# Quantum Physics 2011/12

## Tutorial Sheet 3: More Perturbations

*An asterisk denotes a harder problem, which you are nevertheless encouraged to try!*

1. Quarkonium is a system consisting of a heavy quark of mass $m_Q$ bound to its antiquark, also of mass $m_Q$. The inter-quark potential is of the form

$$V(r) = -\frac{a}{r} + br \ ,$$

where $a, b$ are constants and $r$ is the quark-antiquark separation. Given the Bohr formula for the energy levels of hydrogen

$$E_n^{(0)} = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{n^2} \ ,$$

where $m$ is the reduced mass of the electron-proton system, deduce an expression for the energy levels of quarkonium in the approximation which neglects the second term in $V(r)$.

What are the corresponding degeneracies of the lowest two energy levels?

Use first-order perturbation theory to calculate the corrections to the lowest two energy levels. Why it is not necessary to use degenerate perturbation theory for this problem?

*You may assume that the wavefunctions for hydrogen are:*

$$\begin{aligned}
u_{100} &= (\pi a_0^3)^{-1/2} \exp(-r/a_0) \\
u_{200} &= \left(8\pi a_0^3\right)^{-1/2}\left(1 - \frac{r}{2a_0}\right)\exp\left(-r/2a_0\right) \\
u_{211} &= -\left(\pi a_0^3\right)^{-1/2}\frac{r}{8a_0}\sin\theta\exp\left(i\phi\right)\exp\left(-r/2a_0\right) \\
u_{210} &= \left(8\pi a_0^3\right)^{-1/2}\frac{r}{2a_0}\cos\theta\exp\left(-r/2a_0\right) \\
u_{21-1} &= \left(\pi a_0^3\right)^{-1/2}\frac{r}{8a_0}\sin\theta\exp\left(-i\phi\right)\exp\left(-r/2a_0\right)
\end{aligned}$$

*where the Bohr radius* $a_0 = 4\pi\epsilon_0\hbar^2/me^2$, *and that*

$$\int_0^\infty \exp(-kr)\, r^n \,\mathrm{d}r = n!/k^{n+1}, \quad n > -1$$

2. The isotropic harmonic oscillator in 2 dimensions is described by the Hamiltonian

$$\hat{H}_0 = \sum_i \left\{\frac{\hat{p}_i^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_i^2\right\} \qquad\qquad i = 1,\, 2$$

and has energy eigenvalues

$$E_n \;=\; (n+1)\hbar\omega \;\equiv\; (n_1 + n_2 + 1)\hbar\omega \qquad\qquad n = 0,\, 1,\, 2, \cdots$$

What is the degeneracy of the first excited level? Use degenerate perturbation theory to determine the splitting induced by the perturbation

$$\Delta\hat{V} = C x_1 x_2$$

where $C$ is a constant.

**Hint:** the matrix elements of the perturbation may be computed by using the lowering and raising operators

$$\hat{a}_i \equiv \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x}_i + \frac{i}{\sqrt{2m\omega\hbar}}\,\hat{p}_i \qquad \text{and} \qquad \hat{a}_i^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x}_i - \frac{i}{\sqrt{2m\omega\hbar}}\,\hat{p}_i$$

with the properties that

$$\hat{a}_1|n_1, n_2\rangle = \sqrt{n_1}\,|n_1 - 1, n_2\rangle \qquad \text{and} \qquad \hat{a}_1^\dagger|n_1, n_2\rangle = \sqrt{n_1 + 1}\,|n_1 + 1, n_2\rangle \qquad \text{etc.}$$

3. Look up and write down the n=1 and 2 wavefunctions for a one electron atom (Nuclear charge Z). Evaluate the first-order energy shifts and the diagonal matrix elements for the following perturbations:

   (a) $v\hat{l_z}$
   (b) $vr^2$
   (c) $vr$
   (d) $v\exp(-2r/a_0)$
   (e) $v\exp(-Zr/a_0)$

   In each case comment on the range of $v$ for which perturbation theory is applicable.

4. The change in energy levels in an atom due to the application of an external electric field is known as the *Stark effect.* The perturbation corresponding to a uniform static electric field of magnitude $\mathcal{E}$, applied in the $z$ direction to a hydrogen atom is

   $$\Delta\hat{V} = e\mathcal{E}z = e\mathcal{E}r\cos\theta$$

   Use degenerate perturbation theory to calculate the effect on the 4-fold degenerate $n = 2$ level of atomic hydrogen.

   The relevant unperturbed eigenfunctions are given in Question 1

5. A function has Periodic Boundary Conditions if $\Phi(x) = \Phi(x + L)$ for all $x$: Why is a particle confined to a 1d ring equivalent to periodic boundary conditions?

   $2N$ non-interacting 'electrons' are confined to a 1d ring of length L. Show that their wavefunctions can be written as

   $$\Phi \propto \cos(kx)$$
   $$\Phi \propto \sin(kx)$$

   and explain what $k$ is. Draw a graph of energy against $k = \frac{2\pi}{\lambda}$. Explain why each state is fourfold degenerate

   Assuming that the electrons occupy the lowest energy states according to the Exclusion Principle, show that the highest energy of an electron is independent of $L$, provided the electron density $N/L$ is constant

   The ions in a 1D solid are represented by a perturbation

   $$V(x) = -V_o\cos^2\pi Nx/L$$

   Explain briefly why this represents divalent ions.

   Show that to first order in perturbation theory, this lowers the energy of the highest energy occupied states by $3v_0/4$ or $v_0/4$

   Explain how this perturbation affects conductivity

   *You may use the results that for any integer N*

   $$\int_0^L cos^4(2N\pi x/L)dx = 3L/8; \qquad \int_0^L cos^2(2N\pi x/L)sin^2(2N\pi x/L)dx = L/8$$