a}^2|n\rangle &= \sqrt{n}\,\hat{a}^{\dagger 2}\hat{a}|n-1\rangle \;=\; \sqrt{n(n-1)}\,\hat{a}^{\dagger 2}|n-2\rangle \;=\; (n-1)\sqrt{n}\,\hat{a}^\dagger|n-1\rangle \\
&= n(n-1)|n\rangle
\end{aligned}$$

Collecting up terms:

$$(n+1)(n+2)+(n+1)^2+2n(n+1)+n^2+n(n-1) = 6n^2+6n+3$$

so that the shift in energy to first order is

$$\boxed{\Delta E_n \;=\; 3\gamma\left(\frac{\hbar}{2m\omega}\right)^2\left(2n^2+2n+1\right)\langle n|n\rangle = 3\gamma\left(\frac{\hbar}{2m\omega}\right)^2\left(2n^2+2n+1\right)}$$

As a check, putting $n=0$ reproduces the ground-state shift we obtained above.

4. *A 1-dimensional harmonic oscillator of mass m carries an electric charge, q. A weak, uniform, static electric field of magnitude $\mathcal{E}$ is applied in the x-direction. Show that, to first order in perturbation theory, the oscillator energy levels are unchanged, and calculate the second-order shift. Can you show that the second-order result is in fact exact?*

   Hint: to evaluate matrix elements of $\hat{x}$, write $\hat{x}$ in terms of the harmonic oscillator raising and lowering operators $\hat{a}$ and $\hat{a}^\dagger$ and use the results $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

   The perturbation is just the potential energy of a particle of charge q in a constant electric field of magnitude $\mathcal{E}$:

$$\Delta\hat{V} = -q\mathcal{E}\hat{x}$$

   so that to lowest order, the energy shift of the nth level is

$$\Delta E_n = -q\mathcal{E}\langle n|\hat{x}|n\rangle = -q\mathcal{E}\int_{-\infty}^{\infty} u_n^*(x)\,x\,u_n(x)\,\mathrm{d}x \equiv 0$$

   since the integrand is an odd function of x. So to lowest order, the energy levels are unchanged.

   At second order, the shift is given by

$$\Delta E_n^{(2)} = \sum_{m\neq n}\frac{|V_{mn}|^2}{E_n^{(0)}-E_m^{(0)}} = \sum_{m\neq n}(-q\mathcal{E})^2\frac{|\langle m|\hat{x}|n\rangle|^2}{\hbar\omega(n-m)}$$

   Now the matrix element can be written in terms of matrix elements of $\hat{a}$ and $\hat{a}^\dagger$ using

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right)$$

Thus

$$\langle m|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle m|\left(\hat{a}+\hat{a}^\dagger\right)|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\delta_{m,n-1}+\sqrt{n+1}\,\delta_{m,n+1}\right]$$

which tells us that only the terms with $m = n \pm 1$ contribute to the sum. Plugging this result in we find that

$$\Delta E_n^{(2)} = \frac{q^2\mathcal{E}^2}{\hbar\omega}\frac{\hbar}{2m\omega}\left[\frac{n}{1}+\frac{(n+1)}{-1}\right] = -\frac{q^2\mathcal{E}^2}{2m\omega^2}$$

To see that this is actually the exact solution, we note that the full Hamiltonian can be written

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{1}{2}m\omega^2x^2 - q\mathcal{E}\,x$$

We introduce a new variable

$$\xi \equiv x - \frac{q\mathcal{E}}{m\omega^2}$$

Noting that

$$\xi^2 = x^2 - \frac{2q\mathcal{E}}{m\omega^2}x + \frac{q^2\mathcal{E}^2}{m^2\omega^4}$$

we see that the Hamiltonian can be rewritten as

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}+\frac{1}{2}m\omega^2\xi^2 - \frac{q^2\mathcal{E}^2}{2m\omega^2}$$

and thus the energy of the n^{th} state of the full Hamiltonian has energy eigenvalues.

$$E_n = (n+\frac{1}{2})\hbar\omega - \frac{q^2\mathcal{E}^2}{2m\omega^2}$$

which just differs from the usual 1-d oscillator Hamiltonian by a constant shift. Note that each energy level in the perturbed state corresponds to a level in the unperturbed state, so the quantum number n can still be used to label them.

5. *Starting from the relativistic expression for the total energy of a single particle,* $E = (m^2c^4+p^2c^2)^{1/2}$, *and expanding in powers of* p^2, *obtain the leading relativistic correction to the kinetic energy.*

   For a single relativistic particle

$$\begin{aligned} E = (m^2c^4+p^2c^2)^{1/2} &= mc^2\left[1+\frac{p^2}{m^2c^2}\right]^{1/2} \\ &= mc^2\left[1+\frac{p^2}{2m^2c^2}+\frac{1}{2!}\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\frac{p^4}{m^4c^4}+\ldots\right] \\ &= mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}+\cdots \end{aligned}$$

   Recalling that in special relativity, the kinetic energy is defined to be the difference between the total energy and the rest energy, gives the desired result:

$$T \equiv E - mc^2 = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2}+\ldots$$

for a plane wavefunction $\Phi(x) = A\cos(kx)$, and determine whether $\Phi(x)$ is an eigenstate for a relativistic free particle.

Taking $\Delta T = -\frac{p^4}{8m^3c^2} = -\frac{\hbar^4\nabla^4}{8m^3c^2}$, We need to evaluate:

$$\begin{aligned}
\int \Phi(x)\Delta\hat{T}\Phi(x)dx &= -\int A^2\cos(kx)\frac{\hbar^4}{8m^3c^2}\frac{d^4}{dx^4}\cos(kx)\,\mathrm{dx} \\
&= \int A^2\frac{\hbar^4k^4}{8m^3c^2}\cos^2(kx)\,\mathrm{dx}
\end{aligned}$$

The normalisation is straightforward:

$$A^2 = \int \cos^2(kx)\,\mathrm{dx}$$

Thus for a normalised particle, the integrals cancel and we have:

$$\Delta T = -\frac{\hbar^4k^4}{8m^3c^2}$$

Meanwhile, using the above

$$\hat{T}A\cos(kx) = \left(\frac{\hbar^2k^2}{2m} - \frac{\hbar^4k^4}{8m^3c^2}\right)A\cos(kx)$$

Hence $\cos(kx)$ is still an eigenstate for a free particle, with energy

$$E_k = \frac{\hbar^2k^2}{2m} - \frac{\hbar^4k^4}{8m^3c^2}$$