Quantum Physics 2011/12

Tutorial Sheet 1: Mainly revision

You will need to consult your notes from Junior Honours Quantum Mechanics and/or one of the many textbooks on Quantum Mechanics.

1. Given the expansion of an arbitrary wavefunction or state vector as a linear superposition of eigenstates of the operator $\hat{A}$

$$\Psi(\underline{r},t) = \sum_i c_i(t)\, u_i(\underline{r}) \qquad \text{or} \qquad |\Psi,t\rangle = \sum_i c_i(t)\, |u_i\rangle$$

use the orthonormality properties of the eigenstates to prove that

$$c_i(t) = \int u_i^*(\underline{r})\, \Psi(\underline{r},t)\, \mathrm{d}^3 r \qquad \text{or} \qquad c_i(t) = \langle u_i|\Psi,t\rangle$$

Work through the proof in both wavefunction and Dirac notations.

The state $|\Psi,t\rangle$ is said to be normalised if $\langle\Psi,t|\Psi,t\rangle = 1$. Show that this implies that

$$\sum_i |c_i(t)|^2 = 1$$

Hint: use the expansion $|\Psi,t\rangle = \sum_i c_i(t)|u_i\rangle$ and the corresponding conjugate expansion $\langle\Psi,t| = \sum_j c_j^*(t)\langle u_j|$.

If the expectation value $\langle\hat{A}\rangle_t = \langle\Psi,t|\hat{A}|\Psi,t\rangle$, show by making use of the same expansions that

$$\langle\hat{A}\rangle_t = \sum_i |\langle u_i|\Psi,t\rangle|^2\, A_i$$

and give the physical interpretation of this result.

2. The observables $\mathcal{A}$ and $\mathcal{B}$ are represented by operators $\hat{A}$ and $\hat{B}$ with eigenvalues $\{A_i\}$, $\{B_i\}$ and eigenstates $\{|u_i\rangle\}$, $\{|v_i\rangle\}$ respectively, such that

$$\begin{aligned} |v_1\rangle &= \{\sqrt{3}\,|u_1\rangle + |u_2\rangle\}/2 \\ |v_2\rangle &= \{|u_1\rangle - \sqrt{3}\,|u_2\rangle\}/2 \\ |v_n\rangle &= |u_n\rangle, \quad n \geq 3. \end{aligned}$$

Show that if $\{|u_i\rangle\}$ is an orthonormal basis then so is $\{|v_i\rangle\}$. A certain system is subjected to three successive measurements:

(1) a measurement of $\mathcal{A}$ followed by

(2) a measurement of $\mathcal{B}$ followed by

(3) another measurement of $\mathcal{A}$

Show that if measurement (1) yields any of the values $A_3, A_4, \ldots$ then (3) gives the same result but that if (1) yields the value A_1 there is a probability of $\frac{5}{8}$ that (3) will yield A_1 and a probability of $\frac{3}{8}$ that it will yield A_2. What may be said about the compatibility of $\mathcal{A}$ and $\mathcal{B}$?

3. The normalised energy eigenfunction of the ground state of the hydrogen atom ($Z = 1$) is

$$u_{100}(\underline{r}) = R_{10}(r)Y_{00}(\theta, \phi) = C\ \exp(-r/a_0)$$

where a_0 is the Bohr radius and C is a normalisation constant. For this state

(a) Calculate the normalisation constant, C, by noting the useful integral

$$\int_0^\infty \exp(-br)\, r^n\, \mathrm{d}r = n!/b^{n+1}, \qquad n > -1$$

Alternatively, you can use the computer algebra program Maple if you know how to!

(b) Determine the radial distribution function, $D_{10}(r) \equiv r^2|R_{10}(r)|^2$, and sketch its behaviour; determine the most probable value of the radial coordinate, r, and the probability that the electron is within a sphere of radius a_0; recall that $Y_{00}(\theta, \phi) = 1/\sqrt{4\pi}$; again, you can use Maple to help you if you know how.

(c) Calculate the expectation value of r.

(d) Calculate the expectation value of the potential energy, $V(r)$.

(e) Calculate the uncertainty, Δr, in r (i.e. $\sqrt{\langle r^2\rangle - \langle r\rangle^2}$).

4. At $t = 0$, a particle has a wavefunction $\psi(x, y, z) = A\, z\, \exp[-b(x^2 + y^2 + z^2)]$, where A and b are constants.

(a) Show that this wavefunction is an eigenstate of $\hat{L}^2$ and of $\hat{L}_z$ and find the corresponding eigenvalues.
Hint: express ψ in spherical polars and use the spherical polar expressions for $\hat{L}^2$ and $\hat{L}_z$.

$$\begin{aligned} \hat{L}^2 &= -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] \\ \hat{L}_z &= -i\hbar \frac{\partial}{\partial\phi} \end{aligned}$$

(b) Sketch the function, e.g. with a contour plot in the x=0 plane.

(c) Can you identify the Hamiltonian for which this is an energy eigenstate ?