Quantum Physics 2011/12

Tutorial Sheet 2: Perturbations

An asterisk denotes a harder problem, which you are nevertheless encouraged to try!

The following trigonometric identities may prove useful in the first two questions:

$$\begin{aligned}
\cos^2 A &\equiv 1 - \sin^2 A \\
\sin 2A &\equiv 2 \sin A \cos A \\
\sin A \ \sin B &\equiv \tfrac{1}{2} \left[\cos(A - B) - \cos(A + B)\right] \\
\cos A \ \cos B &\equiv \tfrac{1}{2} \left[\cos(A - B) + \cos(A + B)\right]
\end{aligned}$$

1. A quantum dot is a self assembled nanoparticle in which a single electron state can be confined. A model for such an object is a particle moving in one dimension in the potential

   $$V(x) = \infty, \quad |x| > a, \qquad V(x) = V_0 \cos(\pi x/2a), \quad |x| \le a$$

   Identify an appropriate unperturbed system and perturbation term.

   Calculate the energies of the two lowest states to first order in perturbation theory.

   What is the sign of V_0?

   State two ways in which the colour of a material containing dots can be shifted towards the red.

2. A particle moves in one dimension in the potential

   $$V(x) = \infty, \quad |x| > a, \qquad V(x) = V_0 \sin(\pi x/a), \quad |x| \le a$$

   - show that the first order energy shift is zero;
   - *obtain the expression for the second order correction to the energy of the ground state,

   $$\boxed{\Delta E_1^{(2)} = -\left(\frac{32V_0}{15\pi}\right)^2 \frac{8ma^2}{3\pi^2\hbar^2} - \left(\frac{64V_0}{105\pi}\right)^2 \frac{8ma^2}{15\pi^2\hbar^2} - \ldots}$$

3. The 1-d anharmonic oscillator: a particle of mass m is described by the Hamiltonian

   $$\hat{H} = \frac{\hat{p}^2}{2m} + \tfrac{1}{2} m\omega^2 \hat{x}^2 + \gamma \hat{x}^4$$

   - Assuming that γ is small, use first-order perturbation theory to calculate the ground state energy;
   - *show more generally that the energy eigenvalues are approximately

   $$E_n \simeq (n + \tfrac{1}{2})\hbar\omega + 3\gamma \left(\frac{\hbar}{2m\omega}\right)^2 (2n^2 + 2n + 1)$$

Hint: to evaluate matrix elements of powers of $\hat{x}$, write $\hat{x}$ in terms of the harmonic oscillator raising and lowering operators $\hat{a}$ and $\hat{a}^\dagger$. Recall that the raising and lowering operators are defined by

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2m\omega\hbar}}\,\hat{p} \qquad \text{and} \qquad \hat{a}^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - \frac{i}{\sqrt{2m\omega\hbar}}\,\hat{p}$$

with the properties that

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \text{and} \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

4. A 1-dimensional harmonic oscillator of mass m carries an electric charge, q. A weak, uniform, static electric field of magnitude $\mathcal{E}$ is applied in the x-direction. Write down an expression for the classical electrostatic potential energy for a point particle at x.

   The quantum operator is given by the same expression with $x \to \hat{x}$. By considering the symmetry of the integrals, or otherwise, how that, to first order in perturbation theory, the oscillator energy levels are unchanged, and calculate the second-order shift. Can you show that the second-order result is in fact exact?

   Hint: to evaluate matrix elements of $\hat{x}$, write $\hat{x}$ in terms of the harmonic oscillator raising and lowering operators $\hat{a}$ and $\hat{a}^\dagger$ and use the results $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. To obtain an exact solution, change variables to complete the square in the potential and show it remains a harmonic oscillator.

5. Starting from the relativistic expression for the total energy of a single particle, $E = (m^2c^4 + p^2c^2)^{1/2}$, and expanding in powers of p^2, obtain the leading relativistic correction to the kinetic energy, for a plane wavefunction $\Phi(x) = A\cos(kx)$, and determine whether $\Phi(x)$ is an eigenstate for a relativistic free particle. *Hint* For normalisation of the wavefunction, it helps to keep the integral form for $|A|^{-2} = \int cos^2 kx dx$.