Quantum Physics 2011/12

Tutorial Sheet 6: Time-Dependence and Pseudopotentials

One harder problem, which you are nevertheless encouraged to try, is available online

For the first two problems, you may assume that the hydrogen eigenfunctions are:

$$\begin{aligned}
u_{100} &= (\pi a_0^3)^{-1/2} \exp(-r/a_0) \\
u_{211} &= -\left(\pi a_0^3\right)^{-1/2} \frac{r}{8a_0} \sin\theta \exp(i\phi) \exp(-r/2a_0) \\
u_{210} &= \left(8\pi a_0^3\right)^{-1/2} \frac{r}{2a_0} \cos\theta \exp(-r/2a_0) \\
u_{21-1} &= \left(\pi a_0^3\right)^{-1/2} \frac{r}{8a_0} \sin\theta \exp(-i\phi) \exp(-r/2a_0)
\end{aligned}$$

and

$$\int_0^\infty \exp(-br)\, r^n \, \mathrm{d}r = n!/b^{n+1}, \qquad n > -1$$

1. A hydrogen atom is placed in a uniform but time-dependent electric field of magnitude:

$$\mathcal{E} = 0 \text{ for } t < 0, \qquad \mathcal{E} = \mathcal{E}_0 \exp(-t/\tau) \quad \text{for } t \geq 0 \quad (\tau > 0)$$

where $\mathcal{E}_/$ is a constant. At time $t = 0$, the atom is in the ground ($1s$) state. Show that the probability, to lowest order in perturbation theory, that as $t \to \infty$, the atom is in the $2p$ state in which the component of the orbital angular momentum in the direction of the field is zero, is given by

$$p_{1s\to 2p} = |c(\infty)|^2 = \frac{2^{15}}{3^{10}} \frac{(e\mathcal{E}_0 a_0)^2}{(E_{2p} - E_{1s})^2 + (\hbar/\tau)^2}$$

What is the probability that it is in the $2s$-state?

[**Hint:** take the field direction to be the z-direction. Write down the potential energy of the electron in the given field and treat as a time-dependent perturbation].

2. The neutral kaon states produced via the strong interaction are $|K^0\rangle$ and $|\overline{K}^0\rangle$, eigenvalues of strangeness $\hat{S}$ such that $\hat{S}|K^0\rangle = |K^0\rangle$ and $\hat{S}|\overline{K}^0\rangle = -|\overline{K}^0\rangle$. Acting on these states with a weak force-related operator $\hat{CP}$ gives:

$$\hat{CP}|K^0\rangle = |\overline{K}^0\rangle; \qquad \hat{CP}|\overline{K}^0\rangle = |K^0\rangle$$

Evaluate the CP eigenstates for neutral kaons $|K_1\rangle$ and $|K_2\rangle$ with eigenvalues $CP = \pm 1$.

Kaons decay into pions via the weak force with lifetimes:

$$\tau_1 = 0.9 \times 10^{-10} \text{ (for } |K_1\rangle) \text{ and } \tau_2 = 0.5 \times 10^{-7} \text{ (for } |K_2\rangle).$$

Show that the wavefunction $a_1(t) = a_1(0)e^{-t/2\tau}e^{-iEt/\hbar}|\Phi(\mathbf{r})\rangle$, where E is the energy, represents decay with lifetime τ, and that the amplitudes of the $|K_1\rangle$ and the $|K_2\rangle$ states at rest (i.e. with $E = mc^2$) are:

$$a_1(t) = a_1(0)e^{-t/2\tau_1}e^{-im_1c^2t/\hbar} \qquad \text{and} \qquad a_2(t) = a_2(0)e^{-t/2\tau_2}e^{-im_2c^2t/\hbar}$$

By considering a general state $a|K^0\rangle + b|\overline{K^0}\rangle$ how that the intensity of K^0 (i.e.$|a|^2$) is measured by the operator $\frac{1}{2}(\hat{S}+1)$. What is the operator for the intensity of $|\overline{K}^0\rangle$?

At t=0 a kaon beam is in a pure $|K^0\rangle$ state, with intensity proportional to $|a_0(0)|^2 = 1$ show that at time t

$$|a_0(t)|^2 = \langle \frac{1}{2}(\hat{S}+1)\rangle = \frac{1}{4}\left[e^{-t/\tau_1} + e^{-t/\tau_2} + 2e^{-t/2\tau_1}e^{-t/2\tau_2}\cos(m_{12}t)\right]$$

$$\text{and} \quad I(\overline{K}^0) = \frac{1}{4}\left[e^{-t/\tau_1} + e^{-t/\tau_2} - 2e^{-t/2\tau_1}e^{-t/2\tau_2}\cos(m_{12}t)\right]$$

where $m_{12} = (m_2 - m_1)c^2/\hbar$. What is there a physical reason for the intensities being unchanged if we use $m_{12} = (m_1 - m_2)c^2/\hbar$?

Sketch, as a function of time, the expectation values of: $\frac{1}{2}(\hat{S}+1)$,$\frac{1}{2}(1-\hat{S})$, $\hat{S}$, $\hat{CP}$, $\frac{1}{2}(\hat{CP}+1)$, in a system which began in state $|K^0\rangle$.

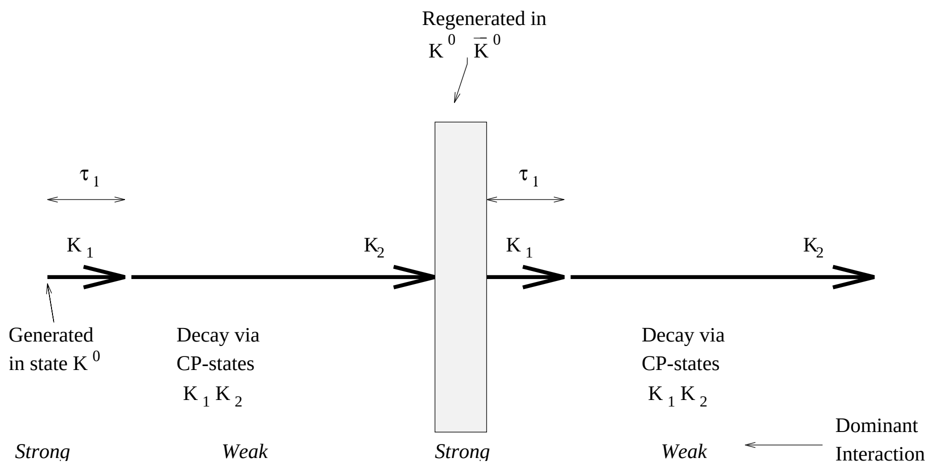

Figure 1 shows an experiment where kaons are generated in state $|K^0\rangle$. After 10^{-9} seconds, assuming that all the $|K_1\rangle$ particles have decayed, the kaons pass through a small region of matter, where they interact via the strong interaction. Very soon after, the kaons leave the matter and move into a region of vacuum where they begin to decay via the weak interaction ($\hat{CP}$). Assume that all coherence between this region and the previous region is lost i.e. the wavefunction is completely collapsed onto its strong interaction eigenstates. Evaluate the appropriate eigenstates and intensities just before the beam enters the matter and just after it leaves the matter. What is the total intensity of kaons and antikaons which survive a further 10^{-9} seconds?

Had it not been for the matter, what would have been the total intensity of kaons and antikaons after 2×10^{-9}?

3. Compare question 3 to a system of circular and plane polarisers and light beams.

4. This question illustrates the principle of the pseudopotential.

   A particle is bound in 1D by a potential which has a complicated form for $|x| < x_c$ but is zero outside this "cut-off" radius. It is known to have a bound eigenstate with energy $-E_0$. Show that in this region of space, the wavefunction can be written as

$$\Phi(x > x_c) = a\exp(-k|x|)$$

   and determine k. What can you say about a?

   Now suppose that we know that the normalisation constant $a = a_0$. Show that the ground state of a finite square well pseudopotential can be used to give exactly the same wavefunction i.e. $\Phi_{PS}(x) = \Phi(x)$, for $x > x_c$.

   Writing the ground state wavefunction of the square well as:

$$\Phi_{PS}(x) = b\cos(k_1 x) \qquad |x| < x_c$$

$$\Phi_{PS}(x) = \Phi(x) = a\exp(-k|x|) \qquad |x| > x_c$$

   where $k_1 = \sqrt{2m(V - E_0)}/\hbar$, determine three simultaneous equations for the required values of the finite well depth V and range x_c, and the normalization constant b.

   When are such pseudopotentials useful?

5. * The electric dipole moment operator is $\underline{\hat{D}} \equiv -e\underline{r}$. The position vector can be written

$$\underline{r} = r\left\{\underline{e}_1 \sin\theta\cos\phi + \underline{e}_2 \sin\theta\sin\phi + \underline{e}_3 \cos\theta\right\}$$

where $\underline{e}_i,\ i = 1, 2, 3,$ are the usual Cartesian unit vectors in the x, y, z directions and θ, ϕ are the polar and azimuthal angles in spherical polar coordinates.

Calculate the dipole matrix elements for the radiative transition from the $n = 2$ states to the $1s$ state of atomic hydrogen.

The Einstein spontaneous transition rate for the $2p \rightarrow 1s$ transition is given by

$$R_{mk}^{\text{spon}} = \frac{\omega_{mk}^3}{3\pi c^3 \hbar \epsilon_0} \left|\langle m|\underline{\hat{D}}|k\rangle\right|^2 = \frac{e^2\omega_{mk}^3}{3\pi c^3 \hbar \epsilon_0} \left|\langle m|\underline{\hat{r}}|k\rangle\right|^2$$

where k and m label initial (2p) and final (1s) states and $\alpha \equiv e^2/(4\pi\epsilon_0)\hbar c$ is the fine structure constant.

Explain in words how this relates to the Fermi Golden Rule.

Assuming that an initial $2p$ state is unpolarised; that is, each of the three possible values of m_ℓ is equally likely, show that this is equal to.

$$R_{2p\rightarrow 1s}^{spon} = \left(\frac{2}{3}\right)^8 \frac{mc^2}{\hbar}\,\alpha^5$$