Quantum Physics 2011/12

Tutorial Sheet 4: Time-dependence

An asterisk denotes a harder problem, which you are nevertheless encouraged to try!

1. An easy one to start with! A particle moving in the infinite 1-d square well potential

$$V(x) = 0 \quad \text{for } |x| < a, \qquad V(x) = \infty \quad \text{for } |x| > a$$

   is set up in the initial state ($t = 0$) described by the wavefunction

$$\Psi(x,0) \equiv \psi(x) = [u_1(x) + u_2(x)]\,/\sqrt{2}$$

   where $u_1(x)$, $u_2(x)$ are the energy eigenfunctions corresponding to the energy eigenvalues E_1 and E_2 respectively. Sketch the probability density at $t = 0$.

   What is the wavefunction at time t?

   Calculate the probabilities P_1 and P_2 that at $t = 0$ a measurement of the total energy yields the results E_1 and E_2 respectively. Do P_1 and P_2 change with time?

   Calculate the probabilities $P_+(t)$ and $P_-(t)$ that at time t the particle is in the intervals $0 < x < a$ and $-a < x < 0$ respectively and try to interpret your results.

2. A system has just two independent states, $|1\rangle$ and $|2\rangle$, represented by the column matrices

$$|1\rangle \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad |2\rangle \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   With respect to these two states, the Hamiltonian has a time-independent matrix representation

$$\begin{pmatrix} E & V \\ V & E \end{pmatrix}$$

   where E and V are both real.

   Show that the probability of a transition from the state $|1\rangle$ to the state $|2\rangle$ in the time interval t is given *without approximation* by

$$p(t) = \sin^2\left(\frac{Vt}{\hbar}\right)$$

   [**Hint:** expand the general state $|\Psi, t\rangle$ in terms of $|1\rangle$ and $|2\rangle$ and substitute in the TDSE. Note that $|1\rangle$ and $|2\rangle$ are not energy eigenstates!]

   Compute the transition probability using first-order time-dependent perturbation theory, taking the unperturbed Hamiltonian matrix to be that for which $|1\rangle$ and $|2\rangle$ *are* energy eigenstates. By comparing with the exact result, deduce the conditions under which you expect the approximation to be good.

3. * A 1-d harmonic oscillator of charge q is acted upon by a uniform electric field which may be considered to be a perturbation and which has time dependence of the form

$$\mathcal{E}(t) = \frac{A}{\sqrt{\pi}\,\tau} \exp\left\{-(t/\tau)^2\right\}$$

Assuming that when $t = -\infty$, the oscillator is in its ground state, evaluate the probability that it is in its first excited state at $t = +\infty$ using time-dependent perturbation theory. You may assume that

$$\begin{aligned}\int_{-\infty}^{\infty} \exp(-y^2)\, dy &= \sqrt{\pi} \\ \langle n+1|\hat{x}|n\rangle &= \sqrt{\frac{(n+1)\hbar}{2m\omega}} \\ \langle n+i|\hat{x}|n\rangle &= 0 \qquad\qquad -1 > i > 1\end{aligned}$$

Discuss the behaviour of the transition probability and the applicability of the perturbation theory result when (a) $\tau \ll \dfrac{1}{\omega}$, and (b) $\tau \gg \dfrac{1}{\omega}$.

4. The Hamiltonian which describes the interaction of a static spin-$\frac{1}{2}$ particle with an external magnetic field, $\underline{B}$, is

$$\hat{H} = -\underline{\hat{\mu}} \cdot \underline{B}$$

When $\underline{B}$ is a static uniform magnetic field in the z-direction, $\underline{B}_0 = (0, 0, B_0)$, the matrix representation of $\hat{H}_0$ is simply

$$-\frac{1}{2}\gamma B_0 \hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with eigenvalues $\mp\frac{1}{2}\gamma B_0 \hbar$ and for this time-independent Hamiltonian, the energy eigenstates are represented by the 2-component column matrices

$$|\uparrow\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\downarrow\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now consider superimposing on the static field $\underline{B}_0$ a time-dependent magnetic field of constant magnitude B_1, rotating in the $x - y$ plane with constant angular frequency ω:

$$\underline{B}_1(t) = (B_1 \cos\omega t, B_1 \sin\omega t, 0)$$

If the Hamiltonian is now written as $\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$, write down a matrix representation of $\hat{V}(t)$.

Any spin state can be written

$$|\Psi, t\rangle = c_1(t) \exp(-iE_{\uparrow}t/\hbar)|\uparrow\rangle + c_2(t) \exp(-iE_{\downarrow}t/\hbar)|\downarrow\rangle$$

Obtain, without approximation, the coupled equations for the amplitudes $c_1(t), c_2(t)$.

$*$ If initially at $t = 0$ the system is in the spin-down state, show that the probability that at time t, the system is in the spin-up state is given without approximation by

$$p_1(t) = |c_1(t)|^2 = A \sin^2 \left\{ \frac{1}{2} \left[(\gamma B_1)^2 + (\omega + \gamma B_0)^2 \right]^{1/2} t \right\}$$

where

$$A = \frac{(\gamma B_1)^2}{\{[(\gamma B_1)^2 + (\omega + \gamma B_0)^2]\}}$$

What is the corresponding probability, $p_2(t)$, that the system is in the spin-down state? Sketch $p_1(t)$ and $p_2(t)$ as functions of time.