Tutorial Sheet 8: Counting states

† denotes an online-only bonus question.

1. † A helium atom has two electrons in states nlm and $n'l'm'$. Write down wavefunctions which are antisymmetric with respect to exchange, and calculate the degeneracy for all combinations with $n = 1$, $n' \leq 3$, ignoring electron-electron interaction.

2. † In muonic helium, one of the electrons is replaced by a muon. Write down the appropriate wavefunctions for states equivalent to those in question 1.

3. The isotropic harmonic oscillator in 3 dimensions is described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{r}^2$$

where $\hat{p}^2 \equiv \underline{\hat{p}} \cdot \underline{\hat{p}}$ and $\hat{r}^2 \equiv \underline{\hat{r}} \cdot \underline{\hat{r}}$. By writing $\hat{H}$ as

$$\hat{H} = \sum_i \left\{ \frac{\hat{p}_i^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_i^2 \right\} \qquad i = 1,\, 2,\, 3$$

show that the eigenfunctions of $\hat{H}$ are simply products of 1-dimensional oscillator eigenfunctions. Assuming the usual formula for the energy eigenvalues of a 1-dimensional harmonic oscillator, show that the eigenvalues of $\hat{H}$ are given by

$$E_n = (n + \frac{3}{2})\,\hbar\omega \qquad n = 0,\, 1,\, 2,\, \ldots$$

If two identical, spin 1/2, fermions are placed in the 3D-SHO potential, what is the degeneracy of the ground, first and second excited state?

By considering the number of ways that a fixed integer, n, can be partitioned into three non-negative integers, show that the eigenvalues are $\frac{1}{2}(n+1)(n+2)$–fold degenerate.

4. The 3-dimensional isotropic harmonic oscillator can also be solved in spherical polar coordinates, since $V(r)$ is a central potential, with energy eigenstates labelled by n, ℓ and m_z, as for the hydrogen atom (except that, by convention, the ground state is labelled by $n = 0$ rather than $n = 1$). By using the parity properties of the eigenfunctions and the degree of degeneracy, try to find the angular momentum quantum number, ℓ, associated with the lowest three energy levels.

5. A 1-d infinite square-well potential between 0 and 2a is occupied by two indistinguishable spin-$\frac{1}{2}$ fermions. Write down the possible degenerate states for the ground state and first two excited states and show that if the fermions are non-interacting, the ground-state energy is the same as for distinguishable particles.

   If the particles repel one another via a delta function potential,

$$V(x_1, x_2) = 2aV_0\delta(x_1 - x_2)$$

use perturbation theory to evaluate the shift in energy of the ground state and the split between singlet and triplet states.

Now evaluate the energy shift and split for the first excited state due to the interaction, comparing the case of indistinguishable particles with that for distinguishable ones.

Comment on the value of the triplet state with reference to the exclusion principle.

You are given that $\int\int \delta(x_1, x_2) f(x_1, x_2) dx_1 dx_2 = \int f(x, x) dx$;
$\int_0^1 \sin^4(\pi y) dy = 3/8$; and $\int_0^{2a} \sin^2(\pi x/2a) \sin^2(\pi x/a) dx = a/2$