Quantum Physics 2011/12

Solutions to Tutorial Sheet 9:

1. This question investigates your understanding of how scattering is used in current physics research in the UK.

   The UK has two major scattering facilities based at Rutherford Appleton Laboratory in Oxford, the Diamond synchrotron (X-rays) and the Isis spallation source (neutrons).

   With reference to the time dependent scattering matrix element

$$\langle \mathbf{k}'|\hat{V}|\mathbf{k}\rangle = \frac{1}{L^3}\int\int\int V(\mathbf{r},t)\,\exp\left(-i\chi.\mathbf{r} - (\omega-\omega')t\right)\mathrm{d}\tau$$

   explain. a) What causes V in each case:

   The potential comes from the interaction of the source beam with the sample which is being studied. X-rays interact with electrons in the sample, hence the scattering can be used to study the electron density. Neutrons interact weakly with nuclei via the strong force, hence the scattering can be used to study the nuclear positions.

   b) Which technique would you use to study (i) the crystal structure of polonium (ii) the crystal structure of deuterated ice (iii) the phonon (vibrational) spectrum of sapphire (Al_2O_3)

   (i) Polonium is a rare, radioactive and expensive element, thus the sample size will be small. Each atom contains many electrons, thus $V(\mathbf{r}$ will be large for X-ray scattering. The crystal structure doesn't change in time, so the time-dependent part is irrelevant. This suggests that X-rays are the appropriate choice.

   (ii) Ice is a common material, so growing large sample is not difficult. To understand the crystal structure, one needs to identify the position of the hydrogen atoms: these have no electrons associated with them (the electron is in the OH bond), but deuterons have a high neutron scattering cross section. Once again, the time dependent part is irrelevant.

   (iii) To study phonons one needs to interact with the time-varying potential of the moving atoms. Thus the interaction time between sample and probe should be long. Neutrons move much slower than the speed of light, and there is no problem obtaining large single crystals of sapphire, thus neutrons are the preferred probe.

2. *Particles of mass m and momentum $\underline{p} \equiv \hbar\underline{k}$ are scattered by the potential*

$$V(r) = V_0\,\exp(-ar)$$

   Show that, in the first Born approximation, the differential and total cross-sections are given by

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right) = \left(\frac{4V_0 ma}{\hbar^2}\right)^2 \frac{1}{(a^2+q^2)^4}$$

   where $q = 2k\sin\theta/2$ is the magnitude of the wave-vector transfer, and

$$\sigma_T = \frac{64\pi m^2 V_0^2}{3a^4\hbar^4}\left\{\frac{16k^4+12a^2k^2+3a^4}{(a^2+4k^2)^3}\right\}$$

Hint: *the required integral may be obtained by parametric differentiation of the integral*

$$\int_0^\infty \sin(qr)\exp(-ar)\,\mathrm{d}r$$

The Born approximation for the differential cross-section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{4m^2}{\hbar^4 q^2}\left|\int_0^\infty r\,V(r)\,\sin qr\,\mathrm{d}r\right|^2$$

In the present case, $V(r) = V_0\exp(-ar)$ and so we need to evaluate the integral

$$\int_0^\infty r\sin qr\;\exp(-ar)\,\mathrm{d}r$$

Following the given hint, we first evaluate the related integral

$$\begin{aligned}
I \equiv \int_0^\infty \sin(qr)\exp(-ar)\,\mathrm{d}r &= \frac{1}{2i}\int_0^\infty \{\exp[(iq-a)r] - \exp[-(iq+a)r]\}\,\mathrm{d}r \\
&= \frac{1}{2i}\left[\frac{\exp[(iq-a)r]}{(iq-a)} + \frac{\exp[-(iq+a)r]}{(iq+a)}\right]_0^\infty \\
&= \frac{1}{2i}\left[\frac{-1}{(iq-a)} - \frac{1}{(iq+a)}\right] = \frac{q}{a^2+q^2}
\end{aligned}$$

Parametric differentiation gives us the required integral:

$$\frac{\partial I}{\partial a} = -\int_0^\infty r\sin qr\;\exp(-ar)\,\mathrm{d}r = \frac{-2aq}{(a^2+q^2)^2}$$

Substituting in the Born formula gives

$$\boxed{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right) = \left(\frac{4V_0 ma}{\hbar^2}\right)^2 \frac{1}{(a^2+q^2)^4}}$$

To calculate the total cross-section we must integrate the differential cross-section over the total solid angle:

$$\sigma_T = \left(\frac{4V_0 ma}{\hbar^2}\right)^2 \int_0^{2\pi}\int_{-1}^{+1} \frac{1}{(a^2+q^2)^4}\,\mathrm{d}(\cos\theta)\,\mathrm{d}\phi$$

The ϕ integration is trivial and just gives 2π. To perform the integral over $\cos\theta$ is simplest if we note that

$$q^2 = 4k^2\sin^2\frac{\theta}{2} = 2k^2(1-cos\theta) \Rightarrow \mathrm{d}q^2 = -2k^2\mathrm{d}(\cos\theta)$$

and change variables from $\cos\theta$ to q^2:

$$\begin{aligned}
\int_{-1}^{+1}\frac{1}{(a^2+q^2)^4}\,\mathrm{d}(\cos\theta) &= \frac{1}{2k^2}\int_0^{4k^2}\frac{1}{(a^2+q^2)^4}\,\mathrm{d}q^2 = \frac{1}{2k^2}\left[-\frac{1}{3(a^2+q^2)^3}\right]_0^{4k^2} \\
&= \frac{1}{6k^2}\left[\frac{1}{a^6} - \frac{1}{(a^2+4k^2)^3}\right] = \frac{2}{3a^6}\left[\frac{16k^4+12a^2k^2+3a^4}{(a^2+4k^2)^3}\right]
\end{aligned}$$

Substituting back into the expression for σ_T gives

$$\boxed{\sigma_T = \frac{64\pi m^2 V_0^2}{3a^4\hbar^4}\left\{\frac{16k^4+12a^2k^2+3a^4}{(a^2+4k^2)^3}\right\}}$$

3. *Particles are scattered by a time varying potential:*

$$V(r) = V_0 L^3 \delta(r)\sin(\alpha t)$$

where δ is a Dirac delta function and the particle take time $t_0 >> 1/\alpha$ to pass through the potential. Show that the differential cross-section in the Born approximation is

Using

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m^2}{4\pi^2\hbar^4}L^6\left|\langle \mathbf{k}',\omega'|\hat{V}|\mathbf{k},\omega\rangle\right|^2$$

We can evaluate the matrix element for plane waves normalised to the volume L^3

$$L^{-3}\int V_0 L^3\delta(r)\exp(i(\mathbf{k}'-\mathbf{k}).\mathbf{r}d\tau\int\exp(i\omega't-i\omega t)\sin(\alpha t)dt$$

$$= V_0\frac{1}{2}\int_0^{t_0}\exp(i\omega't-i\omega t)[e^{i\alpha t}-e^{-i\alpha t}]dt$$

$$= \frac{V_0 t_0}{2}[\delta(\omega'-\omega+\alpha)+\delta(\omega'-\omega-\alpha)]$$

using the representation of the δ function: unless the shift in ω is $\pm\alpha$, we have the integral of a sine wave over many oscillations (i.e. zero). At this resonance the exponentials are just equal to 1.

Now, using the expression for the differential cross section in the notes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \equiv \frac{\text{scattered flux}}{\text{incident flux}} = \frac{mL^3}{\hbar k'}\frac{2\pi}{\hbar}|V_{\mathbf{k}'\mathbf{k}}|^2\frac{L^3}{8\pi^3}\frac{mk}{\hbar^2}$$

gives:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m^2}{4\pi^2\hbar^4}\frac{V_0^2L^6t_0^2}{4}[\delta(\omega'-\omega+\alpha)+\delta(\omega'-\omega-\alpha)]$$

w[5]

Given that this is a model for the interaction of light with a fluctating dipole in a gas molecule, comment on the presence of the factor of L^3 in the potential, the angular dependence, the values of k' and the conservation of energy.

The L^3 term in the potential appears because the gas at given density contains a number of molecules proportional to the volume, hence the larger the normalisation volume, the more scatterers must be considered. Note that although the potential is located at the origin the symmetry of the problem requires that the scattering is the same wherever it is.

There is no angular dependence: scattering is isotropic, although in the Born Approximation most photons are not scattered and continue with wavevector $\mathbf{k}$.

The scattered photons have their frequency shifted by $\pm\alpha$. Since the speed of light is constant, their wavevectors must have $k' = (\omega \pm \alpha)/c$. If you look carefully, you may notice that a factor of k'/k has been ignored. This is OK: recall that the Born Approximationi is valid in the perturbative limit, i.e. $\alpha << \omega$.

The incoming photons have energy $\hbar\omega$, the outgoing ones have energy $\hbar(\omega \pm \alpha$. Energy is not conserved, it is inelastic scattering. The energy is exchanged with the molecule, represented here by the potential, adding one quantum of energy to its oscillating mode. A fuller calculation would treat the molecule as part of the system, not as a boundary condition.

Considering the Born series (lecture 13), can you guess what the second order term would mean here The second order term would imply that there were two interactions with the scattered. Thus the energy shift would be $\pm 2\alpha$.

4. *Two beams of unpolarised electrons are scattered from one another. Taking the scattered wavefunction to be $f(\theta)$, find the scattered intensity for combinations of singlet and triplet spins with appropriate spatial wavefunctions. Show that this is the same scattering cross section given by the combination of* 50% *distinguishable and* 50% *indistinguishable fermion collisions, as discussed in the notes.*

   A scattered electron can arrive at the detector if the scattering angle is either θ or $\pi - \theta$. The intensity must not depend on labelling of these processes as they are indistinguishable. In the region of the interaction, we must therefore combine possible spin states with possible combinations of $f(\theta)$ and $f(\pi - \theta)$ which are eigenstates of the exchange operator (exchanging *processes*). These must give overall antisymmetric wavefunctions:

   Thus all the following have equal weight.

$$(f(\theta) + f(\pi - \theta))(\downarrow\uparrow - \uparrow\downarrow)$$

$$(f(\theta) - f(\pi - \theta)) \uparrow\uparrow$$

$$(f(\theta) - f(\pi - \theta))(\downarrow\uparrow + \uparrow\downarrow)$$

$$(f(\theta) - f(\pi - \theta)) \downarrow\downarrow$$

   Whence, since the scattering is electrostatic and the spin wavefunction is unchanged,

$$\begin{aligned} I(\theta) &= \frac{1}{4}|f(\theta) + f(\pi - \theta)|^2 + \frac{3}{4}|f(\theta) - f(\pi - \theta)|^2 \\ &= |f(\theta)|^2 + |f(\pi - \theta)|^2 - \frac{1}{2}[f(\theta)^* f(\pi - \theta) + f(\theta) f(\pi - \theta)^*] \\ &= \frac{1}{2}|f(\theta)|^2 + \frac{1}{2}|f(\pi - \theta)|^2 + \frac{1}{2}|f(\theta) - f(\pi - \theta)|^2 \end{aligned}$$

   which is the same as the result for scattering of distinguishable particles (first two terms) plus indistinguishable fermions (final term) given in the notes.

   Note the scattering in the $\theta = \pi$ direction. Although the interaction does not affect the spin, all electron pairs scattered in this direction have a spin-entangled singlet state. The electrostatic interaction has filtered out a pure spin state!

5. *Evaluate the differential cross-section in the Born approximation for the potential*

$$V(r) = V_0/r^2$$

What happens to the total cross-section for this potential? You may assume that

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \pi/2.$$

Again we start from the Born formula:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{4m^2}{\hbar^4 q^2}\left|\int_0^\infty r\,V(r)\,\sin qr\,\mathrm{d}r\right|^2$$

The required integral is

$$\int_0^\infty r\,V(r)\,\sin qr\,\mathrm{d}r = V_0\int_0^\infty \frac{\sin qr}{r}\,\mathrm{d}r = V_0\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi V_0}{2}$$

using the given integral with $x \equiv qr$.

Thus

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\pi m V_0}{\hbar^2 q}\right)^2 = \left(\frac{\pi m V_0}{2\hbar^2 k}\right)^2 \operatorname{cosec}^2\frac{\theta}{2}}$$

As before, the total cross-section is obtained by integrating the differential cross-section over the total solid angle:

$$\sigma_T = 2\pi\int_{-1}^{+1}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}(\cos\theta) = \frac{\pi}{k^2}\int_0^{4k^2}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}q^2 = \frac{\pi}{k^2}\int_0^{4k^2}\left(\frac{\pi m V_0}{\hbar^2 q}\right)^2\mathrm{d}q^2$$

Unfortunately, the integral diverges as $q^2 \to 0$ and so the total cross-section is infinite, just as for the Coulomb potential.

6. * *Show that in a classical elastic two-body collision between particles of mass m_1 and m_2, the* LAB *frame scattering angle, θ and the* CM *frame scattering angle, θ^*, are related by*

$$\tan\theta = \frac{\sin\theta^*}{\rho + \cos\theta^*} \quad \text{where} \quad \rho = m_1/m_2$$

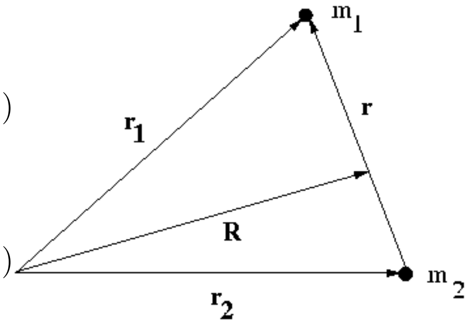
and hence that the LAB *and* CM *frame differential cross-sections are related by*

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_L = \frac{(1+\rho^2+2\rho\cos\theta^*)^{3/2}}{|1+\rho\cos\theta^*|}\cdot\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CM}$$

For the first part of this question, you are referred back to the third-year course *Dynamics & Relativity*.

The position vector of the centre of mass is defined by

$$\underline{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \tag{1}$$



while the relative position of the two particles is

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{2}$$

The position vectors of particles 1 and 2 in terms of the position of the centre of mass and the relative position are

$$\mathbf{r}_1 = \underline{R} + \frac{m_2}{M}\mathbf{r} \qquad \text{and} \qquad \mathbf{r}_2 = \underline{R} - \frac{m_1}{M}\mathbf{r} \tag{3}$$

In the CM frame we take the centre of mass to be at rest at the origin. We adopt the convention that quantities referred to the CM frame are labelled by an asterisk. The relative position $\mathbf{r}$ is independent of the choice of origin.

$$\mathbf{r}_1^* = \frac{m_2}{M}\mathbf{r} \qquad \text{and} \qquad \mathbf{r}_2^* = -\frac{m_1}{M}\mathbf{r} \tag{4}$$

In any other frame, such as, for example, the LAB frame, the centre of mass is moving with velocity $\dot{\underline{R}}$ and the velocities of the two particles are

$$\dot{\mathbf{r}}_1 = \dot{\underline{R}} + \dot{\mathbf{r}}_1^* \qquad \text{and} \qquad \dot{\mathbf{r}}_2 = \dot{\underline{R}} + \dot{\mathbf{r}}_2^* \tag{5}$$

In the CM frame, the momenta of the two particles are equal and opposite:

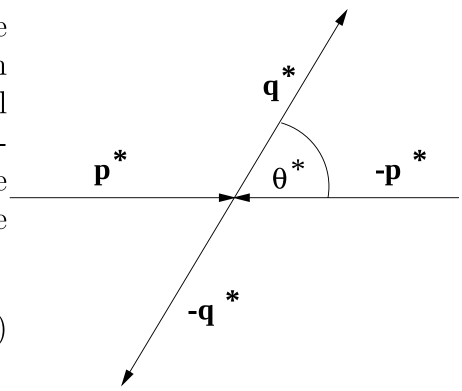
$$m_1\dot{\mathbf{r}}_1^* = -m_2\dot{\mathbf{r}}_2^* = \mu\dot{\mathbf{r}} \equiv \underline{p}^* \tag{6}$$

hence the name *Zero Momentum* frame. In any other frame, their momenta are thus

$$\underline{p}_1 = m_1\dot{\mathbf{r}}_1 = m_1\dot{\underline{R}} + \underline{p}^* \qquad \text{and} \qquad \underline{p}_2 = m_2\dot{\mathbf{r}}_2 = m_2\dot{\underline{R}} - \underline{p}^* \tag{7}$$

In the CM frame the momenta of the two particles are equal and opposite before the collision. Conservation of momentum implies that the momenta are also equal and opposite after the collision, as indicated in the figure. Since the collision is elastic, the kinetic energies are equal before and after, and hence the magnitudes of the momenta before and after are the same:

$$p^* \equiv |\underline{p}^*| = q^* \equiv |\underline{q}^*| \tag{8}$$



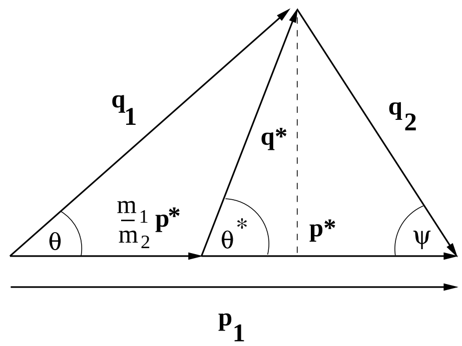
Now in the LAB frame, $\underline{p}_2 = 0$ and hence from equation (7)

$$\dot{\underline{R}} = \frac{1}{m_2}\underline{p}^* \qquad \text{and} \qquad \underline{p}_1 = \frac{m_1}{m_2}\underline{p}^* + \underline{p}^* = \frac{M}{m_2}\underline{p}^* \tag{9}$$

The momenta after the collision are again given by equation (7) but with $\underline{q}^*$ in place of $\underline{p}^*$:

$$\underline{q}_1 = \frac{m_1}{m_2}\underline{p}^* + \underline{q}^* \qquad \text{and} \qquad \underline{q}_2 = \underline{p}^* - \underline{q}^* \tag{10}$$

The relations between the various momentum vectors can be summarised in a vector diagram which also incorporates momentum conservation in the LAB frame: $\underline{p}_1 = \underline{q}_1 + \underline{q}_2$. The vectors $\underline{p}^*$, $\underline{q}^*$ and $\underline{q}_2$ form an isosceles triangle by virtue of the elastic scattering relation, equation (8). We can find relations between the various momenta and angles from this diagram.



We can apply elementary trigonometry to the figure to find the relation between the LAB and CM scattering angles:

$$\tan\theta = \frac{q^*\sin\theta^*}{(m_1/m_2)p^* + q^*\cos\theta^*} = \frac{\sin\theta^*}{(m_1/m_2)+\cos\theta^*} \quad \text{since} \quad p^* = q^*$$

or, in the notation of the question,

$$\tan\theta = \frac{\sin\theta^*}{\rho+\cos\theta^*} \quad \text{where} \quad \rho = m_1/m_2 \tag{11}$$

Note that the relation is independent of the momenta of the two particles and depends only on the ratio of their masses.

The azimuthal angle is unchanged in the transformation from the LAB to the CM frame so that $\phi^* = \phi$.

The number of particles per unit time scattered into $\mathrm{d}\Omega_L = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ is the same as the number per unit time scattered into $\mathrm{d}\Omega_{CM} = \sin\theta^*\,\mathrm{d}\theta^*\,\mathrm{d}\phi^*$ by definition. The incident flux depends only on the relative velocity of projectile and target and hence is the same in either frame. Thus

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_L \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CM} \sin\theta^*\,\mathrm{d}\theta^*\,\mathrm{d}\phi^*$$

with angles related as in equation 11. Differentiating equation 11 gives

$$\sec^2\theta\,\mathrm{d}\theta = \frac{(1+\rho\cos\theta^*)}{(\rho+\cos\theta^*)^2}\,\mathrm{d}\theta^*$$

Multiplying both sides by $\sin\theta\cos^2\theta$ gives

$$\begin{aligned}\sin\theta\,\mathrm{d}\theta &= \frac{(1+\rho\cos\theta^*)}{(\rho+\cos\theta^*)^2}\cos^2\theta\sin\theta\,\mathrm{d}\theta^* \\ &= \frac{(1+\rho\cos\theta^*)}{(\rho+\cos\theta^*)^2}\cos^3\theta\tan\theta\,\mathrm{d}\theta^*\end{aligned}$$

$$
\begin{aligned}
&= \frac{(1+\rho\cos\theta^*)}{(\rho+\cos\theta^*)^3}(\sec^2\theta)^{-3/2}\sin\theta^*\,\mathrm{d}\theta^* \\
&= \frac{(1+\rho\cos\theta^*)}{(\rho+\cos\theta^*)^3}(1+\tan^2\theta)^{-3/2}\sin\theta^*\,\mathrm{d}\theta^* \\
&= \frac{(1+\rho\cos\theta^*)}{(\rho+\cos\theta^*)^3}(\rho+\cos\theta^*)^3(1+2\rho\cos\theta^*+\rho^2)^{-3/2}\sin\theta^*\,\mathrm{d}\theta^* \\
&= \frac{(1+\rho\cos\theta^*)}{(1+2\rho\cos\theta^*+\rho^2)^{3/2}}\sin\theta^*\,\mathrm{d}\theta^*
\end{aligned}
$$

Of course, $\mathrm{d}\Omega_{CM}$ and $\mathrm{d}\Omega_L$ must be taken to be positive, since the number of events is a positive number. Thus

$$
\boxed{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_L = \frac{(1+2\rho\cos\theta^*+\rho^2)^{3/2}}{|(1+\rho\cos\theta^*)|}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CM}}
$$

where it is understood that the positive root is to be taken in the numerator.

Note that all the integrals can be done in two lines using maple:

```
> V:=   (whatever)
> $simplify(int(4.pi.r.V.sin(X.r)/X,r=0..infinity))$
```

You may have to take the $r \to \infty$ limit yourself.

7. *Show that the function* $G(r) = \exp(ikr)/4\pi r$ *is a solution of the Schroedinger equation with a delta function potential:*

$$
\frac{-\hbar^2}{2m}\nabla^2 G(r) + \delta(\boldsymbol{r})G(r) = EG(r)
$$

and evaluate E. You may resort to a verbal argument to explain the delta function without evealuating its strength.

Since G is a function of r only, for $r \neq 0$

$$
\begin{aligned}
\nabla^2(\exp(ikr)/4\pi r) &= \frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\exp(ikr)/4\pi r \\
&= \frac{1}{r^2}\frac{d}{dr}r^2(ik\exp(ikr)/4\pi r - \exp(ikr)/4\pi r^2) \\
&= \frac{1}{r^2}\frac{d}{dr}(ikr\exp(ikr)/4\pi - \exp(ikr)/4\pi) \qquad ** \\
&= \frac{1}{r^2}(-k^2 r\exp(ikr)/4\pi + ik\exp(ikr)/4\pi - ik\exp(ikr)/4\pi) \\
&= -k^2 G(r)
\end{aligned}
$$

Examining the function anywhere but $r = 0$, the TISE is solved only if

$$
E = \frac{\hbar^2 k^2}{2m},
$$

the usual expression for the kinetic energy of a wave.

For $r = 0$ we have a problem at ** with division by zero: it is clear that $G(\mathbf{r})$ diverges and has a cusp at the origin, some curvature and therefore some finite kinetic energy associated with integrating over this cusp. Thus its second derivative must be a delta function at the origin. If we replace the cusp by a continuous function (e.g. a quadratic) with zero slope at the origin in the region $0 - \epsilon$, and taking the limit as $\epsilon \to 0$, we obtain the strength of the delta function as the additional contribution to the kinetic energy.

Show that for a potential $V(r) = \hbar^2 U(r)/2m$ the time independent Schroedinger equation gives:

$$\Phi(\boldsymbol{r}) = Ae^{i\boldsymbol{k}.\boldsymbol{r}} + \int G(r-r')U(r')Ae^{i\boldsymbol{k}.\boldsymbol{r}'}d^3\boldsymbol{r}' + \int G(r-r')U(r')G(r'-r'')U(r'')\Phi(\boldsymbol{r})d^3\boldsymbol{r}'d^3\boldsymbol{r}''$$

Under what circumstances is this useful?

we show first the relation

$$\Phi(r) = Ae^{ik.r} + \int G(r-r')U(r')\Phi(r')d^3r'$$

With this potential and defining $k^2 = 2mE/\hbar^2$ the TISE becomes

$$(-\nabla^2 + U(r))\Phi(r) = k^2\Phi(r)$$

Any function can be written in the form of an integral:

$$F(r) = \int \delta(r-r')F(r')d^3r'$$

where the delta function picks out the value of $F(r)$.

Applying this $U(r)\Phi(r)$to gives us

$$(\nabla^2 + k^2)\Phi(r) - \int \delta(r-r')U(r')\Phi(r')d^3r' = 0$$

substituting from above for the delta function

$$(\nabla^2 + k^2)\Phi(r) - \int (\nabla^2 + k^2)G(r-r')U(r')\Phi(r')d^3r' = 0$$

Substituting the given expression for $\Phi(r)$, and noting $(\nabla^2 + k^2)\exp(-ikr) = 0$ gives

$$(\nabla^2 + k^2)\int G(r-r')U(r')\Phi(r')d^3r' - \int (\nabla^2 + k^2)G(r-r')U(r')\Phi(r')d^3r' = 0$$

Since ∇^2 acts on the unprimed coordinate, we can take it out of the integral and the above equation is clearly satisfied.

Having proved

$$\Phi(\boldsymbol{r}) = Ae^{i\boldsymbol{k}.\boldsymbol{r}} + \int G(r-r')U(r')\Phi(\boldsymbol{r}')d^3\boldsymbol{r}'$$

we can now find the required relation by substituting the RHS into itself:

$$\Phi(\boldsymbol{r}') = Ae^{i\boldsymbol{k}.\boldsymbol{r}'} + \int G(r' - r'')U(r'')\Phi(\boldsymbol{r}'')d^3\boldsymbol{r}''$$

Physically, the wavefunction is in a form consistent with boundary conditions for a scattering experiment: the first term is a plane wave, representing an incoming particle beam, the second represents the scattering effect of a potential.

Mathematically, by repeated substitution, we create a series of terms with progressively higher powers of $U(\boldsymbol{r})$. This will be useful if it converges - i.e. $U(\boldsymbol{r})$ is small. This is the condition for the Born Approximation, which considers only the first two terms.