# Quantum Physics 2011/12

## Solutions to Tutorial Sheet 4: Time-dependence

1. *An easy one to start with! A particle moving in the infinite 1-d square well potential*

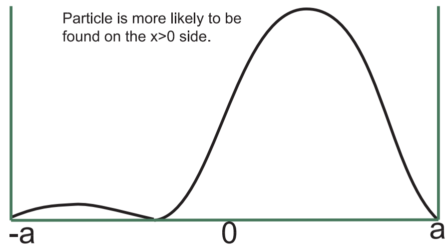
$$V(x) = 0 \quad \text{for } |x| < a, \quad V(x) = \infty \quad \text{for } |x| > a$$

*is set up in the initial state ($t = 0$) described by the wavefunction*

$$\Psi(x, 0) \equiv \psi(x) = [u_1(x) + u_2(x)] / \sqrt{2}$$

*where $u_1(x)$, $u_2(x)$ are the energy eigenfunctions corresponding to the energy eigenvalues $E_1$ and $E_2$ respectively. Sketch the probability density at $t = 0$.*

The wavefunction has positive interference for $x > a$, and negative for $x < a$. It is zero. at $\cos(\pi x/2a) - \sin(\pi x/a) = 0$, i.e. $-\pi/6$




*What is the wavefunction at time $t$ ?*

Recall that the expansion in energy eigenfunctions of a general solution of the TDSE is

$$\Psi(x, t) = \sum_n c_n \exp(-iE_n t/\hbar)\, u_n(x)$$

where the coefficients $c_n$ are constant in time if the Hamiltonian is time-independent, which is the case here, so that

$$\Psi(x, 0) = \sum_n c_n\, u_n(x)$$

In the present case, $c_1 = c_2 = 1/\sqrt{2}$ and $c_n = 0, n \geq 3$ and thus

$$\Psi(x, t) = \frac{1}{\sqrt{2}} \exp(-iE_1 t/\hbar)\, u_1(x) + \frac{1}{\sqrt{2}} \exp(-iE_2 t/\hbar)\, u_2(x)$$

*Calculate the probabilities $P_1$ and $P_2$ that at $t = 0$ a measurement of the total energy yields the results $E_1$ and $E_2$ respectively. Do $P_1$ and $P_2$ change with time?*

The probability of getting the result $E_n$ in a measurement of the energy at $t = 0$ is $P(E_n) = |c_n|^2$. Thus

$$\boxed{P(E_1) = P(E_2) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}}$$

At time $t$, the corresponding probabilities are:

$$\begin{aligned} |c_1 \exp(-iE_1 t/\hbar)|^2 &= |c_1|^2 = \frac{1}{2} \\ |c_2 \exp(-iE_2 t/\hbar)|^2 &= |c_2|^2 = \frac{1}{2} \end{aligned}$$

so that $P(E_1)$ and $P(E_2)$ are time independent.

*Calculate the probabilities $P_+(t)$ and $P_-(t)$ that at time $t$ the particle is in the intervals $0 < x < a$ and $-a < x < 0$ respectively and try to interpret your results.*

The probability that, at time $t$, the particle is in the interval $0 < x < a$ is

$$P_+(t) = \int_0^a |\Psi(x,t)|^2 \, \mathrm{d}x$$

since $|\Psi(x,t)|^2 \, \mathrm{d}x$ is the probability of finding the particle in the infinitesimal interval $x \to x + \mathrm{d}x$.

Now

$$\begin{aligned}
|\Psi(x,t)|^2 &= \frac{1}{2} \left|\exp(-iE_1 t/\hbar)\, u_1(x) + \exp(-iE_2 t/\hbar)\, u_2(x)\right|^2 \\
&= \frac{1}{2}|u_1(x)|^2 + \frac{1}{2}|u_2(x)|^2 + \frac{1}{2}\exp[-i(E_1 - E_2)t/\hbar] u_1(x) u_2^*(x) \\
&+ \frac{1}{2}\exp[i(E_1 - E_2)t/\hbar] u_1^*(x) u_2(x)
\end{aligned}$$

Defining $\omega \equiv (E_2 - E_1)/\hbar$, and noting that $u_1$ and $u_2$ are real, we obtain

$$P_+(t) = \frac{1}{2}\int_0^a |u_1(x)|^2 \mathrm{d}x + \frac{1}{2}\int_0^a |u_2(x)|^2 \mathrm{d}x + \cos\omega t \int_0^a u_1(x) u_2(x) \, \mathrm{d}x$$

We can write down the values of the first two integrals without explicit evaluation if we observe that both integrands are even functions of $x$, so that

$$\int_0^a |u_1(x)|^2 \mathrm{d}x = \frac{1}{2}\int_{-a}^a |u_1(x)|^2 \mathrm{d}x = \frac{1}{2}$$

from the normalisation, with a similar result for $u_2$. This tells us that particles in the eigenstates $u_1$ and $u_2$ have equal chance of being on either side of zero.

We still have to evaluate

$$\int_0^a u_1(x) u_2(x) \, \mathrm{d}x = \frac{1}{a}\int_0^a \cos\left(\frac{\pi x}{2a}\right) \sin\left(\frac{\pi x}{a}\right) \, \mathrm{d}x$$

on substituting in the explicit form of the lowest two eigenfunctions for the symmetric infinite square well.

Using the trignometric identity

$$\sin A \cos B \equiv \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$$

we obtain

$$\int_0^a u_1(x) u_2(x) \, \mathrm{d}x = \frac{1}{2a}\int_0^a \left\{\sin\left(\frac{3\pi x}{2a}\right) + \sin\left(\frac{\pi x}{2a}\right)\right\} \mathrm{d}x$$

$$\begin{aligned}
&= \frac{1}{2a}\left[-\frac{2a}{3\pi}\cos\left(\frac{3\pi x}{2a}\right) - \frac{2a}{\pi}\cos\left(\frac{\pi x}{2a}\right)\right]_0^a \\
&= \frac{1}{2a}\left[\frac{2a}{3\pi} + \frac{2a}{\pi}\right] \\
&= \frac{4}{3\pi}
\end{aligned}$$

Putting it all together, we find

$$\boxed{\begin{aligned}
P_+(t) &= \frac{1}{2} + \frac{4}{3\pi}\cos\omega t \\
P_-(t) &= 1 - P_+(t) = \frac{1}{2} - \frac{4}{3\pi}\cos\omega t
\end{aligned}}$$

Which implies that over time the probability distribution moves from side to side!

2. *A system has just two independent states, $|1\rangle$ and $|2\rangle$, represented by the column matrices*

$$|1\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad |2\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

*With respect to these two states, the Hamiltonian has a time-independent matrix representation*

$$\begin{pmatrix} E & V \\ V & E \end{pmatrix}$$

*where $E$ and $V$ are both real.*

*Show that the probability of a transition from the state $|1\rangle$ to the state $|2\rangle$ in the time interval $t$ is given* without approximation *by*

$$p(t) = \sin^2\left(\frac{Vt}{\hbar}\right)$$

*[**Hint:** expand the general state $|\Psi, t\rangle$ in terms of $|1\rangle$ and $|2\rangle$ and substitute in the* TDSE*. Note that $|1\rangle$ and $|2\rangle$ are not energy eigenstates!]*

The states $|1\rangle$ and $|2\rangle$ are NOT energy eigenstates, since the perturbed Hamiltonian is not diagonal! Nevertheless, they form a complete set and we can expand an arbitrary state as follows:

$$|\Psi, t\rangle = \sum_{r=1}^{2} c_r(t)|r\rangle$$

which has the matrix representation

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = c_1(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Substituting in the TDSE gives

$$i\hbar\begin{pmatrix} \dot{c}_1(t) \\ \dot{c}_2(t) \end{pmatrix} = \begin{pmatrix} E & V \\ V & E \end{pmatrix}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

which yields the coupled equations

$$\begin{aligned}
i\hbar\dot{c}_1 &= E\,c_1 + V\,c_2 \\
i\hbar\dot{c}_2 &= V\,c_1 + E\,c_2
\end{aligned}$$

The neatest way to solve these equations is to take the sum and difference:

$$\begin{aligned} i\hbar\,(\dot{c}_1+\dot{c}_2) &= (E+V)(c_1+c_2) \quad\Rightarrow\quad c_1+c_2 = A\exp[-i(E+V)t/\hbar] \\ i\hbar\,(\dot{c}_1-\dot{c}_2) &= (E-V)(c_1-c_2) \quad\Rightarrow\quad c_1-c_2 = B\exp[-i(E-V)t/\hbar] \end{aligned}$$

where $A$ and $B$ are constants of integration.

Alternatively one can find the eigenvalues and eigenvectors of the Hamiltonian $(1,1)/\sqrt{2}$ and $(1,-1)/\sqrt{2}$ and write down the time evolution of state vector in the usual way.

If the system is in the state $|1\rangle$ at $t=0$, then $c_1(0)=1$ and $c_2(0)=0$. Imposing these initial conditions gives $A=B=1$, so that

$$c_2(t) = \frac{1}{2}\exp(-iEt/\hbar)\left\{\exp(-iVt/\hbar)-\exp(iVt/\hbar)\right\} = -i\exp(-iEt/\hbar)\sin\left(\frac{Vt}{\hbar}\right)$$

Thus the probability that the system is in state $|2\rangle$ at time $t$ is

$$\boxed{p(t) = |c_2(t)|^2 = \sin^2\left(\frac{Vt}{\hbar}\right)}$$

*Compute the transition probability using first-order time-dependent perturbation theory, taking the unperturbed Hamiltonian matrix to be that for which $|1\rangle$ and $|2\rangle$ are energy eigenstates. By comparing with the exact result, deduce the conditions under which you expect the approximation to be good.*

We can treat the problem by perturbation theory by writing $\hat{H} = \hat{H}_0 + \hat{V}$, where

$$\hat{H}_0 \to \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \qquad \text{and} \qquad \hat{V} \to \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix}$$

so that $|1\rangle$ and $|2\rangle$ are eigenstates of $\hat{H}_0$ with eigenvalue $E \equiv E_1 = E_2$.

Time-dependent perturbation theory for the constant perturbation $\hat{V}$ then gives

$$p_{21}(t) = \frac{1}{\hbar^2}\left|\int_0^t V_{21}\,\exp[i(E_2-E_1)t'/\hbar]\,\mathrm{d}t'\right|^2 = \frac{V^2}{\hbar^2}\left|\int_0^t \mathrm{d}t'\right|^2 = \frac{V^2t^2}{\hbar^2}$$

This is the leading term in the series expansion of $\sin^2$ and so agrees with the exact result provided that $V^2t^2/\hbar^2 \ll 1$, that is, $Vt \ll \hbar$.

3. * *A 1-d harmonic oscillator of charge $q$ is acted upon by a uniform electric field which may be considered to be a perturbation and which has time dependence of the form*

$$\mathcal{E}(t) = \frac{A}{\sqrt{\pi}\,\tau}\,\exp\left\{-(t/\tau)^2\right\}$$

*Assuming that when $t=-\infty$, the oscillator is in its ground state, evaluate the probability that it is in its first excited state at $t=+\infty$ using time-dependent perturbation theory. You may assume that*

$$\begin{aligned} \int_{-\infty}^{\infty}\exp(-y^2)\,dy &= \sqrt{\pi} \\ \langle n+1|\hat{x}|n\rangle &= \sqrt{\frac{(n+1)\hbar}{2m\omega}} \\ \langle n+i|\hat{x}|n\rangle &= 0 \qquad\qquad -1>i>1 \end{aligned}$$

First-order time-dependent perturbation theory for small $\hat{V}(t)$ gives

$$p_{mk}(\infty) = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} V_{mk}(t)\exp(i\omega_{mk}t)\,\mathrm{d}t\right|^2$$

where

$$V_{mk} = \langle m|\hat{V}(t)|k\rangle$$

In the present case,

$$\hat{V}(t) = -q\mathcal{E}(t)\hat{x}$$

Since $\hat{x}$ is proportional to $(\hat{a} + \hat{a}^\dagger)$, it only connects "nearest-neighbour" states in the energy eigenvalue spectrum. Thus if the initial state is the oscillator ground state, $n = 0$, the only possible final state (at first order) is the $n = 1$ state.

$$V_{10} = \langle 1|\hat{V}(t)|0\rangle = -q\mathcal{E}(t)\langle 1|\hat{x}|0\rangle = -q\left(\frac{\hbar}{2m\omega}\right)^{1/2}\mathcal{E}(t) \quad \text{and} \quad \omega_{10} = \omega$$

Thus

$$p_{10} = \frac{q^2A^2}{2\pi\tau^2 m\hbar\omega}\left|\int_{-\infty}^{\infty}\exp\left\{i\omega t - t^2/\tau^2\right\}\,\mathrm{d}t\right|^2$$

The integral may be evaluated by 'completing the square':

$$i\omega t - t^2/\tau^2 = -\left(t/\tau - \frac{1}{2}i\omega\tau\right)^2 - \tfrac{1}{4}\omega^2\tau^2$$

We now change variables:

$$y \equiv t/\tau - \frac{1}{2}i\omega\tau, \qquad \mathrm{d}y = \mathrm{d}t/\tau$$

and the desired integral becomes a standard Gaussian integral:

$$\int_{-\infty}^{\infty}\exp\left\{i\omega t - t^2/\tau^2\right\}\,\mathrm{d}t = \exp(-\tfrac{1}{4}\omega^2\tau^2)\tau\int_{-\infty}^{\infty}\exp(-y^2)\,\mathrm{d}y = \exp(-\tfrac{1}{4}\omega^2\tau^2)\tau\sqrt{\pi}$$

Substituting in the expression for the transition probability yields:

$$\boxed{p_{10} = \frac{q^2A^2}{2m\hbar\omega}\exp(-\frac{1}{2}\omega^2\tau^2)}$$

*Discuss the behaviour of the transition probability and the applicability of the perturbation theory result when (a)* $\tau \ll \dfrac{1}{\omega}$*, and (b)* $\tau \gg \dfrac{1}{\omega}$*.*

(a) if $\tau \ll \dfrac{1}{\omega}$, then $\exp(-\tfrac{1}{2}\omega^2\tau^2) \simeq 1$ and we have

$$p_{10} \simeq \frac{q^2A^2}{2m\hbar\omega} = \text{constant}$$

We expect perturbation theory to be good if $p_{10} \ll 1$, that is, if

$$\frac{q^2A^2}{2m} \ll \hbar\omega$$

**Remark:** In the limit $\tau \to 0$, the electric field becomes proportional to a Dirac $\delta$-function:

$$\lim_{\tau \to 0} \mathcal{E}(t) = A\delta(t)$$

and we obtain the stated result for $\tau \ll \dfrac{1}{\omega}$. This is the *impulse approximation.*

(b) if $\tau \gg \dfrac{1}{\omega}$, then $\exp(-\frac{1}{2}\omega^2\tau^2) \to 0$ and thus the transition probability tends to zero, even in the case $q^2A^2/2m \approx \hbar\omega$. This is the *adiabatic approximation.*

4. *The Hamiltonian which describes the interaction of a static spin-$\frac{1}{2}$ particle with an external magnetic field, $\underline{B}$, is*

$$\hat{H} = -\underline{\hat{\mu}}.\underline{B}$$

*When $\underline{B}$ is a static uniform magnetic field in the $z$-direction, $\underline{B}_0 = (0, 0, B_0)$, the matrix representation of $\hat{H}_0$ is simply*

$$-\frac{1}{2}\gamma B_0\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

*with eigenvalues $\mp\frac{1}{2}\gamma B_0\hbar$ and for this time-independent Hamiltonian, the energy eigenstates are represented by the 2-component column matrices*

$$|\uparrow\rangle \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\downarrow\rangle \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

*Now consider superimposing on the static field $\underline{B}_0$ a time-dependent magnetic field of constant magnitude $B_1$, rotating in the $x-y$ plane with constant angular frequency $\omega$:*

$$\underline{B}_1(t) = (B_1 \cos\omega t, B_1 \sin\omega t, 0)$$

*If the Hamiltonian is now written as $\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$, write down a matrix representation of $\hat{V}(t)$.*

$$\hat{V}(t) = -\frac{1}{2}\gamma\hbar B_1 (\sigma_x \cos\omega t + \sigma_y \sin\omega t) = -\frac{1}{2}\gamma\hbar B_1 \begin{pmatrix} 0 & \exp(-i\omega t) \\ \exp(i\omega t) & 0 \end{pmatrix}$$

*Any spin state can be written*

$$|\Psi, t\rangle = c_1(t)\exp(-iE_\uparrow t/\hbar)|\uparrow\rangle + c_2(t)\exp(-iE_\downarrow t/\hbar)|\downarrow\rangle$$

*Obtain, without approximation, the coupled equations for the amplitudes $c_1(t), c_2(t)$.*

The full Hamiltonian has the representation

$$\hat{H}(t) = -\frac{1}{2}\gamma\hbar \begin{pmatrix} B_0 & B_1\exp(-i\omega t) \\ B_1\exp(i\omega t) & -B_0 \end{pmatrix}$$

The matrix representation of the state vector at time $t$ is

$$|\Psi, t\rangle \;\to\; \begin{pmatrix} c_1(t) \\ 0 \end{pmatrix}\exp(-iE_\uparrow t/\hbar) + \begin{pmatrix} 0 \\ c_2(t) \end{pmatrix}\exp(-iE_\downarrow t/\hbar)$$

Substituting this into the TDSE

$$i\hbar\frac{\partial}{\partial t}|\Psi, t\rangle = \hat{H}(t)|\Psi, t\rangle$$

gives the coupled first-order equations for the amplitudes:

$$\begin{aligned} i\hbar\dot{c}_1(t) &= -\frac{1}{2}\gamma\hbar B_1 \exp[i(\omega_{\uparrow\downarrow} - \omega)t]c_2(t) \\ i\hbar\dot{c}_2(t) &= -\frac{1}{2}\gamma\hbar B_1 \exp[i(\omega_{\downarrow\uparrow} + \omega)t]c_1(t) \end{aligned}$$

where we have used the fact that the unperturbed energy eigenvalues are

$$E_{\uparrow} = -\frac{1}{2}\gamma\hbar B_0 \quad \text{and} \quad E_{\downarrow} = \frac{1}{2}\gamma\hbar B_0$$

and defined

$$\omega_{\uparrow\downarrow} = -\omega_{\downarrow\uparrow} \equiv \frac{(E_{\uparrow} - E_{\downarrow})}{\hbar} = -\gamma B_0$$

*
*If initially at $t = 0$ the system is in the spin-down state, show that the probability that at time $t$, the system is in the spin-up state is given without approximation by*

$$p_1(t) = |c_1(t)|^2 = A\sin^2\left\{\frac{1}{2}\left[(\gamma B_1)^2 + (\omega + \gamma B_0)^2\right]^{1/2} t\right\}$$

*where*

$$A = \frac{(\gamma B_1)^2}{\{(\gamma B_1)^2 + (\omega + \gamma B_0)^2\}}$$

*What is the corresponding probability, $p_2(t)$, that the system is in the spin-down state? Sketch $p_1(t)$ and $p_2(t)$ as functions of time.*

The given initial conditions correspond to $c_1(0) = 0$ and $c_2(0) = 1$.

We can obtain an exact solution to the above coupled equations by first eliminating say $c_2(t)$ between the two equations.

The first equation gives

$$c_2(t) = -\frac{2i}{\gamma B_1}\exp[-i(\omega_{\uparrow\downarrow} - \omega)t]\,\dot{c}_1(t)$$

Differentiating with respect to $t$, we find that

$$\dot{c}_2(t) = -\frac{2i}{\gamma B_1}\exp[-i(\omega_{\uparrow\downarrow} - \omega)t]\,\ddot{c}_1(t) - \frac{2(\omega_{\uparrow\downarrow} - \omega)}{\gamma B_1}\exp[-i(\omega_{\uparrow\downarrow} - \omega)t]\,\dot{c}_1(t)$$

but from the second of the coupled equations,

$$\dot{c}_2(t) = \frac{i\gamma B_1}{2}\exp[i(\omega_{\downarrow\uparrow} + \omega)t]\,c_1(t)$$

Equating these expressions for $\dot{c}_2(t)$, noting that $\omega_{\uparrow\downarrow} = -\omega_{\downarrow\uparrow} = -\gamma B_0$, and simplifying gives the following second-order differential equation for $c_1(t)$:

$$\ddot{c}_1 + i\Omega\dot{c}_1 + (\frac{1}{2}\gamma B_1)^2 c_1 = 0$$

where $\Omega \equiv \omega + \gamma B_0$.

To solve this, try a solution of the form $c_1(t) = A\exp(ipt)$, which yields the auxiliary equation

$$-p^2 - p\Omega + (\frac{1}{2}\gamma B_1)^2 = 0$$

with roots

$$p = -\frac{1}{2}\Omega \pm \frac{1}{2}\sqrt{\Omega^2 + (\gamma B_1)^2} \equiv -\frac{1}{2}\Omega \pm \frac{1}{2}\Delta$$

where we have defined $\Delta = \sqrt{\Omega^2 + (\gamma B_1)^2}$.
Thus the general solution is

$$c_1(t) = \exp(-\frac{1}{2}i\Omega t)\left[A_1 \cos(\frac{1}{2}\Delta t) + A_2 \sin(\frac{1}{2}\Delta t)\right]$$

The first initial condition $c_1(0) = 0$ fixes the value of $A_1$ to be zero. Thus

$$c_1(t) = \exp(-\frac{1}{2}i\Omega t) \times A_2 \sin(\frac{1}{2}\Delta t)$$

To determine $A_2$, we need to use the second initial condition, $c_2(0) = 1$. From the first of the coupled equations, we have

$$c_2(t) = -\frac{2i}{\gamma B_1}\exp(i\Omega t)\dot{c}_1(t) = -\frac{2iA_2}{\gamma B_1}\exp(\frac{1}{2}i\Omega t)\left[-\frac{i\Omega}{2}\sin(\frac{1}{2}\Delta t) + \frac{\Delta}{2}\cos(\frac{1}{2}\Delta t)\right]$$

Setting $t = 0$ gives

$$c_2(0) = 1 = -\frac{2iA_2}{\gamma B_1}\frac{\Delta}{2}$$

so that

$$A_2 = \frac{i\gamma B_1}{\Delta}$$

Finally, then we have for the probability amplitude

$$c_1(t) = \frac{i\gamma B_1}{\Delta}\exp(-\frac{1}{2}i\Omega t)\sin(\frac{1}{2}\Delta t)$$

so that the probability of finding the system in the spin up state at time $t$ is

$$\boxed{p_1(t) = |c_1(t)|^2 = \left|\frac{\gamma B_1}{\Delta}\right|^2 \sin^2(\frac{1}{2}\Delta t) = A\sin^2\left\{\frac{1}{2}\left[(\gamma B_1)^2 + (\omega + \gamma B_0)^2\right]^{1/2} t\right\}}$$

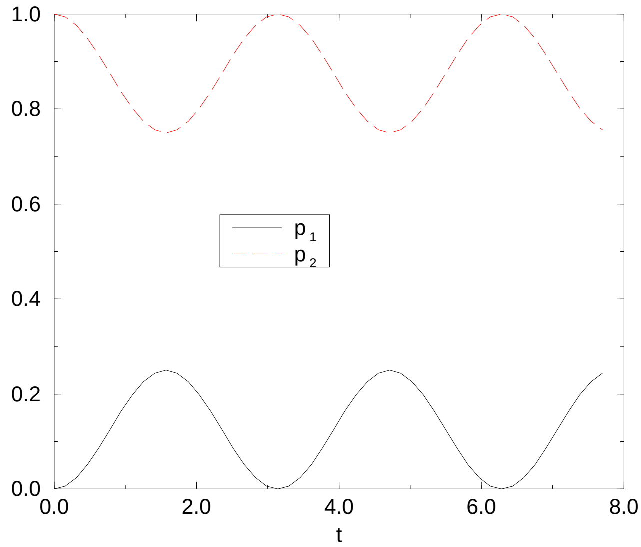
where

$$A = \frac{(\gamma B_1)^2}{\{[(\gamma B_1)^2 + (\omega + \gamma B_0)^2]\}}$$

The probability that the system is in the spin down state at time $t$ is simply

$$p_2(t) = 1 - p_1(t) = 1 - A\sin^2\left\{\frac{1}{2}\left[(\gamma B_1)^2 + (\omega + \gamma B_0)^2\right]^{1/2} t\right\}$$

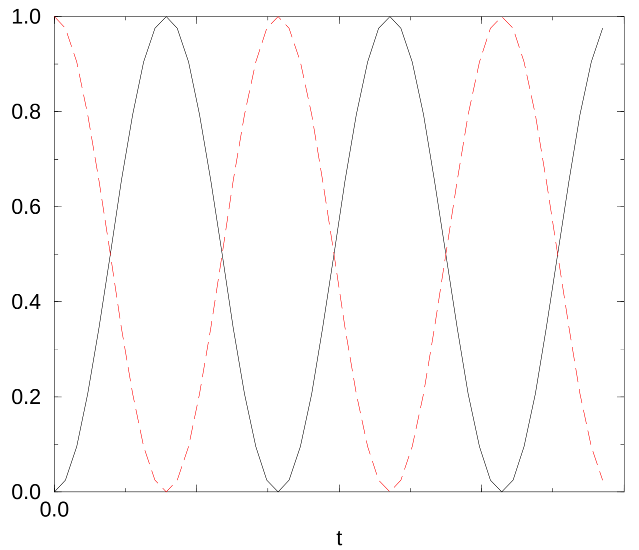
Thus the two probabilities are periodic; $p_1(t)$ oscillates between 0 and its maximum value, $A$, whilst $p_2(t)$ oscillates between its maximum value of 1 and its minimum value of $(1 - A)$.




Usually in practice, one has $B_1 \ll B_0$, so that the amplitude $A$ is quite small, being of order $(B_1/B_0)^2$. However, something interesting happens if the angular frequency of the rotating field, $\omega$ is tuned to the Bohr frequency $\omega_{\uparrow\downarrow}$ of the transition. In this case, $A = 1$ and the probabilities

$$p_1(t) = \sin^2(\frac{1}{2}\gamma B_1 t), \qquad p_2(t) = \cos^2(\frac{1}{2}\gamma B_1 t)$$

are large, oscillating between 0 and 1, even for $B_1 \ll B_0$. This is a *resonance phenomenon.*




The analysis of this problem forms the basis for high-precision methods for determining magnetic moments of atoms and subatomic particles, and for techniques such as nuclear magnetic resonance (NMR).