Quantum Physics 2011/12

Tutorial Sheet 1: Mainly revision

You will need to consult your notes from Junior Honours Quantum Mechanics and/or one of the many textbooks on Quantum Mechanics.

1. Given the expansion of an arbitrary wavefunction or state vector as a linear superposition of eigenstates of the operator $\hat{A}$

$$\Psi(r, t) = \sum_i c_i(t) \, u_i(r) \quad \text{or} \quad |\Psi, t\rangle = \sum_i c_i(t) \, |u_i\rangle$$

use the orthonormality properties of the eigenstates to prove that

$$c_i(t) = \int u_i^*(r) \, \Psi(r, t) \, d^3r \quad \text{or} \quad c_i(t) = \langle u_i | \Psi, t \rangle$$

Work through the proof in both wavefunction and Dirac notations.

The state $|\Psi, t\rangle$ is said to be normalised if $\langle \Psi, t | \Psi, t \rangle = 1$. Show that this implies that

$$\sum_i |c_i(t)|^2 = 1$$

Hint: use the expansion $|\Psi, t\rangle = \sum_i c_i(t) |u_i\rangle$ and the corresponding conjugate expansion $\langle \Psi, t | = \sum_j c_j^* (t) \langle u_j |$.

If the expectation value $\langle \hat{A} \rangle_t = \langle \Psi, t | \hat{A} | \Psi, t \rangle$, show by making use of the same expansions that

$$\langle \hat{A} \rangle_t = \sum_i |\langle u_i | \Psi, t \rangle|^2 \, A_i$$

and give the physical interpretation of this result.

2. The observables $\mathcal{A}$ and $\mathcal{B}$ are represented by operators $\hat{A}$ and $\hat{B}$ with eigenvalues $\{A_i\}$, $\{B_i\}$ and eigenstates $\{ |u_i\rangle \}$, $\{ |v_i\rangle \}$ respectively, such that

$|v_1\rangle = \left( \sqrt{3} |u_1\rangle + |u_2\rangle \right)/2$

$|v_2\rangle = \left( |u_1\rangle - \sqrt{3} |u_2\rangle \right)/2$

$|v_n\rangle = |u_n\rangle, \quad n \geq 3.$

Show that if $\{ |u_i\rangle \}$ is an orthonormal basis then so is $\{ |v_i\rangle \}$. A certain system is subjected to three successive measurements:

(1) a measurement of $\mathcal{A}$ followed by
(2) a measurement of $\mathcal{B}$ followed by
(3) another measurement of $\mathcal{A}$

Show that if measurement (1) yields any of the values $A_3, A_4, \ldots$ then (3) gives the same result but that if (1) yields the value $A_1$ there is a probability of $\frac{5}{8}$ that (3) will yield $A_1$ and a probability of $\frac{3}{8}$ that it will yield $A_2$. What may be said about the compatibility of $\mathcal{A}$ and $\mathcal{B}$?
3. The normalised energy eigenfunction of the ground state of the hydrogen atom \((Z = 1)\) is
\[
u_{100}(r) = R_{10}(r)Y_{00}(\theta, \phi) = C \exp(-r/a_0)
\]
where \(a_0\) is the Bohr radius and \(C\) is a normalisation constant. For this state

(a) Calculate the normalisation constant, \(C\), by noting the useful integral
\[
\int_0^\infty \exp(-br) r^n \, dr = n! / b^{n+1}, \quad n > -1
\]
Alternatively, you can use the computer algebra program Maple if you know how to!

(b) Determine the radial distribution function, \(D_{10}(r) \equiv r^2 |R_{10}(r)|^2\), and sketch its behaviour; determine the most probable value of the radial coordinate, \(r\), and the probability that the electron is within a sphere of radius \(a_0\); recall that \(Y_{00}(\theta, \phi) = 1/\sqrt{4\pi}\); again, you can use Maple to help you if you know how.

(c) Calculate the expectation value of \(r\).

(d) Calculate the expectation value of the potential energy, \(V(r)\).

(e) Calculate the uncertainty, \(\Delta r\), in \(r\) (i.e. \(\sqrt{\langle r^2 \rangle - \langle r \rangle^2} \)).

4. At \(t = 0\), a particle has a wavefunction \(\psi(x, y, z) = Az \exp[-b(x^2 + y^2 + z^2)]\), where \(A\) and \(b\) are constants.

(a) Show that this wavefunction is an eigenstate of \(\hat{L}^2\) and of \(\hat{L}_z\) and find the corresponding eigenvalues.

Hint: express \(\psi\) in spherical polars and use the spherical polar expressions for \(\hat{L}^2\) and \(\hat{L}_z\).

\[
\hat{L}^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]
\]
\[
\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}
\]

(b) Sketch the function, e.g. with a contour plot in the \(x=0\) plane.

(c) Can you identify the Hamiltonian for which this is an energy eigenstate?